# COEFFICIENT INEQUALITIES FOR CERTAIN MEROMORPHICALLY $p$-VALENT FUNCTIONS

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*Abstract:* The aim of this paper is to prove some inequalities for $p$-valent meromorphic functions in the punctured unit disk $\Delta^*$ and find important corollaries.

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# 1. Introduction

Let $\Sigma_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^{-p} + \sum_{k=p}^{\infty} a_k z^k \tag{1.1}$$

which are analytic meromorphic multivalent in the punctured unit disk

$$\Delta^* = \{z : 0 < |z| < 1\}.$$

We say that $f(z)$ is $p$-valently starlike of order $\gamma(0 \leq \gamma < p)$ if and only if for $z \in \Delta^*$

$$-\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma. \tag{1.2}$$

Also $f(z)$ is $p$-valently convex of order $\gamma(0 \leq \gamma < p)$ if and only if

$$-\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma, \qquad z \in \Delta^*. \tag{1.3}$$

**Definition 1.1.** *A function $f(z) \in \Sigma_p$ is said to be in the subclass $X_p^*(j)$ if it satisfies the inequality*

$$\left|\frac{(p-1)!}{(-1)^j(p+j-1)!} \cdot \frac{f^{(j)}(z)}{z^{-p-j}} - 1\right| < 1, \tag{1.4}$$

*where*

$$f^{(j)}(z) = (-1)^j \frac{(p+j-1)!}{(p-1)!\, z^{p+j}} + \sum_{k=p}^{\infty} \frac{k!}{(k-j)!} a_k z^{k-j} \tag{1.5}$$


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*is the $j$-th differential of $f(z)$ and a function $f(z) \in \Sigma_p$ is said to be in the subclass $Y_p^*(j)$ if it satisfies the inequality*

$$\left| -\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} - (p+j) \right| < p+j. \tag{1.6}$$

To establish our main results we need the following lemma due to Jack [5].

**Lemma 1.2.** *Let $w(z)$ be analytic in $\Delta = \{z : |z| < 1\}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0$ then*

$$z_0 w'(z_0) = c w(z_0),$$

*where $c$ is a real number and $c \geq 1$.*

Some different inequalities on $p$-valent holomorphic and $p$-valent meromorphic functions by using operators were studied in [1], [2], [3] and [4].


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## 2. Main Results

**Theorem 2.1.** *If $f(z) \in \Sigma_p$ satisfies the inequality*

$$\operatorname{Re}\left\{ \frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} + p + j \right\} > 1 - \frac{1}{2p}, \tag{2.1}$$

*then $f(z) \in X_p^*(j)$.*

*Proof.* Letting $f(z) \in \Sigma_p$, we define the function $w(z)$ by

$$\frac{(p-1)!}{(-1)^j(p+j-1)!} \frac{f^{(j)}(z)}{z^{-p-j}} = 1 - w(z), \qquad (z \in \Delta^*). \tag{2.2}$$

It is easy to verify that $w(0) = 0$.

From (2.2) we obtain

$$f^{(j)}(z) = -\frac{(-1)^j(p+j-1)!}{(p-1)!} z^{-p-j} + \frac{(p+j-1)!}{(p-1)!} z^{-p-j} w(z)$$

or

$$\begin{aligned}[f^{(j)}(z)]' &= (-1)^j(p+j)z^{-p-j-1}\frac{(p+j-1)!}{(p-1)!} \\ &\quad + (-1)^{j+1}(p+j)z^{-p-j-1}\frac{(p+j-1!}{(p-1)!}w(z) + (-1)^j\frac{(p+j-1)!}{(p-1)!}z^{-p-j}w'(z).\end{aligned}$$

After a simple calculation we obtain

$$\frac{zw'(z)}{1-w(z)} = -\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} - (p+j). \tag{2.3}$$

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Now, suppose that there exists a point $z_0 \in \Delta^*$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1.$$

Then, by letting $w(z_0) = e^{i\theta}(w(z_0) \neq 1)$ and using Jack's lemma in the equation (2.3), we have

$$\begin{aligned} -\operatorname{Re}\left\{\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} + p + j\right\} &= \operatorname{Re}\left\{\frac{z_0 w'(z_0)}{1-w(z_0)}\right\} = \operatorname{Re}\left\{\frac{cw(z_0)}{1-w(z_0)}\right\} \\ &= c\operatorname{Re}\left\{\frac{e^{i\theta}}{1-e^{i\theta}}\right\} = \frac{-c}{2} < \frac{-1}{2} \end{aligned}$$

which contradicts the hypothesis (2.1). Hence we conclude that for all $z$, $|w(z)| < 1$ and from (2.2) we have

$$\left|\frac{(p-1)!f^{(j)}(z)}{(-1)^j(p+j-1)!z^{-p-j}} - 1\right| = |w(z)| < 1$$

and this gives the result. □

**Theorem 2.2.** *If $f(z) \in \Sigma_p$ satisfies the inequality*

$$\operatorname{Re}\left\{\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} - \left(1 + \frac{z[f^{(j)}(z)]''}{[f^{(j)}(z)]'}\right)\right\} > \frac{2p+1}{2(p+1)}, \tag{2.4}$$

*then $f(z) \in Y_p^*(j)$.*

*Proof.* Let $f(z) \in \Sigma_p$. We consider the function $w(z)$ as follows:

$$-\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} = (p+j)(1-w(z)). \tag{2.5}$$


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It is easy to see that $w(0) = 0$. Furthermore, by differentiating both sides of (2.5) we get

$$-\left[1+\frac{z[f^{(j)}(z)]''}{[f^{(j)}(z)]'}\right] = (p+j)(1-w(z)) + \frac{zw'(z)}{1-w(z)}.$$

Now suppose that there exists a point $z_0 \in \Delta^*$ such that $\max\limits_{|z|\le|z_0|} |w(z)| = |w(z_0)| = 1$.

Then, by letting $w(z_0) = e^{i\theta}$ and using Jack's lemma we have

$$\begin{aligned}\mathrm{Re}\left\{\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} - \left(1+\frac{z[f^{(j)}(z)]''}{f^{(j)}(z)]'}\right)\right\} &= \mathrm{Re}\left\{\frac{z_0w'(z_0)}{1-w(z_0)}\right\} \\ &= c\ \mathrm{Re}\left\{\frac{e^{i\theta}}{1-e^{i\theta}}\right\} = -\frac{c}{2} < -\frac{1}{2},\end{aligned}$$

which contradicts the condition (2.4). So we conclude that $|w(z)| < 1$ for all $z \in \Delta^*$. Hence from (2.5) we obtain

$$\left|-\frac{z[f^{(j)}(z)]'}{[f^{(j)}(z)]} - (p+j)\right| < p+j.$$

This completes the proof. $\square$

By taking $j = 0$ in Theorems 2.1 and 2.2, we obtain the following corollaries.

**Corollary 2.3.** *If $f(z) \in \Sigma_p$ satisfies the inequality*

$$-\mathrm{Re}\left\{\frac{zf'}{f} + p\right\} > 1 - \frac{1}{2p},$$

*then*

$$\left|\frac{f(z)}{z^{-p}} - 1\right| < 1.$$




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**Corollary 2.4.** *If $f(z) \in \Sigma_p$ satisfies the inequality*

$$-\operatorname{Re}\left\{\frac{zf'}{f} - \left(1 + \frac{zf''}{f'}\right)\right\} > \frac{2p+1}{2(p+1)},$$

*then $\left|-\frac{zf'}{f} - p\right| < p$ or equivalently $f(z)$ is meromorphically $p$-valent starlike with respect to the origin.*

By taking $j = 1$ in Theorems 2.1 and 2.2, we obtain the following corollaries.

**Corollary 2.5.** *If $f(z) \in \Sigma_p$ satisfies the inequality*

$$-\operatorname{Re}\left\{\frac{zf''}{f'} + p + 1\right\} > 1 - \frac{1}{2p},$$

*then $\left|-\frac{f'(z)}{z^{-p-1}} - p\right| < p$ or equivalently $f(z)$ is meromorphically $p$-valent close-to-convex with respect to the origin.*

**Corollary 2.6.** *If $f(z) \in \Sigma_p$ satisfies the inequality*

$$-\operatorname{Re}\left\{\frac{zf''}{f'} - \left(1 + \frac{zf'''}{f''}\right)\right\} > \frac{2p+1}{2(p+1)},$$

*then*

$$\left|-\frac{zf''}{f'} - (p+1)\right| < p+1,$$

*or equivalently $f(z)$ is meromorphically multivalent convex.*



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