

COEFFICIENT INEQUALITIES FOR CERTAIN MEROMORPHICALLY p -VALENT FUNCTIONS

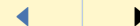
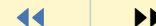


Meromorphically p -valent
Functions

A. Ebadian and Sh. Najafzadeh
vol. 9, iss. 4, art. 114, 2008

[Title Page](#)

[Contents](#)



Page 1 of 9

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

A. EBADIAN

Department of Mathematics
Faculty of Science, Urmia University
Urmia, Iran
EMail: aebadian@yahoo.com

Sh. NAJAFZADEH

Department of Mathematics
Maragheh University
Maragheh, Iran
EMail: najafzadeh1234@yahoo.ie

Received: 12 November, 2007

Accepted: 21 August, 2008

Communicated by: S.S. Dragomir

2000 AMS Sub. Class.: 30C45, 30C50.

Key words: p -Valent, Meromorphic, Starlike, Convex, Close-to-convex.

Abstract: The aim of this paper is to prove some inequalities for p -valent meromorphic functions in the punctured unit disk Δ^* and find important corollaries.

Acknowledgements: The authors are grateful to the referees for their valuable suggestions and comments. The paper was revised according to their suggestions.

Contents

1 Introduction	3
2 Main Results	5



**Meromorphically p -valent
Functions**

A. Ebadian and Sh. Najafzadeh

vol. 9, iss. 4, art. 114, 2008

[Title Page](#)

[Contents](#)



Page 2 of 9

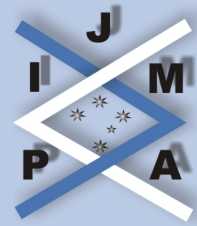
[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 3 of 9

[Go Back](#)

[Full Screen](#)

[Close](#)

1. Introduction

Let Σ_p denote the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = z^{-p} + \sum_{k=p}^{\infty} a_k z^k$$

which are analytic meromorphic multivalent in the punctured unit disk

$$\Delta^* = \{z : 0 < |z| < 1\}.$$

We say that $f(z)$ is p -valently starlike of order γ ($0 \leq \gamma < p$) if and only if for $z \in \Delta^*$

$$(1.2) \quad -\operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > \gamma.$$

Also $f(z)$ is p -valently convex of order γ ($0 \leq \gamma < p$) if and only if

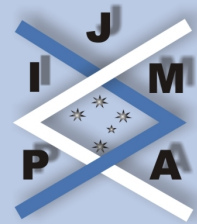
$$(1.3) \quad -\operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > \gamma, \quad z \in \Delta^*.$$

Definition 1.1. A function $f(z) \in \Sigma_p$ is said to be in the subclass $X_p^*(j)$ if it satisfies the inequality

$$(1.4) \quad \left| \frac{(p-1)!}{(-1)^j (p+j-1)!} \cdot \frac{f^{(j)}(z)}{z^{-p-j}} - 1 \right| < 1,$$

where

$$(1.5) \quad f^{(j)}(z) = (-1)^j \frac{(p+j-1)!}{(p-1)! z^{p+j}} + \sum_{k=p}^{\infty} \frac{k!}{(k-j)!} a_k z^{k-j}$$



Title Page

Contents



Page 4 of 9

Go Back

Full Screen

Close

is the j -th differential of $f(z)$ and a function $f(z) \in \Sigma_p$ is said to be in the subclass $Y_p^*(j)$ if it satisfies the inequality

$$(1.6) \quad \left| -\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} - (p+j) \right| < p+j.$$

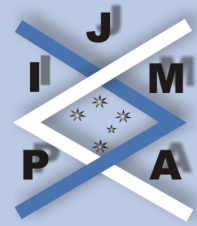
To establish our main results we need the following lemma due to Jack [5].

Lemma 1.2. *Let $w(z)$ be analytic in $\Delta = \{z : |z| < 1\}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0 then*

$$z_0 w'(z_0) = c w(z_0),$$

where c is a real number and $c \geq 1$.

Some different inequalities on p -valent holomorphic and p -valent meromorphic functions by using operators were studied in [1], [2], [3] and [4].



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 5 of 9

Go Back

Full Screen

Close

2. Main Results

Theorem 2.1. If $f(z) \in \Sigma_p$ satisfies the inequality

$$(2.1) \quad \operatorname{Re} \left\{ \frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} + p + j \right\} > 1 - \frac{1}{2p},$$

then $f(z) \in X_p^*(j)$.

Proof. Letting $f(z) \in \Sigma_p$, we define the function $w(z)$ by

$$(2.2) \quad \frac{(p-1)!}{(-1)^j(p+j-1)!} \frac{f^{(j)}(z)}{z^{-p-j}} = 1 - w(z), \quad (z \in \Delta^*).$$

It is easy to verify that $w(0) = 0$.

From (2.2) we obtain

$$f^{(j)}(z) = -\frac{(-1)^j(p+j-1)!}{(p-1)!} z^{-p-j} + \frac{(p+j-1)!}{(p-1)!} z^{-p-j} w(z)$$

or

$$\begin{aligned} [f^{(j)}(z)]' &= (-1)^j(p+j)z^{-p-j-1} \frac{(p+j-1)!}{(p-1)!} \\ &+ (-1)^{j+1}(p+j)z^{-p-j-1} \frac{(p+j-1)!}{(p-1)!} w(z) + (-1)^j \frac{(p+j-1)!}{(p-1)!} z^{-p-j} w'(z). \end{aligned}$$

After a simple calculation we obtain

$$(2.3) \quad \frac{zw'(z)}{1-w(z)} = -\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} - (p+j).$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 6 of 9

Go Back

Full Screen

Close

Now, suppose that there exists a point $z_0 \in \Delta^*$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then, by letting $w(z_0) = e^{i\theta}$ ($w(z_0) \neq 1$) and using Jack's lemma in the equation (2.3), we have

$$\begin{aligned} -\operatorname{Re} \left\{ \frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} + p + j \right\} &= \operatorname{Re} \left\{ \frac{z_0 w'(z_0)}{1 - w(z_0)} \right\} = \operatorname{Re} \left\{ \frac{c w(z_0)}{1 - w(z_0)} \right\} \\ &= c \operatorname{Re} \left\{ \frac{e^{i\theta}}{1 - e^{i\theta}} \right\} = \frac{-c}{2} < \frac{-1}{2} \end{aligned}$$

which contradicts the hypothesis (2.1). Hence we conclude that for all z , $|w(z)| < 1$ and from (2.2) we have

$$\left| \frac{(p-1)! f^{(j)}(z)}{(-1)^j (p+j-1)! z^{-p-j}} - 1 \right| = |w(z)| < 1$$

and this gives the result. □

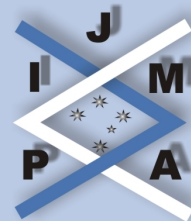
Theorem 2.2. *If $f(z) \in \Sigma_p$ satisfies the inequality*

$$(2.4) \quad \operatorname{Re} \left\{ \frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} - \left(1 + \frac{z[f^{(j)}(z)]''}{[f^{(j)}(z)]'} \right) \right\} > \frac{2p+1}{2(p+1)},$$

then $f(z) \in Y_p^*(j)$.

Proof. Let $f(z) \in \Sigma_p$. We consider the function $w(z)$ as follows:

$$(2.5) \quad -\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} = (p+j)(1-w(z)).$$



[Title Page](#)

[Contents](#)

◀◀ ▶▶

◀ ▶

Page 7 of 9

[Go Back](#)

[Full Screen](#)

[Close](#)

It is easy to see that $w(0) = 0$. Furthermore, by differentiating both sides of (2.5) we get

$$-\left[1 + \frac{z[f^{(j)}(z)]''}{[f^{(j)}(z)]'}\right] = (p+j)(1-w(z)) + \frac{zw'(z)}{1-w(z)}.$$

Now suppose that there exists a point $z_0 \in \Delta^*$ such that $\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1$.

Then, by letting $w(z_0) = e^{i\theta}$ and using Jack's lemma we have

$$\begin{aligned} \operatorname{Re} \left\{ \frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} - \left(1 + \frac{z[f^{(j)}(z)]''}{f^{(j)}(z)'}\right) \right\} &= \operatorname{Re} \left\{ \frac{z_0 w'(z_0)}{1-w(z_0)} \right\} \\ &= c \operatorname{Re} \left\{ \frac{e^{i\theta}}{1-e^{i\theta}} \right\} = -\frac{c}{2} < -\frac{1}{2}, \end{aligned}$$

which contradicts the condition (2.4). So we conclude that $|w(z)| < 1$ for all $z \in \Delta^*$. Hence from (2.5) we obtain

$$\left| -\frac{z[f^{(j)}(z)]'}{[f^{(j)}(z)]'} - (p+j) \right| < p+j.$$

This completes the proof. □

By taking $j = 0$ in Theorems 2.1 and 2.2, we obtain the following corollaries.

Corollary 2.3. *If $f(z) \in \Sigma_p$ satisfies the inequality*

$$-\operatorname{Re} \left\{ \frac{zf'}{f} + p \right\} > 1 - \frac{1}{2p},$$

then

$$\left| \frac{f(z)}{z^{-p}} - 1 \right| < 1.$$



Title Page

Contents



Page 8 of 9

Go Back

Full Screen

Close

Corollary 2.4. If $f(z) \in \Sigma_p$ satisfies the inequality

$$-\operatorname{Re} \left\{ \frac{zf'}{f} - \left(1 + \frac{zf''}{f'} \right) \right\} > \frac{2p+1}{2(p+1)},$$

then $\left| -\frac{zf'}{f} - p \right| < p$ or equivalently $f(z)$ is meromorphically p -valent starlike with respect to the origin.

By taking $j = 1$ in Theorems 2.1 and 2.2, we obtain the following corollaries.

Corollary 2.5. If $f(z) \in \Sigma_p$ satisfies the inequality

$$-\operatorname{Re} \left\{ \frac{zf''}{f'} + p + 1 \right\} > 1 - \frac{1}{2p},$$

then $\left| -\frac{f'(z)}{z^{-p-1}} - p \right| < p$ or equivalently $f(z)$ is meromorphically p -valent close-to-convex with respect to the origin.

Corollary 2.6. If $f(z) \in \Sigma_p$ satisfies the inequality

$$-\operatorname{Re} \left\{ \frac{zf''}{f'} - \left(1 + \frac{zf'''}{f''} \right) \right\} > \frac{2p+1}{2(p+1)},$$

then

$$\left| -\frac{zf''}{f'} - (p+1) \right| < p+1,$$

or equivalently $f(z)$ is meromorphically multivalent convex.

References

- [1] H. IRMAK AND O.F. CETIN, Some theorems involving inequalities on p -valent functions, *Turk. J. Math.*, **23** (1999), 453–459.
- [2] H. IRMAK, N.E. CHO, O.F. CETIN AND R.K. RAINA, Certain inequalities involving meromorphically functions, *Hacet. Bull. Nat. Sci. Eng. Ser. B*, **30** (2001), 39–43.
- [3] H. IRMAK AND R.K. RAINA, On certain classes of functions associated with multivalently analytic and multivalently meromorphic functions, *Soochow J. Math.*, **32**(3) (2006), 413–419.
- [4] H. IRMAK AND R.K. RAINA, New classes of non-normalized meromorphically multivalent functions, *Sarajevo J. Math.*, **3**(16)(2) (2007), 157–162.
- [5] I.S. JACK, Functions starlike and convex of order α , *J. London Math. Soc.*, **3** (1971), 469–474.



Meromorphically p -valent
Functions

A. Ebadian and Sh. Najafzadeh
vol. 9, iss. 4, art. 114, 2008

Title Page

Contents



Page 9 of 9

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756