journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 9 (2008), Issue 1, Article 5, 10 pp.

ON SOME INEQUALITIES IN NORMED ALGEBRAS

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Received 24 October, 2007; accepted 29 October, 2007
Communicated by S. Saitoh

ABSTRACT. Some inequalities in normed algebras that provides lower and upper bounds for the norm of $\sum_{j=1}^{n} a_j x_j$ are obtained. Applications for estimating the quantities $\left\| \left\| x^{-1} \right\| x \pm \left\| y^{-1} \right\| y \right\|$ and $\left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\|$ for invertible elements x, y in unital normed algebras are also given.

Key words and phrases: Normed algebras, Unital algebras, Generalised triangle inequality.

2000 *Mathematics Subject Classification.* Primary 46H05; Secondary 47A10.

1. INTRODUCTION

In [1], in order to provide a generalisation of a norm inequality for n vectors in a normed linear space obtained by Pečarić and Rajić in [2], the author obtained the following result:

$$\max_{k\in\{1,\dots,n\}} \left\{ |\alpha_k| \left\| \sum_{j=1}^{n} x_j \right\| - \sum_{j=1}^{n} |\alpha_j - \alpha_k| \|x_j\| \right\} \leq \left\| \sum_{j=1}^{n} \alpha_j x_j \right\| \leq \min_{k\in\{1,\dots,n\}} \left\{ |\alpha_k| \left\| \sum_{j=1}^{n} x_j \right\| + \sum_{j=1}^{n} |\alpha_j - \alpha_k| \|x_j\| \right\}, \tag{1.1}$$

where x_j, $j \in \{1, \dots, n\}$ are vectors in the normed linear space $(X, \|\cdot\|)$ over $\mathbb{K}$ while α_j, $j \in \{1, \dots, n\}$ are scalars in $\mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$.

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For $\alpha_k = \frac{1}{\|x_k\|}$, with $x_k \neq 0$, $k \in \{1, \dots, n\}$ the above inequality produces the following result established by Pečarić and Rajić in [2]:

$$\max_{k\in\{1,\dots,n\}} \left\{ \frac{1}{\|x_k\|} \left[\left\| \sum_{j=1}^{n} x_j \right\| - \sum_{j=1}^{n} |\|x_j\| - \|x_k\|| \right] \right\} \leq \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \leq \min_{k\in\{1,\dots,n\}} \left\{ \frac{1}{\|x_k\|} \left[\left\| \sum_{j=1}^{n} x_j \right\| + \sum_{j=1}^{n} |\|x_j\| - \|x_k\|| \right] \right\}, \tag{1.2}$$

which implies the following refinement and reverse of the generalised triangle inequality due to M. Kato et al. [3]:

$$\min_{k\in\{1,\dots,n\}} \{\|x_k\|\} \left[n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right] \leq \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\| \leq \max_{k\in\{1,\dots,n\}} \{\|x_k\|\} \left[n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right]. \tag{1.3}$$

The other natural choice, $\alpha_k = \|x_k\|$, $k \in \{1, \dots, n\}$ in (1.1) produces the result

$$\max_{k\in\{1,\dots,n\}} \left\{ \|x_k\| \left\| \sum_{j=1}^{n} x_j \right\| - \sum_{j=1}^{n} |\|x_j\| - \|x_k\|| \, \|x_j\| \right\} \leq \left\| \sum_{j=1}^{n} \|x_j\| \, x_j \right\| \leq \min_{k\in\{1,\dots,n\}} \left\{ \|x_k\| \left\| \sum_{j=1}^{n} x_j \right\| + \sum_{j=1}^{n} |\|x_j\| - \|x_k\|| \, \|x_j\| \right\}, \tag{1.4}$$

which in its turn implies another refinement and reverse of the generalised triangle inequality:

$$(0 \leq) \frac{\sum_{j=1}^{n} \|x_j\|^2 - \left\| \sum_{j=1}^{n} \|x_j\| \, x_j \right\|}{\max_{k\in\{1,\dots,n\}} \{\|x_k\|\}} \leq \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\| \leq \frac{\sum_{j=1}^{n} \|x_j\|^2 - \left\| \sum_{j=1}^{n} \|x_j\| \, x_j \right\|}{\min_{k\in\{1,\dots,n\}} \{\|x_k\|\}}, \tag{1.5}$$

provided $x_k \neq 0$, $k \in \{1, \dots, n\}$.

In [2], the authors have shown that the case $n = 2$ in (1.2) produces the *Maligranda-Mercer inequality*:

$$\frac{\|x - y\| - |\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}} \leq \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{\|x - y\| + |\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}}, \tag{1.6}$$

for any $x, y \in X \setminus \{0\}$.

We notice that Maligranda proved the right inequality in [5] while Mercer proved the left inequality in [4].

We have shown in [1] that the following dual result for two vectors is also valid:

$$(0 \leq) \frac{\|x - y\|}{\min\{\|x\|, \|y\|\}} - \frac{|\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}} \leq \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \leq \frac{\|x - y\|}{\max\{\|x\|, \|y\|\}} + \frac{|\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}}, \tag{1.7}$$

for any $x, y \in X \setminus \{0\}$.

Motivated by the above results, the aim of the present paper is to establish lower and upper bounds for the norm of $\sum_{j=1}^{n} a_j x_j$, where $a_j, x_j, j \in \{1, \dots, n\}$ are elements in a normed algebra $(A, \|\cdot\|)$ over the real or complex number field $\mathbb{K}$. In the case where $(A, \|\cdot\|)$ is a unital algebra and x, y are invertible, lower and upper bounds for the quantities

$$\left\| \left\| x^{-1} \right\| x \pm \left\| y^{-1} \right\| y \right\| \quad \text{and} \quad \left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\|$$

are provided as well.

2. Inequalities for n Pairs of Elements

Let $(A, \|\cdot\|)$ be a normed algebra over the real or complex number field $\mathbb{K}$.

Theorem 2.1. *If $(a_j, x_j) \in A^2$, $j \in \{1, \dots, n\}$, then*

$$\begin{aligned} \max_{k \in \{1,\dots,n\}} &\left\{ \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| - \sum_{j=1}^{n} \|a_j - a_k\| \|x_j\| \right\} \\ &\leq \max_{k \in \{1,\dots,n\}} \left\{ \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| - \sum_{j=1}^{n} \|(a_j - a_k) x_j\| \right\} \\ &\leq \left\| \sum_{j=1}^{n} a_j x_j \right\| \\ &\leq \min_{k \in \{1,\dots,n\}} \left\{ \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \|(a_j - a_k) x_j\| \right\} \\ &\leq \min_{k \in \{1,\dots,n\}} \left\{ \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \|a_j - a_k\| \|x_j\| \right\}. \end{aligned} \tag{2.1}$$

Proof. Observe that for any $k \in \{1, \dots, n\}$ we have

$$\sum_{j=1}^{n} a_j x_j = a_k \left(\sum_{j=1}^{n} x_j \right) + \sum_{j=1}^{n} (a_j - a_k) x_j.$$

Taking the norm and utilising the triangle inequality and the normed algebra properties, we have

$$\begin{aligned} \left\| \sum_{j=1}^{n} a_j x_j \right\| &\leq \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \left\| \sum_{j=1}^{n} (a_j - a_k) x_j \right\| \\ &\leq \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \|(a_j - a_k) x_j\| \\ &\leq \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \|a_j - a_k\| \|x_j\|, \end{aligned}$$

for any $k \in \{1, \dots, n\}$, which implies the second part in (2.1).

Observing that

$$\sum_{j=1}^{n} a_j x_j = a_k \left(\sum_{j=1}^{n} x_j \right) - \sum_{j=1}^{n} (a_k - a_j) x_j$$

and utilising the continuity of the norm, we have

$$\begin{aligned}\left\|\sum_{j=1}^{n} a_j x_j\right\| &\geq \left\|a_k\left(\sum_{j=1}^{n} x_j\right) - \sum_{j=1}^{n}(a_k - a_j)\, x_j\right\| \\ &\geq \left\|a_k\left(\sum_{j=1}^{n} x_j\right)\right\| - \left\|\sum_{j=1}^{n}(a_k - a_j)\, x_j\right\| \\ &\geq \left\|a_k\left(\sum_{j=1}^{n} x_j\right)\right\| - \sum_{j=1}^{n}\left\|(a_k - a_j)\, x_j\right\| \\ &\geq \left\|a_k\left(\sum_{j=1}^{n} x_j\right)\right\| - \sum_{j=1}^{n}\left\|a_k - a_j\right\|\left\|x_j\right\|\end{aligned}$$

for any $k \in \{1, \dots, n\}$, which implies the first part in (2.1). □

Remark 2.2. If there exists $r > 0$ so that $\|a_j - a_k\| \leq r\,\|a_k\|$ for any $j, k \in \{1, \dots, n\}$, then, by the second part of (2.1), we have

$$\left\|\sum_{j=1}^{n} a_j x_j\right\| \leq \min_{k\in\{1,\dots,n\}}\{\|a_k\|\}\left[\left\|\sum_{j=1}^{n} x_j\right\| + r\sum_{j=1}^{n}\|x_j\|\right]. \tag{2.2}$$

Corollary 2.3. *If $x_j \in A$, $j \in \{1, \dots, n\}$, then*

$$\begin{aligned}&\max_{k\in\{1,\dots,n\}}\left\{\left\|x_k\left(\sum_{j=1}^{n} x_j\right)\right\| - \sum_{j=1}^{n}\|x_j - x_k\|\,\|x_j\|\right\} \\ &\leq \max_{k\in\{1,\dots,n\}}\left\{\left\|x_k\left(\sum_{j=1}^{n} x_j\right)\right\| - \sum_{j=1}^{n}\|(x_j - x_k)\,x_j\|\right\} \leq \left\|\sum_{j=1}^{n} x_j^2\right\| \\ &\leq \min_{k\in\{1,\dots,n\}}\left\{\left\|x_k\left(\sum_{j=1}^{n} x_j\right)\right\| + \sum_{j=1}^{n}\|(x_j - x_k)\,x_j\|\right\} \\ &\leq \min_{k\in\{1,\dots,n\}}\left\{\left\|x_k\left(\sum_{j=1}^{n} x_j\right)\right\| + \sum_{j=1}^{n}\|x_j - x_k\|\,\|x_j\|\right\}.\end{aligned} \tag{2.3}$$

Corollary 2.4. *Assume that A is a unital normed algebra. If $x_j \in A$ are invertible for any $j \in \{1, \dots, n\}$, then*

$$\begin{aligned}&\min_{k\in\{1,\dots,n\}}\left\|x_k^{-1}\right\|\left[\sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\|\right] \\ &\leq \sum_{j=1}^{n}\left\|x_j^{-1}\right\|\|x_j\| - \left\|\sum_{j=1}^{n}\left\|x_j^{-1}\right\| x_j\right\| \\ &\leq \max_{k\in\{1,\dots,n\}}\left\|x_k^{-1}\right\|\left[\sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\|\right].\end{aligned} \tag{2.4}$$

Proof. If $1 \in A$ is the unity, then on choosing $a_k = \left\|x_k^{-1}\right\| \cdot 1$ in (2.1) we get

$$\begin{aligned}
\text{(2.5)}\qquad & \max_{k\in\{1,\dots,n\}}\left\{\left\|x_k^{-1}\right\|\left\|\sum_{j=1}^{n}x_j\right\| - \sum_{j=1}^{n}\left|\left\|x_j^{-1}\right\| - \left\|x_k^{-1}\right\|\right|\left\|x_j\right\|\right\} \\
& \le \left\|\sum_{j=1}^{n}\left\|x_j^{-1}\right\| x_j\right\| \\
& \le \min_{k\in\{1,\dots,n\}}\left\{\left\|x_k^{-1}\right\|\left\|\sum_{j=1}^{n}x_j\right\| + \sum_{j=1}^{n}\left|\left\|x_j^{-1}\right\| - \left\|x_k^{-1}\right\|\right|\left\|x_j\right\|\right\}.
\end{aligned}$$

Now, assume that $\min_{k\in\{1,\dots,n\}}\left\{\left\|x_k^{-1}\right\|\right\} = \left\|x_{k_0}^{-1}\right\|$. Then

$$\begin{aligned}
&\left\|x_{k_0}^{-1}\right\|\left\|\sum_{j=1}^{n}x_j\right\| + \sum_{j=1}^{n}\left|\left\|x_j^{-1}\right\| - \left\|x_{k_0}^{-1}\right\|\right|\left\|x_j\right\| \\
&\qquad = -\left\|x_{k_0}^{-1}\right\|\left(\sum_{j=1}^{n}\left\|x_j\right\| - \left\|\sum_{j=1}^{n}x_j\right\|\right) + \sum_{j=1}^{n}\left\|x_j^{-1}\right\|\left\|x_j\right\|.
\end{aligned}$$

Utilising the second inequality in (2.5), we deduce

$$\left\|x_{k_0}^{-1}\right\|\left(\sum_{j=1}^{n}\left\|x_j\right\| - \left\|\sum_{j=1}^{n}x_j\right\|\right) \le \sum_{j=1}^{n}\left\|x_j^{-1}\right\|\left\|x_j\right\| - \left\|\sum_{j=1}^{n}\left\|x_j^{-1}\right\| x_j\right\|$$

and the first inequality in (2.4) is proved.

The second part of (2.4) can be proved in a similar manner, however, the details are omitted. □

Remark 2.5. An equivalent form of (2.4) is:

$$\begin{aligned}
\text{(2.6)}\qquad & \frac{\sum_{j=1}^{n}\left\|x_j^{-1}\right\|\left\|x_j\right\| - \left\|\sum_{j=1}^{n}\left\|x_j^{-1}\right\| x_j\right\|}{\max_{k\in\{1,\dots,n\}}\left\|x_k^{-1}\right\|} \\
&\qquad \le \sum_{j=1}^{n}\left\|x_j\right\| - \left\|\sum_{j=1}^{n}x_j\right\| \le \frac{\sum_{j=1}^{n}\left\|x_j^{-1}\right\|\left\|x_j\right\| - \left\|\sum_{j=1}^{n}\left\|x_j^{-1}\right\| x_j\right\|}{\min_{k\in\{1,\dots,n\}}\left\|x_k^{-1}\right\|},
\end{aligned}$$

which provides both a refinement and a reverse inequality for the generalised triangle inequality.

3. Inequalities for Two Pairs of Elements

The following particular case of Theorem 2.1 is of interest for applications.

Lemma 3.1. *If* $(a,b),(x,y) \in A^2$, *then*

$$\begin{aligned}
\text{(3.1)}\qquad & \max\left\{\left\|a(x\pm y)\right\| - \left\|(b-a)y\right\|, \left\|b(x\pm y)\right\| - \left\|(b-a)x\right\|\right\} \\
& \le \left\|ax \pm by\right\| \le \min\left\{\left\|a(x\pm y)\right\| + \left\|(b-a)y\right\|, \left\|b(x\pm y)\right\| + \left\|(b-a)x\right\|\right\}
\end{aligned}$$

or, equivalently,

$$\begin{aligned}(3.2)\qquad &\frac{1}{2}\left\{\|a(x\pm y)\|+\|b(x\pm y)\|-[\|(b-a)y\|+\|(b-a)x\|]\right\}\\ &\qquad+\frac{1}{2}\left|\|a(x\pm y)\|-\|b(x\pm y)\|+\|(b-a)y\|-\|(b-a)x\|\right|\\ &\leq\|ax\pm by\|\\ &\leq\frac{1}{2}\left\{\|a(x\pm y)\|+\|b(x\pm y)\|+[\|(b-a)y\|+\|(b-a)x\|]\right\}\\ &\qquad-\frac{1}{2}\left|\|a(x\pm y)\|+\|b(x\pm y)\|-\|(b-a)y\|-\|(b-a)x\|\right|.\end{aligned}$$

Proof. The inequality (3.1) follows from Theorem 2.1 for $n=2$, $a_1=a$, $a_2=b$, $x_1=x$ and $x_2=\pm y$.

Utilising the properties of real numbers,

$$\min\{\alpha,\beta\}=\frac{1}{2}[\alpha+\beta-|\alpha-\beta|],\quad \max\{\alpha,\beta\}=\frac{1}{2}[\alpha+\beta+|\alpha-\beta|];\quad \alpha,\beta\in\mathbb{R};$$

the inequality (3.1) is clearly equivalent with (3.2). □

The following result contains some upper bounds for $\|ax\pm by\|$ that are perhaps more useful for applications.

Theorem 3.2. *If* $(a,b),(x,y)\in A^2$, *then*

$$\begin{aligned}(3.3)\qquad \|ax\pm by\| &\leq \min\{\|a(x\pm y)\|,\|b(x\pm y)\|\}+\|b-a\|\max\{\|x\|,\|y\|\}\\ &\leq\|x\pm y\|\min\{\|a\|,\|b\|\}+\|b-a\|\max\{\|x\|,\|y\|\}\end{aligned}$$

and

$$\begin{aligned}(3.4)\qquad \|ax\pm by\| &\leq \|x\pm y\|\max\{\|a\|,\|b\|\}+\min\{\|(b-a)x\|,\|(b-a)y\|\}\\ &\leq\|x\pm y\|\max\{\|a\|,\|b\|\}+\|b-a\|\min\{\|x\|,\|y\|\}.\end{aligned}$$

Proof. Observe that $\|(b-a)x\|\leq\|b-a\|\|x\|$ and $\|(b-a)y\|\leq\|b-a\|\|y\|$, and then

$$(3.5)\qquad \|(b-a)x\|,\|(b-a)y\|\leq\|b-a\|\max\{\|x\|,\|y\|\},$$

which implies that

$$\begin{aligned}&\min\{\|a(x\pm y)\|+\|(b-a)y\|,\|b(x\pm y)\|+\|(b-a)x\|\}\\ &\leq\min\{\|a(x\pm y)\|,\|b(x\pm y)\|\}+\|b-a\|\max\{\|x\|,\|y\|\}\\ &\leq\|x\pm y\|\min\{\|a\|,\|b\|\}+\|b-a\|\max\{\|x\|,\|y\|\}.\end{aligned}$$

Utilising the second inequality in (3.1), we deduce (3.3).

Also, since $\|a(x\pm y)\|\leq\|a\|\|x\pm y\|$ and $\|b(x\pm y)\|\leq\|b\|\|x\pm y\|$, hence

$$\|a(x\pm y)\|,\|b(x\pm y)\|\leq\|x\pm y\|\max\{\|a\|,\|b\|\},$$

which implies that

$$\begin{aligned}&\min\{\|a(x\pm y)\|+\|(b-a)y\|,\|b(x\pm y)\|+\|(b-a)x\|\}\\ &\leq\|x\pm y\|\max\{\|a\|,\|b\|\}+\min\{\|(b-a)x\|,\|(b-a)y\|\}\\ &\leq\|x\pm y\|\max\{\|a\|,\|b\|\}+\|b-a\|\min\{\|x\|,\|y\|\},\end{aligned}$$

and the inequality (3.4) is also proved. □

The following corollary may be more useful for applications.

Corollary 3.3. *If* $(a,b),(x,y)\in A^2$, *then*

$$\|ax\pm by\|\le\|x\pm y\|\cdot\frac{\|a\|+\|b\|}{2}+\|b-a\|\cdot\frac{\|x\|+\|y\|}{2}. \tag{3.6}$$

Proof. Follows from Theorem 3.2 by adding the last inequality in (3.3) to the last inequality (3.4) and utilising the property that $\min\{\alpha,\beta\}+\max\{\alpha,\beta\}=\alpha+\beta$, $\alpha,\beta\in\mathbb{R}$. □

The following lower bounds for $\|ax\pm by\|$ can be stated as well:

Theorem 3.4. *For any* (a,b) *and* $(x,y)\in A^2$, *we have:*

$$\begin{aligned}(3.7)\qquad &\max\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\|b-a\|\max\{\|x\|,\|y\|\}\\ &\le\max\{\|a(x\pm y)\|,\|b(x\pm y)\|\}-\|b-a\|\max\{\|x\|,\|y\|\}\\ &\le\|ax\pm by\|\end{aligned}$$

and

$$\begin{aligned}(3.8)\qquad &\min\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\|b-a\|\min\{\|x\|,\|y\|\}\\ &\le\min\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\min\{\|(b-a)x\|,\|(b-a)y\|\}\\ &\le\|ax\pm by\|.\end{aligned}$$

Proof. Observe that, by (3.5) we have that

$$\begin{aligned}&\max\{\|a(x\pm y)\|-\|(b-a)y\|,\|b(x\pm y)\|-\|(b-a)x\|\}\\ &\ge\max\{\|ax\pm ay\|,\|bx\pm by\|\}-\|b-a\|\max\{\|x\|,\|y\|\}\\ &\ge\max\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\|b-a\|\max\{\|x\|,\|y\|\}\end{aligned}$$

and on utilising the first inequality in (3.1), the inequality (3.7) is proved.

Observe also that, since

$$\|a(x\pm y)\|,\|b(x\pm y)\|\ge\min\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\},$$

then

$$\begin{aligned}&\max\{\|a(x\pm y)\|-\|(b-a)y\|,\|b(x\pm y)\|-\|(b-a)x\|\}\\ &\ge\min\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\min\{\|(b-a)x\|,\|(b-a)y\|\}\\ &\ge\min\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\|b-a\|\min\{\|x\|,\|y\|\}.\end{aligned}$$

Then, by the first inequality in (3.1), we deduce (3.8). □

Corollary 3.5. *For any* $(a,b),(x,y)\in A^2$, *we have*

$$\frac{1}{2}\cdot[|\|ax\|-\|ay\||+|\|bx\|-\|by\||]-\|b-a\|\cdot\frac{\|x\|+\|y\|}{2}\le\|ax\pm by\|. \tag{3.9}$$

The proof follows from Theorem 3.4 by adding (3.7) to (3.8). The details are omitted.

4. APPLICATIONS FOR TWO INVERTIBLE ELEMENTS

In this section we assume that A is a unital algebra with the unity 1. The following results provide some upper bounds for the quantity $\|\|x^{-1}\|x\pm\|y^{-1}\|y\|$, where x and y are invertible in A.

Proposition 4.1. *If* $(x,y)\in A^2$ *are invertible, then*

$$\begin{aligned}(4.1)\qquad &\|\|x^{-1}\|x\pm\|y^{-1}\|y\|\\ &\le\|x\pm y\|\min\{\|x^{-1}\|,\|y^{-1}\|\}+|\|x^{-1}\|-\|y^{-1}\||\max\{\|x\|,\|y\|\}\end{aligned}$$

and

$$\left\| \left\| x^{-1} \right\| x \pm \left\| y^{-1} \right\| y \right\| \leq \| x \pm y \| \max \left\{ \left\| x^{-1} \right\|, \left\| y^{-1} \right\| \right\} + \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \min \left\{ \| x \|, \| y \| \right\}. \tag{4.2}$$

Proof. Follows by Theorem 3.2 on choosing $a = \|x^{-1}\| \cdot 1$ and $b = \|y^{-1}\| \cdot 1$. □

Corollary 4.2. *With the above assumption for x and y, we have*

$$\left\| \left\| x^{-1} \right\| x \pm \left\| y^{-1} \right\| y \right\| \leq \| x \pm y \| \cdot \frac{\| x^{-1} \| + \| y^{-1} \|}{2} + \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \cdot \frac{\| x \| + \| y \|}{2}. \tag{4.3}$$

Lower bounds for $\|\|x^{-1}\| x \pm \|y^{-1}\| y\|$ are provided below:

Proposition 4.3. *If $(x, y) \in A^2$ are invertible, then*

$$\| x \pm y \| \max \left\{ \left\| x^{-1} \right\|, \left\| y^{-1} \right\| \right\} - \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \max \left\{ \| x \|, \| y \| \right\} \leq \left\| \left\| x^{-1} \right\| x \pm \left\| y^{-1} \right\| y \right\| \tag{4.4}$$

and

$$\| x \pm y \| \min \left\{ \left\| x^{-1} \right\|, \left\| y^{-1} \right\| \right\} - \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \min \left\{ \| x \|, \| y \| \right\} \leq \left\| \left\| x^{-1} \right\| x \pm \left\| y^{-1} \right\| y \right\|. \tag{4.5}$$

Proof. The first inequality in (4.4) follows from the second inequality in (3.7) on choosing $a = \|x^{-1}\| \cdot 1$ and $b = \|y^{-1}\| \cdot 1$.

We know from the proof of Theorem 3.4 that

$$\max \left\{ \| a (x \pm y) \| - \| (b - a) y \|, \| b (x \pm y) \| - \| (b - a) x \| \right\} \leq \| a x \pm b y \|. \tag{4.6}$$

If in this inequality we choose $a = \|x^{-1}\| \cdot 1$ and $b = \|y^{-1}\| \cdot 1$, then we get

$$\begin{aligned} &\left\| \left\| x^{-1} \right\| x \pm \left\| y^{-1} \right\| y \right\| \\ &\geq \max \left\{ \left\| x^{-1} \right\| \| x \pm y \| - \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \| y \|, \left\| y^{-1} \right\| \| x \pm y \| - \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \| x \| \right\} \\ &\geq \| x \pm y \| \min \left\{ \left\| x^{-1} \right\|, \left\| y^{-1} \right\| \right\} - \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \min \left\{ \| x \|, \| y \| \right\} \end{aligned}$$

and the inequality (4.5) is obtained. □

Corollary 4.4. *If $(x, y) \in A^2$ are invertible, then*

$$\| x \pm y \| \cdot \frac{\| x^{-1} \| + \| y^{-1} \|}{2} - \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \cdot \frac{\| x \| + \| y \|}{2} \leq \left\| \left\| x^{-1} \right\| x \pm \left\| y^{-1} \right\| y \right\|. \tag{4.7}$$

Remark 4.5. We observe that the inequalities (4.3) and (4.7) are in fact equivalent with:

$$\left| \left\| \left\| x^{-1} \right\| x \pm \left\| y^{-1} \right\| y \right\| - \| x \pm y \| \cdot \frac{\| x^{-1} \| + \| y^{-1} \|}{2} \right| \leq \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \cdot \frac{\| x \| + \| y \|}{2}. \tag{4.8}$$

Now we consider the dual expansion $\|\|y^{-1}\| x \pm \|x^{-1}\| y\|$, for which the following upper bounds can be stated.

Proposition 4.6. *If (x, y) are invertible in A, then*

$$\left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\| \leq \| x \pm y \| \min \left\{ \left\| x^{-1} \right\|, \left\| y^{-1} \right\| \right\} + \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \max \left\{ \| x \|, \| y \| \right\} \tag{4.9}$$

and

$$\left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\| \leq \| x \pm y \| \max \left\{ \left\| x^{-1} \right\|, \left\| y^{-1} \right\| \right\} + \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \min \left\{ \| x \|, \| y \| \right\}. \tag{4.10}$$

In particular,

$$\left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\| \leq \| x \pm y \| \cdot \frac{\| x^{-1} \| + \| y^{-1} \|}{2} + \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \cdot \frac{\| x \| + \| y \|}{2}. \tag{4.11}$$

The proof follows from Theorem 3.2 on choosing $a = \| y^{-1} \| \cdot 1$ and $b = \| x^{-1} \| \cdot 1$.

The lower bounds for the quantity $\| \| y^{-1} \| x \pm \| x^{-1} \| y \|$ are incorporated in:

Proposition 4.7. *If (x, y) are invertible in A, then*

$$\| x \pm y \| \max \left\{ \left\| x^{-1} \right\|, \left\| y^{-1} \right\| \right\} - \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \max \left\{ \| x \|, \| y \| \right\} \leq \left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\| \tag{4.12}$$

and

$$\| x \pm y \| \min \left\{ \left\| x^{-1} \right\|, \left\| y^{-1} \right\| \right\} - \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \min \left\{ \| x \|, \| y \| \right\} \leq \left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\|. \tag{4.13}$$

In particular,

$$\| x \pm y \| \cdot \frac{\| x^{-1} \| + \| y^{-1} \|}{2} - \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \cdot \frac{\| x \| + \| y \|}{2} \leq \left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\|. \tag{4.14}$$

Remark 4.8. We observe that the inequalities (4.11) and (4.14) are equivalent with

$$\left| \left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\| - \| x \pm y \| \cdot \frac{\| x^{-1} \| + \| y^{-1} \|}{2} \right| \leq \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \cdot \frac{\| x \| + \| y \|}{2}. \tag{4.15}$$

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