

# SOME RETARDED NONLINEAR INTEGRAL INEQUALITIES IN TWO VARIABLES AND APPLICATIONS

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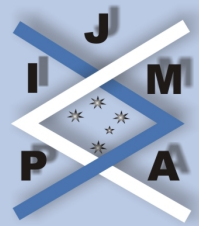
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*Key words:* Integral inequality, Nonlinear, Two variables, Retarded.

*Abstract:* In this paper, some retarded nonlinear integral inequalities in two variables with more than one distinct nonlinear term are established. Our results are also applied to show the boundedness of the solutions of certain partial differential equations.



Retarded Nonlinear Integral  
Inequalities

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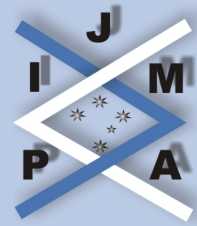
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## 1. Introduction

The Gronwall-Bellman integral inequality plays an important role in the qualitative analysis of the solutions of differential and integral equations. During the past few years, many retarded inequalities have been discovered (see in [1, 2, 4, 5, 6, 10, 11]). Lipovan [4] investigated the following retarded inequality

$$(1.1) \quad u(t) \leq a + \int_{b(t_0)}^{b(t)} f(s)w(u(s))ds, \quad t_0 \leq t \leq t_1,$$

and Agarwal et al. [6] generalized (1.1) to a more general case as follows

$$(1.2) \quad u(t) \leq a(t) + \sum_{i=1}^n \int_{b_i(t_0)}^{b_i(t)} f_i(s)w_i(u(s))ds, \quad t_0 \leq t \leq t_1.$$

Recently, many people such as Wang [10], Cheung [9] and Dragomir [8] established some new integral inequalities involving functions of two independent variables and Zhao et al. [11] also established advanced integral inequalities.

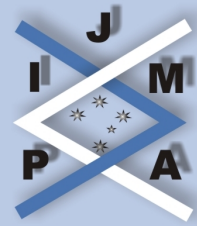
The purpose of this paper, motivated by the works of Agarwal [6], Cheung [9] and Zhao [11], is to discuss more general integral inequalities with  $n$  nonlinear terms

$$(1.3) \quad u(x, y) \leq a(x, y) + \sum_{i=1}^n \int_{\alpha_i(0)}^{\alpha_i(x)} \int_{\beta_i(y)}^{\infty} f_i(x, y, s, t)w_i(u(s, t))dtds$$

and

$$(1.4) \quad u(x, y) \leq a(x, y) + \sum_{i=1}^n \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} f_i(x, y, s, t)w_i(u(s, t))dtds.$$

Our results can be used more effectively to study the boundedness and uniqueness of the solutions of certain partial differential equations. Moreover, at the end of this paper, an example is presented to show the applications of our results.



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## 2. Statement of Main Results

Let  $\mathbb{R} = (-\infty, \infty)$  and  $\mathbb{R}_+ = [0, \infty)$ .  $D_1z(x, y)$  and  $D_2z(x, y)$  denote the first-order partial derivatives of  $z(x, y)$  with respect to  $x$  and  $y$  respectively.

As in [6], define  $w_1 \propto w_2$  for  $w_1, w_2 : A \subset \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$  if  $\frac{w_2}{w_1}$  is nondecreasing on  $A$ . Assume that

(B<sub>1</sub>)  $w_i(u)$  ( $i = 1, \dots, n$ ) is a nonnegative, nondecreasing and continuous function for  $u \in \mathbb{R}_+$  with  $w_i(u) > 0$  for  $u > 0$  such that  $w_1 \propto w_2 \propto \dots \propto w_n$ ;

(B<sub>2</sub>)  $a(x, y)$  is a nonnegative and continuous function for  $x, y \in \mathbb{R}_+$ ;

(B<sub>3</sub>)  $f_i(x, y, s, t)$  ( $i = 1, \dots, n$ ) is a continuous and nonnegative function for  $x, y, s, t \in \mathbb{R}_+$ .

Take the notation  $W_i(u) := \int_{u_i}^u \frac{dz}{w_i(z)}$  for  $u \geq u_i$ , where  $u_i > 0$  is a given constant. Clearly,  $W_i$  is strictly increasing, so its inverse  $W_i^{-1}$  is well defined, continuous and increasing in its corresponding domain.

**Theorem 2.1.** *Under the assumptions (B<sub>1</sub>), (B<sub>2</sub>) and (B<sub>3</sub>), suppose  $a(x, y)$  and  $f_i(x, y, s, t)$  are bounded in  $y \in \mathbb{R}_+$ . Let  $\alpha_i(x), \beta_i(y)$  be nonnegative, continuously differentiable and nondecreasing functions with  $\alpha_i(x) \leq x$  and  $\beta_i(y) \geq y$  on  $\mathbb{R}_+$  for  $i = 1, 2, \dots, n$ . If  $u(x, y)$  is a continuous and nonnegative function satisfying (1.3), then*

$$(2.1) \quad u(x, y) \leq W_n^{-1} \left[ W_n(b_n(x, y)) + \int_{\alpha_n(0)}^{\alpha_n(x)} \int_{\beta_n(y)}^{\infty} \tilde{f}_n(x, y, s, t) dt ds \right]$$



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for all  $0 \leq x \leq x_1, y_1 \leq y < \infty$ , where  $b_n(x, y)$  is determined recursively by

$$(2.2) \quad b_1(x, y) = \sup_{0 \leq \tau \leq x} \sup_{y \leq \mu < \infty} a(\tau, \mu),$$

$$b_{i+1}(x, y) = W_i^{-1} \left[ W_i(b_i(x, y)) + \int_{\alpha_i(0)}^{\alpha_i(x)} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(x, y, s, t) dt ds \right],$$

$$\tilde{f}_i(x, y, s, t) = \sup_{0 \leq \tau \leq x} \sup_{y \leq \mu < \infty} f_i(\tau, \mu, s, t),$$

$W_1(0) := 0$ , and  $x_1, y_1 \in \mathbb{R}_+$  are chosen such that

$$(2.3) \quad W_i(b_i(x_1, y_1)) + \int_{\alpha_i(0)}^{\alpha_i(x_1)} \int_{\beta_i(y_1)}^{\infty} \tilde{f}_i(x, y, s, t) dt ds \leq \int_{u_i}^{\infty} \frac{dz}{w_i(z)}$$

for  $i = 1, \dots, n$ .

The proof of Theorem 2.1 will be given in the next section.

*Remark 1.* As in [6], different choices of  $u_i$  in  $W_i$  do not affect our results. If all  $w_i$  ( $i = 1, \dots, n$ ) satisfy  $\int_{u_i}^{\infty} \frac{dz}{w_i(z)} = \infty$ , then (2.1) is true for all  $x, y \in \mathbb{R}_+$ .

*Remark 2.* As in [10], if  $w_i(u)$  ( $i = 1, \dots, n$ ) are continuous functions on  $\mathbb{R}_+$  and positive on  $(0, \infty)$  but the sequence of  $\{w_i(u)\}$  does not satisfy  $w_1 \propto w_2 \propto \dots \propto w_n$ , we can use a technique of monotonization of the sequence of functions  $w_i(u)$ , calculated by

$$(2.4) \quad \tilde{w}_1(u) := \max_{\theta \in [0, u]} w_1(\theta),$$

$$\tilde{w}_{i+1}(u) := \max_{\theta \in [0, u]} \left\{ \frac{w_{i+1}(\theta)}{\tilde{w}_i(\theta)} \right\} \tilde{w}_i(u), \quad i = 1, \dots, n-1.$$



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Clearly,  $\tilde{w}_i(u) \geq w_i(u)$  ( $i = 1, \dots, n$ ). (1.3) and (1.4) can also become

$$(2.5) \quad u(x, y) \leq a(x, y) + \sum_{i=1}^n \int_{\alpha_i(0)}^{\alpha_i(x)} \int_{\beta_i(y)}^{\infty} f_i(x, y, s, t) \tilde{w}_i(u(s, t)) dt ds$$

and

$$(2.6) \quad u(x, y) \leq a(x, y) + \sum_{i=1}^n \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} f_i(x, y, s, t) \tilde{w}_i(u(s, t)) dt ds,$$

where the function sequence  $\{\tilde{w}_i(u)\}$  satisfies the assumption  $(B_1)$ .

**Theorem 2.2.** *Under the assumptions  $(B_1)$ ,  $(B_2)$  and  $(B_3)$ , suppose  $a(x, y)$  and  $f_i(x, y, s, t)$  are bounded in  $x, y \in \mathbb{R}_+$ . Let  $\alpha_i(x)$ ,  $\beta_i(y)$  be nonnegative, continuously differentiable and nondecreasing functions with  $\alpha_i(x) \geq x$  and  $\beta_i(y) \geq y$  on  $\mathbb{R}_+$  for  $i = 1, 2, \dots, n$ . If  $u(x, y)$  is a continuous and nonnegative function satisfying (1.4), then*

$$(2.7) \quad u(x, y) \leq W_n^{-1} \left[ W_n(b_n(x, y)) + \int_{\alpha_n(x)}^{\infty} \int_{\beta_n(y)}^{\infty} \hat{f}_n(x, y, s, t) dt ds \right]$$

for all  $\hat{x}_1 \leq x < \infty$ ,  $\hat{y}_1 \leq y < \infty$ , where  $b_n(x, y)$  is determined recursively by

$$(2.8) \quad \begin{aligned} b_1(x, y) &= \sup_{x \leq \tau < \infty} \sup_{y \leq \mu < \infty} a(\tau, \mu), \\ b_{i+1}(x, y) &= W_i^{-1} \left[ W_i(b_i(x, y)) + \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \hat{f}_i(x, y, s, t) dt ds \right], \\ \hat{f}_i(x, y, s, t) &= \sup_{x \leq \tau < \infty} \sup_{y \leq \mu < \infty} f_i(\tau, \mu, s, t), \end{aligned}$$



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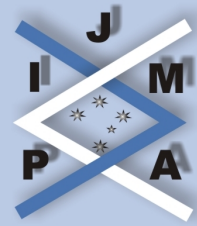
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$W_1(0) := 0$ , and  $\hat{x}_1, \hat{y}_1 \in \mathbb{R}_+$  are chosen such that

$$(2.9) \quad W_i(b_i(\hat{x}_1, \hat{y}_1)) + \int_{\alpha_i(\hat{x}_1)}^{\infty} \int_{\beta_i(\hat{y}_1)}^{\infty} \hat{f}_i(x, y, s, t) dt ds \leq \int_{u_i}^{\infty} \frac{dz}{w_i(z)}$$

for  $i = 1, \dots, n$ .

The proof is similar to the argument in the proof of Theorem 2.1 with suitable modifications. In the next section, we omit its proof.



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### 3. Proof of Theorem 2.1

From the assumptions, we know that  $b_1(x, y)$  and  $\tilde{f}_i(x, y, s, t)$  are well defined. Moreover,  $\tilde{a}(x, y)$  and  $\tilde{f}_i(x, y, s, t)$  are nonnegative, nondecreasing in  $x$  and non-increasing in  $y$  and satisfy  $b_1(x, y) \geq a(x, y)$  and  $\tilde{f}_i(x, y, s, t) \geq f_i(x, y, s, t)$  for each  $i = 1, \dots, n$ .

We first discuss the case  $a(x, y) > 0$  for all  $x, y \in \mathbb{R}_+$ . From (1.3), we have

$$(3.1) \quad u(x, y) \leq b_1(x, y) + \sum_{i=1}^n \int_{\alpha_i(0)}^{\alpha_i(x)} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(x, y, s, t) w_i(u(s, t)) dt ds.$$

Choose arbitrary  $\tilde{x}_1, \tilde{y}_1$  such that  $0 \leq \tilde{x}_1 \leq x_1, y_1 \leq \tilde{y}_1 < \infty$ . From (3.1), we obtain

$$(3.2) \quad u(x, y) \leq b_1(\tilde{x}_1, \tilde{y}_1) + \sum_{i=1}^n \int_{\alpha_i(0)}^{\alpha_i(x)} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, s, t) w_i(u(s, t)) dt ds$$

for all  $0 \leq x \leq \tilde{x}_1 \leq x_1, y_1 \leq \tilde{y}_1 \leq y < \infty$ .

We claim that

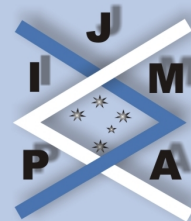
$$(3.3) \quad u(x, y) \leq W_n^{-1} \left[ W_n(\tilde{b}_n(\tilde{x}_1, \tilde{y}_1, x, y)) + \int_{\alpha_n(0)}^{\alpha_n(x)} \int_{\beta_n(y)}^{\infty} \tilde{f}_n(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right]$$

for all  $0 \leq x \leq \min\{\tilde{x}_1, x_2\}, \max\{\tilde{y}_1, y_2\} \leq y < \infty$ , where

$$\tilde{b}_1(\tilde{x}_1, \tilde{y}_1, x, y) = b_1(\tilde{x}_1, \tilde{y}_1),$$

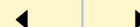
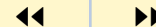
$$(3.4) \quad \begin{aligned} & \tilde{b}_{i+1}(\tilde{x}_1, \tilde{y}_1, x, y) \\ &= W_i^{-1} \left[ W_i(\tilde{b}_i(\tilde{x}_1, \tilde{y}_1, x, y)) + \int_{\alpha_i(0)}^{\alpha_i(x)} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right] \end{aligned}$$





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for  $i = 1, \dots, n - 1$  and  $x_2, y_2 \in \mathbb{R}_+$  are chosen such that

$$(3.5) \quad W_i(\tilde{b}_i(\tilde{x}_1, \tilde{y}_1, x_2, y_2)) + \int_{\alpha_i(0)}^{\alpha_i(x_2)} \int_{\beta_i(y_2)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \leq \int_{u_i}^{\infty} \frac{dz}{w_i(z)}$$

for  $i = 1, \dots, n$ .

Note that we may take  $x_2 = x_1$  and  $y_2 = y_1$ . In fact,  $\tilde{b}_i(\tilde{x}_1, \tilde{y}_1, x, y)$  and  $\tilde{f}_i(\tilde{x}_1, \tilde{y}_1, x, y)$  are nondecreasing in  $\tilde{x}_1$  and nonincreasing in  $\tilde{y}_1$  for fixed  $x, y$ . Furthermore, it is easy to check that  $\tilde{b}_i(\tilde{x}_1, \tilde{y}_1, \tilde{x}_1, \tilde{y}_1) = b_i(\tilde{x}_1, \tilde{y}_1)$  for  $i = 1, \dots, n$ . If  $x_2$  and  $y_2$  are replaced by  $x_1$  and  $y_1$  on the left side of (3.5) respectively, from (2.3) we have

$$\begin{aligned} W_i(\tilde{b}_i(\tilde{x}_1, \tilde{y}_1, x_1, y_1)) + \int_{\alpha_i(0)}^{\alpha_i(x_1)} \int_{\beta_i(y_1)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \\ \leq W_i(\tilde{b}_i(x_1, y_1, x_1, y_1)) + \int_{\alpha_i(0)}^{\alpha_i(x_1)} \int_{\beta_i(y_1)}^{\infty} \tilde{f}_i(x_1, y_1, s, t) dt ds \\ = W_i(b_i(x_1, y_1)) + \int_{\alpha_i(0)}^{\alpha_i(x_1)} \int_{\beta_i(y_1)}^{\infty} \tilde{f}_i(x_1, y_1, s, t) dt ds \\ \leq \int_{u_i}^{\infty} \frac{dz}{w_i(z)}. \end{aligned}$$

Thus, we can take  $x_2 = x_1, y_2 = y_1$ .

In the following, we will use mathematical induction to prove (3.3).

For  $n = 1$ , let

$$z(x, y) = b_1(\tilde{x}_1, \tilde{y}_1) + \int_{\alpha_1(0)}^{\alpha_1(x)} \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, s, t) w_1(u(s, t)) dt ds.$$



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Then  $z(x, y)$  is differentiable, nonnegative, nondecreasing for  $x \in [0, \tilde{x}_1]$  and nonincreasing for  $y \in [\tilde{y}_1, \infty]$  and  $z(0, y) = z(x, \infty) = b_1(\tilde{x}_1, \tilde{y}_1)$ . From (3.2), we have

$$(3.6) \quad u(x, y) \leq z(x, y).$$

Considering  $\alpha_1(x) \leq x$  and  $\alpha_1'(x) \geq 0$  for  $x \in \mathbb{R}_+$ , we have

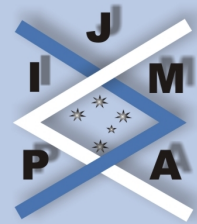
$$(3.7) \quad \begin{aligned} D_1 z(x, y) &= \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, \alpha_1(x), t) w_1(u(\alpha_1(x), t)) dt \alpha_1'(x) \\ &\leq \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, \alpha_1(x), t) w_1(z(\alpha_1(x), t)) dt \alpha_1'(x) \\ &\leq w_1(z(x, y)) \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, \alpha_1(x), t) dt \alpha_1'(x). \end{aligned}$$

Since  $w_1$  is nondecreasing and  $z(x, y) > 0$ , we get

$$(3.8) \quad \frac{D_1(z(x, y))}{w_1(z(x, y))} \leq \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, \alpha_1(x), t) dt \alpha_1'(x).$$

Integrating both sides of the above inequality from 0 to  $x$ , we obtain

$$(3.9) \quad \begin{aligned} W_1(z(x, y)) &\leq W_1(z(0, y)) + \int_0^x \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, \alpha_1(s), t) \alpha_1'(s) dt ds \\ &= W_1(b_1(\tilde{x}_1, \tilde{y}_1)) + \int_{\alpha_1(0)}^{\alpha_1(x)} \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, s, t) dt ds. \end{aligned}$$



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Thus the monotonicity of  $W_1^{-1}$  and (3.5) imply

$$u(x, y) \leq z(x, y) \leq W_1^{-1} \left[ W_1(b_1(\tilde{x}_1, \tilde{y}_1)) + \int_{\alpha_1(0)}^{\alpha_1(x)} \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right],$$

namely, (3.3) is true for  $n = 1$ .

Assume that (3.3) is true for  $n = m$ . Consider

$$u(x, y) \leq b_1(\tilde{x}_1, \tilde{y}_1) + \sum_{i=1}^{m+1} \int_{\alpha_i(0)}^{\alpha_i(x)} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, s, t) w_i(u(s, t)) dt ds$$

for all  $0 \leq x \leq \tilde{x}_1, \tilde{y}_1 \leq y < \infty$ . Let

$$z(x, y) = b_1(\tilde{x}_1, \tilde{y}_1) + \sum_{i=1}^{m+1} \int_{\alpha_i(0)}^{\alpha_i(x)} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, s, t) w_i(u(s, t)) dt ds.$$

Then  $z(x, y)$  is differentiable, nonnegative, nondecreasing for  $x \in [0, \tilde{x}_1]$  and nonincreasing for  $y \in [\tilde{y}_1, \infty]$ . Obviously,  $z(0, y) = z(x, 0) = b_1(\tilde{x}_1, \tilde{y}_1)$  and  $u(x, y) \leq z(x, y)$ . Since  $w_1$  is nondecreasing and  $z(x, y) > 0$ , noting that  $\alpha_i(x) \leq x$  and  $\alpha_i'(x) \geq 0$  for  $x \in \mathbb{R}_+$ , we have

$$\begin{aligned} \frac{D_1(z(x, y))}{w_1(z(x, y))} &\leq \frac{\sum_{i=1}^{m+1} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, \alpha_i(x), t) w_i(u(\alpha_i(x), t)) dt \alpha_i'(x)}{w_1(z(x, y))} \\ &\leq \frac{\sum_{i=1}^{m+1} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, \alpha_i(x), t) w_i(z(\alpha_i(x), t)) dt \alpha_i'(x)}{w_1(z(x, y))} \end{aligned}$$



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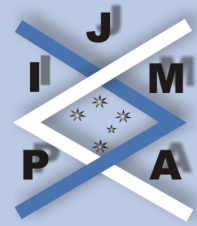
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$$\begin{aligned}
 &\leq \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, \alpha_1(x), t) dt \alpha'_1(x) \\
 &\quad + \sum_{i=2}^{m+1} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, \alpha_i(x), t) \phi_i(z(\alpha_i(x), t)) dt \alpha'_i(x) \\
 &\leq \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, \alpha_1(x), t) dt \alpha'_1(x) \\
 &\quad + \sum_{i=1}^m \int_{\beta_{i+1}(y)}^{\infty} \tilde{f}_{i+1}(\tilde{x}_1, \tilde{y}_1, \alpha_{i+1}(x), t) \phi_{i+1}(z(\alpha_{i+1}(x), t)) dt \alpha'_{i+1}(x),
 \end{aligned}$$

where  $\phi_{i+1}(u) = \frac{w_{i+1}(u)}{w_i(u)}$ ,  $i = 1, \dots, m$ . Integrating the above inequality from 0 to  $x$ , we obtain

$$\begin{aligned}
 &W_1(z(x, y)) \\
 &\leq W_1(b_1(\tilde{x}_1, \tilde{y}_1)) + \int_0^x \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, \alpha_1(s), t) \alpha'_1(s) dt ds \\
 &\quad + \sum_{i=1}^m \int_0^x \int_{\beta_{i+1}(y)}^{\infty} \tilde{f}_{i+1}(\tilde{x}_1, \tilde{y}_1, \alpha_{i+1}(s), t) \phi_{i+1}(z(\alpha_{i+1}(s), t)) \alpha'_{i+1}(s) dt ds \\
 &\leq W_1(b_1(\tilde{x}_1, \tilde{y}_1)) + \int_{\alpha_1(0)}^{\alpha_1(x)} \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \\
 &\quad + \sum_{i=1}^m \int_{\alpha_{i+1}(0)}^{\alpha_{i+1}(x)} \int_{\beta_{i+1}(y)}^{\infty} \tilde{f}_{i+1}(\tilde{x}_1, \tilde{y}_1, s, t) \phi_{i+1}(z(s, t)) dt ds,
 \end{aligned}$$



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or

$$\xi(x, y) \leq c_1(x, y) + \sum_{i=1}^m \int_{\alpha_{i+1}(0)}^{\alpha_{i+1}(x)} \int_{\beta_{i+1}(y)}^{\infty} \tilde{f}_{i+1}(\tilde{x}_1, \tilde{y}_1, s, t) \phi_{i+1}(W_1^{-1}(\xi(s, t))) dt ds$$

for  $0 \leq x \leq \tilde{x}_1$ ,  $\tilde{y}_1 \leq y < \infty$ . This is the same as (3.3) for  $n = m$ , where  $\xi(x, y) = W_1(z(x, y))$  and

$$c_1(x, y) = W_1(b_1(\tilde{x}_1, \tilde{y}_1)) + \int_{\alpha_1(0)}^{\alpha_1(x)} \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, s, t) dt ds.$$

From the assumption  $(B_1)$ , each  $\phi_{i+1}(W_1^{-1}(u))$  ( $i = 1, \dots, m$ ) is continuous and nondecreasing for  $u$ . Moreover,  $\phi_2(W_1^{-1}) \propto \phi_3(W_1^{-1}) \propto \dots \propto \phi_{m+1}(W_1^{-1})$ . By the inductive assumption, we have

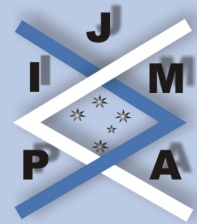
$$(3.10) \quad \xi(x, y) \leq \Phi_{m+1}^{-1} \left[ \Phi_{m+1}(c_m(x, y)) + \int_{\alpha_{m+1}(0)}^{\alpha_{m+1}(x)} \int_{\beta_{m+1}(y)}^{\infty} \tilde{f}_{m+1}(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right]$$

for all  $0 \leq x \leq \min\{\tilde{x}_1, x_3\}$ ,  $\max\{\tilde{y}_1, y_3\} \leq y < \infty$ , where

$$\Phi_{i+1}(u) = \int_{\tilde{u}_{i+1}}^u \frac{dz}{\phi_{i+1}(W_1^{-1}(z))},$$

$u > 0$ ,  $\tilde{u}_{i+1} = W_1(u_{i+1})$ ,  $\Phi_{i+1}^{-1}$  is the inverse of  $\Phi_{i+1}$ ,  $i = 1, \dots, m$ ,

$$c_{i+1}(x, y) = \Phi_{i+1}^{-1} \left[ \Phi_{i+1}(c_i(x, y)) + \int_{\alpha_{i+1}(0)}^{\alpha_{i+1}(x)} \int_{\beta_{i+1}(y)}^{\infty} \tilde{f}_{i+1}(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right], \quad i = 1, \dots, m,$$



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and  $x_3, y_3 \in \mathbb{R}_+$  are chosen such that

$$(3.11) \quad \Phi_{i+1}(c_i(x_3, y_3)) + \int_{\alpha_{i+1}(0)}^{\alpha_{i+1}(x_3)} \int_{\beta_{i+1}(y_3)}^{\infty} \tilde{f}_{i+1}(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \leq \int_{\tilde{u}_{i+1}}^{W_1(\infty)} \frac{dz}{\phi_{i+1}(W_1^{-1}(z))}$$

for  $i = 1, \dots, m$ .

Note that

$$\begin{aligned} \Phi_i(u) &= \int_{\tilde{u}_i}^u \frac{dz}{\phi_i(W_1^{-1}(z))} \\ &= \int_{W_1(u_i)}^u \frac{w_1(W_1^{-1}(z)) dz}{w_i(W_1^{-1}(z))} \\ &= \int_{u_i}^{W_1^{-1}(u)} \frac{dz}{w_i(z)} = W_i \circ W_1^{-1}(u), \quad i = 2, \dots, m + 1. \end{aligned}$$

From (3.10), we have

$$(3.12) \quad \begin{aligned} u(x, y) &\leq z(x, y) = W_1^{-1}(\xi(x, y)) \\ &\leq W_{m+1}^{-1} \left[ W_{m+1}(W_1^{-1}(c_m(x, y))) \right. \\ &\quad \left. + \int_{\alpha_{m+1}(0)}^{\alpha_{m+1}(x)} \int_{\beta_{m+1}(y)}^{\infty} \tilde{f}_{m+1}(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right] \end{aligned}$$

for all  $0 \leq x \leq \min\{\tilde{x}_1, x_3\}$ ,  $\max\{\tilde{y}_1, y_3\} \leq y < \infty$ . Let  $\tilde{c}_i(x, y) = W_1^{-1}(c_i(x, y))$ .



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Then,

$$\begin{aligned}\tilde{c}_1(x, y) &= W_1^{-1}(c_1(x, y)) \\ &= W_1^{-1} \left[ W_1(b_1(\tilde{x}_1, \tilde{y}_1)) + \int_{\alpha_1(0)}^{\alpha_1(x)} \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right] \\ &= \tilde{b}_2(\tilde{x}_1, \tilde{y}_1, x, y).\end{aligned}$$

Moreover, with the assumption that  $\tilde{c}_m(x, y) = \tilde{b}_{m+1}(\tilde{x}_1, \tilde{y}_1, x, y)$ , we have

$$\begin{aligned}\tilde{c}_{m+1}(x, y) &= W_1^{-1} \left[ \Phi_{m+1}^{-1}(\Phi_{m+1}(c_m(x, y))) + \int_{\alpha_{m+1}(0)}^{\alpha_{m+1}(x)} \int_{\beta_{m+1}(y)}^{\infty} \tilde{f}_{m+1}(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right] \\ &= W_{m+1}^{-1} \left[ W_{m+1}(W_1^{-1}(c_m(x, y))) + \int_{\alpha_{m+1}(0)}^{\alpha_{m+1}(x)} \int_{\beta_{m+1}(y)}^{\infty} \tilde{f}_{m+1}(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right] \\ &= W_{m+1}^{-1} \left[ W_{m+1}(\tilde{c}_m(x, y)) + \int_{\alpha_{m+1}(0)}^{\alpha_{m+1}(x)} \int_{\beta_{m+1}(y)}^{\infty} \tilde{f}_{m+1}(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right] \\ &= W_{m+1}^{-1} \left[ W_{m+1}(\tilde{b}_{m+1}(\tilde{x}_1, \tilde{y}_1, x, y)) + \int_{\alpha_{m+1}(0)}^{\alpha_{m+1}(x)} \int_{\beta_{m+1}(y)}^{\infty} \tilde{f}_{m+1}(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right] \\ &= \tilde{b}_{m+2}(\tilde{x}_1, \tilde{y}_1, x, y).\end{aligned}$$

This proves that

$$\tilde{c}_i(x, y) = \tilde{b}_{i+1}(\tilde{x}_1, \tilde{y}_1, x, y), \quad i = 1, \dots, m.$$



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Therefore, (3.11) becomes

$$\begin{aligned} & W_{i+1}(\tilde{b}_{i+1}(\tilde{x}_1, \tilde{y}_1, x_3, y_3)) + \int_{\alpha_{i+1}(0)}^{\alpha_{i+1}(x_3)} \int_{\beta_{i+1}(y_3)}^{\infty} \tilde{f}_{i+1}(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \\ & \leq \int_{\tilde{u}_{i+1}}^{W_1(\infty)} \frac{dz}{\phi_{i+1}(W_1^{-1}(z))} \\ & = \int_{u_{i+1}}^{\infty} \frac{dz}{w_{i+1}(z)}, \quad i = 1, \dots, m. \end{aligned}$$

The above inequalities and (3.5) imply that we may take  $x_2 = x_3$ ,  $y_2 = y_3$ . From (3.12) we get

$$\begin{aligned} u(x, y) \leq W_{m+1}^{-1} & \left[ W_{m+1}(\tilde{b}_{m+1}(\tilde{x}_1, \tilde{y}_1, x, y)) \right. \\ & \left. + \int_{\alpha_{m+1}(0)}^{\alpha_{m+1}(x)} \int_{\beta_{m+1}(y)}^{\infty} \tilde{f}_{m+1}(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right] \end{aligned}$$

for all  $0 \leq x \leq \tilde{x}_1 \leq x_2, y_2 \leq \tilde{y}_1 \leq y < \infty$ . This proves (3.3) by mathematical induction.

Taking  $x = \tilde{x}_1, y = \tilde{y}_1, x_2 = x_1$  and  $y_2 = y_1$ , we have

$$\begin{aligned} (3.13) \quad u(\tilde{x}_1, \tilde{y}_1) \leq W_n^{-1} & \left[ W_n(\tilde{b}_n(\tilde{x}_1, \tilde{y}_1, \tilde{x}_1, \tilde{y}_1)) \right. \\ & \left. + \int_{\alpha_n(0)}^{\alpha_n(\tilde{x}_1)} \int_{\beta_n(\tilde{y}_1)}^{\infty} \tilde{f}_n(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right] \end{aligned}$$





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for  $0 \leq \tilde{x}_1 \leq x_1, y_1 \leq \tilde{y}_1 < \infty$ . It is easy to verify that  $\tilde{b}_n(\tilde{x}_1, \tilde{y}_1, \tilde{x}_1, \tilde{y}_1) = b_n(\tilde{x}_1, \tilde{y}_1)$ . Thus, (3.13) can be written as

u(x-tilde, y-tilde) <= W\_n^-1 [ W\_n(b\_n(x-tilde, y-tilde)) + integral from alpha\_n(0) to alpha\_n(x-tilde) integral from beta\_n(y-tilde) to infinity f-tilde\_n(x-tilde, y-tilde, s, t) dt ds ]

Since x-tilde, y-tilde are arbitrary, replace x-tilde and y-tilde by x and y respectively and we have

u(x, y) <= W\_n^-1 [ W\_n(b\_n(x, y)) + integral from alpha\_n(0) to alpha\_n(x) integral from beta\_n(y) to infinity f-tilde\_n(x, y, s, t) dt ds ]

for all 0 <= x <= x\_1, y\_1 <= y <= infinity.

In case a(x, y) = 0 for some x, y in R+. Let b\_1,epsilon(x, y) := b\_1(x, y) + epsilon for all x, y in R+, where epsilon > 0 is arbitrary, and then b\_1,epsilon(x, y) > 0. Using the same arguments as above, where b\_1(x, y) is replaced with b\_1,epsilon(x, y) > 0, we get

u(x, y) <= W\_n^-1 [ W\_n(b\_n,epsilon(x, y)) + integral from alpha\_n(0) to alpha\_n(x) integral from beta\_n(y) to infinity d-tilde\_n(x, y, s, t) dt ds ]

Letting epsilon -> 0+, we obtain (2.1) by the continuity of b\_1,epsilon in epsilon and the continuity of W\_i and W\_i^-1 under the notation W\_1(0) := 0. square



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## 4. Applications

Consider the retarded partial differential equation

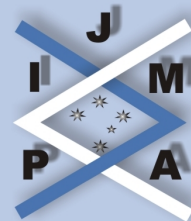
$$(4.1) \quad D_1 D_2 v(x, y) = \frac{1}{(x+1)^2(y+1)^2} + \exp(-x) \exp(-y) \sqrt{|v(x, y)|} + \frac{3}{4} x \exp\left(-\frac{x}{2}\right) \exp(-3y) v\left(\frac{x}{2}, 3y\right),$$

$$(4.2) \quad v(x, \infty) = \sigma(x), v(0, y) = \tau(y), v(0, \infty) = k,$$

for  $x, y \in \mathbb{R}_+$ , where  $\sigma, \tau \in C(\mathbb{R}_+, \mathbb{R})$ ,  $\sigma(x)$  is nondecreasing in  $x$ ,  $\tau(y)$  is nonincreasing in  $y$ , and  $k$  is a real constant. Integrating (4.1) with respect to  $x$  and  $y$  and using the initial conditions (4.2), we get

$$\begin{aligned} v(x, y) &= \sigma(x) + \tau(y) - k - \frac{x}{(x+1)(y+1)} \\ &\quad - \int_0^x \int_y^\infty \exp(-s) \exp(-t) \sqrt{|v(s, t)|} dt ds \\ &\quad - \frac{3}{4} \int_0^x \int_y^\infty s \exp\left(-\frac{s}{2}\right) \exp(-3t) v\left(\frac{s}{2}, 3t\right) dt ds \\ &= \sigma(x) + \tau(y) - k - \frac{x}{(x+1)(y+1)} \\ &\quad - \int_0^x \int_y^\infty \exp(-s) \exp(-t) \sqrt{|v(s, t)|} dt ds \\ &\quad - \int_0^{\frac{x}{2}} \int_{3y}^\infty s \exp(-s) \exp(-t) v(s, t) dt ds. \end{aligned}$$

Thus,



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$$|v(x, y)| \leq |\sigma(x) + \tau(y) - k| + \frac{x}{(x+1)(y+1)} + \int_0^x \int_y^\infty \exp(-s) \exp(-t) \sqrt{|v(s, t)|} dt ds + \int_0^{\frac{x}{2}} \int_{3y}^\infty s \exp(-s) \exp(-t) |v(s, t)| dt ds.$$

Letting  $u(x, y) = |v(x, y)|$ , we have

$$u(x, y) \leq a(x, y) + \int_{\alpha_1(0)}^{\alpha_1(x)} \int_{\beta_1(y)}^\infty f_1(x, y, s, t) w_1(u) dt ds + \int_{\alpha_2(0)}^{\alpha_2(x)} \int_{\beta_2(y)}^\infty f_2(x, y, s, t) w_2(u) dt ds,$$

where

$$a(x, y) = |\sigma(x) + \tau(y) - k| + \frac{x}{(x+1)(y+1)},$$

$$\alpha_1(x) = x, \quad \beta_1(y) = y, \quad \alpha_2(x) = \frac{x}{2}, \quad \beta_2(y) = 3y, \quad w_1(u) = \sqrt{u}, \quad w_2(u) = u,$$

$$f_1(x, y, s, t) = \exp(-s) \exp(-t), \quad f_2(x, y, s, t) = s \exp(-s) \exp(-t).$$

Clearly,  $\frac{w_2(u)}{w_1(u)} = \frac{u}{\sqrt{u}} = \sqrt{u}$  is nondecreasing for  $u > 0$ , that is,  $w_1 \propto w_2$ . Then for  $u_1, u_2 > 0$

$$b_1(x, y) = a(x, y), \quad \tilde{f}_1(x, y, s, t) = f_1(x, y, s, t), \quad \tilde{f}_2(x, y, s, t) = f_2(x, y, s, t),$$

$$W_1(u) = \int_{u_1}^u \frac{dz}{\sqrt{z}} = 2(\sqrt{u} - \sqrt{u_1}), \quad W_1^{-1}(u) = \left(\frac{u}{2} + \sqrt{u_1}\right)^2,$$

$$W_2(u) = \int_{u_2}^u \frac{dz}{z} = \ln \frac{u}{u_2}, \quad W_2^{-1}(u) = u_2 \exp(u),$$

$$b_2(x, y) = W_1^{-1} \left[ W_1(b_1(x, y)) + \int_0^x \int_y^\infty \tilde{f}_1(x, y, s, t) dt ds \right]$$

$$= W_1^{-1} \left[ 2 \left( \sqrt{b_1(x, y)} - \sqrt{u_1} \right) + (1 - \exp(-x)) \exp(-y) \right]$$

$$= \left[ \sqrt{b_1(x, y)} + \frac{1}{2} (1 - \exp(-x)) \exp(-y) \right]^2.$$

By Theorem 2.1, we have

$$|v(x, y)| \leq W_2^{-1} \left[ W_2(b_2(x, y)) + \int_0^{\frac{x}{2}} \int_{3y}^\infty \tilde{d}_2(x, y, s, t) dt ds \right]$$

$$= W_2^{-1} \left[ \ln \frac{b_2(x, y)}{u_2} + \left( 1 - \left( \frac{x}{2} + 1 \right) \exp\left(-\frac{x}{2}\right) \right) \exp(-3y) \right]$$

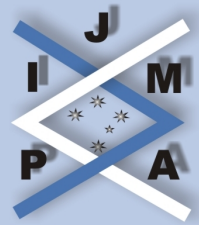
$$= u_2 \exp \left[ \ln \frac{b_2(x, y)}{u_2} + \left( 1 - \left( \frac{x}{2} + 1 \right) \exp\left(-\frac{x}{2}\right) \right) \exp(-3y) \right]$$

$$= b_2(x, y) \exp \left[ \left( 1 - \left( \frac{x}{2} + 1 \right) \exp\left(-\frac{x}{2}\right) \right) \exp(-3y) \right]$$

$$= \left( \sqrt{|\sigma(x) + \tau(y) - k| + \frac{x}{(x+1)(y+1)}} + \frac{1}{2} (1 - \exp(-x)) \exp(-y) \right)^2$$

$$\times \exp \left[ \left( 1 - \left( \frac{x}{2} + 1 \right) \exp\left(-\frac{x}{2}\right) \right) \exp(-3y) \right].$$

This implies that the solution of (4.1) is bounded for  $x, y \in \mathbb{R}_+$  provided that  $\sigma(x) + \tau(y) - k$  is bounded for all  $x, y \in \mathbb{R}_+$ .



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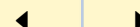
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Retarded Nonlinear Integral  
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