journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 10 (2009), Issue 2, Article 40, 8 pp.

ON A CERTAIN SUBCLASS OF STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

A. Y. LASHIN

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
MANSOURA UNIVERSITY
MANSOURA, 35516, EGYPT.
aylashin@yahoo.com

Received 11 December, 2007; accepted 29 May, 2009
Communicated by S.S. Dragomir

ABSTRACT. We introduce the class $\overline{H}(\alpha, \beta)$ of analytic functions with negative coefficients. In this paper we give some properties of functions in the class $\overline{H}(\alpha, \beta)$ and we obtain coefficient estimates, neighborhood and integral means inequalities for the function $f(z)$ belonging to the class $\overline{H}(\alpha, \beta)$. We also establish some results concerning the partial sums for the function $f(z)$ belonging to the class $\overline{H}(\alpha, \beta)$.

Key words and phrases: Univalent functions, Starlike functions, Integral means, Neighborhoods, Partial sums.

2000 *Mathematics Subject Classification.* 30C45.

1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. And let S denote the subclass of A consisting of univalent functions $f(z)$ in U.

A function $f(z)$ in S is said to be starlike of order α if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in U),$$

for some α $(0 \leq \alpha < 1)$. We denote by $S^*(\alpha)$ the class of all functions in S which are starlike of order α. It is well-known that

$$S^*(\alpha) \subseteq S^*(0) \equiv S^*.$$

The author would like to thank the referee of the paper for his helpful suggestions.
368-07

Further, a function $f(z)$ in S is said to be convex of order α in U if and only if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \qquad (z\in U),$$

for some α $(0\le\alpha<1)$. We denote by $K(\alpha)$ the class of all functions in S which are convex of order α.

The classes $S^*(\alpha)$, and $K(\alpha)$ were first introduced by Robertson [8], and later were studied by Schild [10], MacGregor [4], and Pinchuk [7].

Let T denote the subclass of S whose elements can be expressed in the form:

$$f(z)=z-\sum_{k=2}^{\infty}a_kz^k \qquad (a_k\ge 0). \tag{1.2}$$

We denote by $T^*(\alpha)$ and $C(\alpha)$, respectively, the classes obtained by taking the intersections of $S^*(\alpha)$ and $K(\alpha)$ with T,

$$T^*(\alpha)=S^*(\alpha)\cap T \quad\text{and}\quad C(\alpha)=K(\alpha)\cap T.$$

The classes $T^*(\alpha)$ and $C(\alpha)$ were introduced by Silverman [11].

Let $H(\alpha,\beta)$ denote the class of functions $f(z)\in A$ which satisfy the condition

$$\operatorname{Re}\left(\frac{\alpha z^2f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\right)>\beta$$

for some $\alpha\ge 0$, $0\le\beta<1$, $\frac{f(z)}{z}\neq 0$ and $z\in U$.

The classes $H(\alpha,\beta)$ and $H(\alpha,0)$ were introduced and studied by Obraddovic and Joshi [5], Padmanabhan [6], Li and Owa [2], Xu and Yang [14], Singh and Gupta [13], and others.

Further, we denote by $\overline{H}(\alpha,\beta)$ the class obtained by taking intersections of the class $H(\alpha,\beta)$ with T, that is

$$\overline{H}(\alpha,\beta)=H(\alpha,\beta)\cap T.$$

We note that

$$\overline{H}(0,\beta)=T^*(\beta) \qquad \text{(Silverman [11])}.$$

2. COEFFICIENT ESTIMATES

Theorem 2.1. *A function $f(z)\in T$ is in the class $\overline{H}(\alpha,\beta)$ if and only if*

$$\sum_{k=2}^{\infty}[(k-1)(\alpha k+1)+(1-\beta)]a_k\le 1-\beta. \tag{2.1}$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds and let $|z|<1$. Then we have

$$\begin{aligned}\left|\frac{\alpha z^2f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}-1\right| &= \left|\frac{-\sum_{k=2}^{\infty}(k-1)(\alpha k+1)a_kz^{k-1}}{1-\sum_{k=2}^{\infty}a_kz^{k-1}}\right| \\ &\le \frac{\sum_{k=2}^{\infty}(k-1)(\alpha k+1)a_k}{1-\sum_{k=2}^{\infty}a_k}\le 1-\beta.\end{aligned}$$

This shows that the values of $\frac{\alpha z^2f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}$ lie in the circle centered at $w=1$ whose radius is $1-\beta$. Hence $f(z)$ is in the class $\overline{H}(\alpha,\beta)$.

To prove the converse, assume that $f(z)$ defined by (1.2) is in the class $\overline{H}(\alpha,\beta)$. Then

$$\operatorname{Re}\left(\frac{\alpha z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\right)=\operatorname{Re}\left(\frac{1-\sum_{k=2}^{\infty}[\alpha k(k-1)+k)]a_k z^{k-1}}{1-\sum_{k=2}^{\infty}a_k z^{k-1}}\right)>\beta \tag{2.2}$$

for $z\in U$. Choose values of z on the real axis so that $\frac{\alpha z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}$ is real. Upon clearing the denominator in (2.2) and letting $z\to 1^-$ through real values, we have

$$\beta\left(1-\sum_{k=2}^{\infty}a_k\right)\le 1-\sum_{k=2}^{\infty}[\alpha k(k-1)+k]a_k,$$

which obviously is the required result (2.1).

Finally, we note that the assertion (2.1) of Theorem 2.1 is sharp, with the extremal function being

$$f(z)=z-\frac{1-\beta}{[(k-1)(\alpha k+1)+(1-\beta)]}z^k \qquad (k\ge 2). \tag{2.3}$$

□

Corollary 2.2. *Let $f(z)\in T$ be in the class $\overline{H}(\alpha,\beta)$. Then we have*

$$a_k\le\frac{1-\beta}{[(k-1)(\alpha k+1)+(1-\beta)]} \qquad (k\ge 2). \tag{2.4}$$

Equality in (2.4) holds true for the function $f(z)$ given by (2.3).

3. SOME PROPERTIES OF THE CLASS $\overline{H}(\alpha,\beta)$

Theorem 3.1. *Let $0\le\alpha_1<\alpha_2$ and $0\le\beta<1$. Then $\overline{H}(\alpha_2,\beta)\subset\overline{H}(\alpha_1,\beta)$.*

Proof. It follows from Theorem 2.1. That

$$\sum_{k=2}^{\infty}[(k-1)(\alpha_1 k+1)+(1-\beta)]a_k<\sum_{k=2}^{\infty}[(k-1)(\alpha_2 k+1)+(1-\beta)]a_k\le 1-\beta$$

for $f(z)\in\overline{H}(\alpha_2,\beta)$. Hence $f(z)\in\overline{H}(\alpha_1,\beta)$. □

Corollary 3.2. $\overline{H}(\alpha,\beta)\subseteq T^*(\beta)$.

The proof is now immediate because $\alpha\ge 0$.

4. NEIGHBORHOOD RESULTS

Following the earlier investigations of Goodman [1] and Ruscheweyh [9], we define the $\delta-$neighborhood of function $f(z)\in T$ by:

$$N_\delta(f)=\left\{g\in T: g(z)=z-\sum_{k=2}^{\infty}b_k z^k,\ \sum_{k=2}^{\infty}k\,|a_k-b_k|\le\delta\right\}.$$

In particular, for the identity function

$$e(z)=z,$$

we immediately have

$$N_\delta(e)=\left\{g\in T: g(z)=z-\sum_{k=2}^{\infty}b_k z^k,\ \sum_{k=2}^{\infty}k\,|b_k|\le\delta\right\}. \tag{4.1}$$

Theorem 4.1. $\overline{H}(\alpha,\beta)\subseteq N_\delta(e)$, *where* $\delta=\frac{2(1-\beta)}{(2\alpha+2-\beta)}$.

Proof. Let $f(z) \in \overline{H}(\alpha,\beta)$. Then, in view of Theorem 2.1, since $[(k-1)(\alpha k+1)+(1-\beta)]$ is an increasing function of k $(k \geq 2)$, we have

$$(2\alpha+2-\beta)\sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty}[(k-1)(\alpha k+1)+(1-\beta)]a_k \leq 1-\beta,$$

which immediately yields

$$\sum_{k=2}^{\infty} a_k \leq \frac{1-\beta}{(2\alpha+2-\beta)}. \tag{4.2}$$

On the other hand, we also find from (2.1)

$$\begin{aligned}(\alpha+1)\sum_{k=2}^{\infty} ka_k - \beta\sum_{k=2}^{\infty} a_k &\leq \sum_{k=2}^{\infty}[(\alpha(k-1)+1)k-\beta)]a_k \\ &= \sum_{k=2}^{\infty}[(k-1)(\alpha k+1)+(1-\beta)]a_k \leq 1-\beta. \end{aligned} \tag{4.3}$$

From (4.3) and (4.2), we have

$$\begin{aligned}(\alpha+1)\sum_{k=2}^{\infty} ka_k &\leq (1-\beta)+\beta\sum_{k=2}^{\infty} a_k \\ &\leq (1-\beta)+\beta\frac{1-\beta}{(2\alpha+2-\beta)} \\ &\leq \frac{2(\alpha+1)(1-\beta)}{(2\alpha+2-\beta)},\end{aligned}$$

that is,

$$\sum_{k=2}^{\infty} ka_k \leq \frac{2\,(1-\beta)}{(2\alpha+2-\beta)} = \delta, \tag{4.4}$$

which in view of the definition (4.1), prove Theorem 4.1. □

Letting $\alpha = 0$, in the above theorem, we have:

Corollary 4.2. *$T^*(\beta) \subseteq N_\delta(e)$, where $\delta = \frac{2(1-\beta)}{(2-\beta)}$.*

5. INTEGRAL MEANS INEQUALITIES

We need the following lemma.

Lemma 5.1 ([3])**.** *If f and g are analytic in U with $f \prec g$, then*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^{\delta} d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^{\delta} d\theta,$$

where $\delta > 0$, $z = re^{i\theta}$ and $0 < r < 1$.

Applying Lemma 5.1, and (2.1), we prove the following theorem.

Theorem 5.2. *Let $\delta > 0$. If $f(z) \in \overline{H}(\alpha,\beta)$, then for $z = re^{i\theta}$, $0 < r < 1$, we have*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\delta} d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^{\delta} d\theta,$$

where

$$f_2(z) = z - \frac{(1-\beta)}{(2\alpha+2-\beta)} z^2. \tag{5.1}$$

Proof. Let $f(z)$ defined by (1.2) and $f_2(z)$ be given by (5.1). We must show that

$$\int_0^{2\pi} \left| 1 - \sum_{k=2}^{\infty} a_k z^{k-1} \right|^{\delta} d\theta \le \int_0^{2\pi} \left| 1 - \frac{(1-\beta)}{(2\alpha+2-\beta)} z \right|^{\delta} d\theta.$$

By Lemma 5.1, it suffices to show that

$$1 - \sum_{k=2}^{\infty} a_k z^{k-1} \prec 1 - \frac{(1-\beta)}{(2\alpha+2-\beta)} z.$$

Setting

$$1 - \sum_{k=2}^{\infty} a_k z^{k-1} = 1 - \frac{(1-\beta)}{(2\alpha+2-\beta)} w(z). \tag{5.2}$$

From (5.2) and (2.1), we obtain

$$\begin{aligned} |w(z)| &= \left| \sum_{k=2}^{\infty} \frac{(2\alpha+2-\beta)}{(1-\beta)} a_k z^{k-1} \right| \\ &\le |z| \sum_{k=2}^{\infty} \frac{[(k-1)(\alpha k+1)+(1-\beta)]}{1-\beta} a_k \le |z|. \end{aligned}$$

This completes the proof of the theorem. □

Letting $\alpha = 0$ in the above theorem, we have:

Corollary 5.3. *Let $\delta > 0$. If $f(z) \in T^*(\beta)$, then for $z = re^{i\theta}$, $0 < r < 1$, we have*

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^{\delta} d\theta \le \int_0^{2\pi} \left| f_2(re^{i\theta}) \right|^{\delta} d\theta,$$

where

$$f_2(z) = z - \frac{(1-\beta)}{(2-\beta)} z^2.$$

6. PARTIAL SUMS

In this section we will examine the ratio of a function of the form (1.2) to its sequence of partial sums defined by $f_1(z) = z$ and $f_n(z) = z - \sum_{k=2}^{n} a_k z^k$ when the coefficients of f are sufficiently small to satisfy the condition (2.1). We will determine sharp lower bounds for $\operatorname{Re}\left(\frac{f(z)}{f_n(z)}\right)$, $\operatorname{Re}\left(\frac{f_n(z)}{f(z)}\right)$, $\operatorname{Re}\left(\frac{f'(z)}{f_n'(z)}\right)$ and $\operatorname{Re}\left(\frac{f_n'(z)}{f'(z)}\right)$.

In what follows, we will use the well known result that

$$\operatorname{Re} \frac{1-w(z)}{1+w(z)} > 0, \qquad z \in U,$$

if and only if

$$w(z) = \sum_{k=1}^{\infty} c_k z^k$$

satisfies the inequality $|w(z)| \le |z|$.

Theorem 6.1. *If $f(z) \in \overline{H}(\alpha,\beta)$, then*

$$\operatorname{Re}\frac{f(z)}{f_n(z)} \geq 1-\frac{1}{c_{n+1}} \qquad (z\in U,\ n\in N) \tag{6.1}$$

and

$$\operatorname{Re}\left(\frac{f_n(z)}{f(z)}\right) \geq \frac{c_{n+1}}{1+c_{n+1}} \qquad (z\in U,\ n\in N)\,, \tag{6.2}$$

where $\left(c_k =: \frac{[(k-1)(\alpha k+1)+(1-\beta)]}{1-\beta}\right)$. *The estimates in (6.1) and (6.2) are sharp.*

Proof. We employ the same technique used by Silverman [12]. The function $f(z) \in \overline{H}(\alpha,\beta)$, if and only if $\sum_{k=2}^{\infty} c_k a_k \leq 1$. It is easy to verify that $c_{k+1} > c_k > 1$. Thus,

$$\sum_{k=2}^{n} a_k + c_{n+1}\sum_{k=n+1}^{\infty} a_k \leq \sum_{k=2}^{\infty} c_k a_k \leq 1. \tag{6.3}$$

We may write

$$c_{n+1}\left\{\frac{f(z)}{f_n(z)}-\left(1-\frac{1}{c_{n+1}}\right)\right\} = \frac{1-\sum_{k=2}^{n} a_k z^{k-1} - c_{n+1}\sum_{k=n+1}^{\infty} a_k z^{k-1}}{1-\sum_{k=2}^{n} a_k z^{k-1}} = \frac{1+D(z)}{1+E(z)}.$$

Set

$$\frac{1+D(z)}{1+E(z)} = \frac{1-w(z)}{1+w(z)},$$

so that

$$w(z) = \frac{E(z)-D(z)}{2+D(z)+E(z)}.$$

Then

$$w(z) = \frac{c_{n+1}\sum_{k=n+1}^{\infty} a_k z^{k-1}}{2-2\sum_{k=2}^{n} a_k z^{k-1} - c_{n+1}\sum_{k=n+1}^{\infty} a_k z^{k-1}}$$

and

$$|w(z)| \leq \frac{c_{n+1}\sum_{k=n+1}^{\infty} a_k}{2-2\sum_{k=2}^{n} a_k - c_{n+1}\sum_{k=n+1}^{\infty} a_k}.$$

Now $|w(z)| \leq 1$ if and only if

$$\sum_{k=2}^{n} a_k + c_{n+1}\sum_{k=n+1}^{\infty} a_k \leq 1,$$

which is true by (6.3). This readily yields the assertion (6.1) of Theorem 6.1.

To see that

$$f(z) = z - \frac{z^{n+1}}{c_{n+1}} \tag{6.4}$$

gives sharp results, we observe that

$$\frac{f(z)}{f_n(z)} = 1-\frac{z^n}{c_{n+1}}.$$

Letting $z \to 1^-$, we have

$$\frac{f(z)}{f_n(z)} = 1-\frac{1}{c_{n+1}},$$

which shows that the bounds in (6.1) are the best possible for each $n \in N$. Similarly, we take

$$(1+c_{n+1})\left(\frac{f_n(z)}{f(z)} - \frac{c_{n+1}}{1+c_{n+1}}\right) = \frac{1-\sum_{k=2}^{n} a_k z^{k-1} + c_{n+1}\sum_{k=n+1}^{\infty} a_k z^{k-1}}{1-\sum_{k=2}^{\infty} a_k z^{k-1}} := \frac{1-w(z)}{1+w(z)},$$

where

$$|w(z)| \le \frac{(1+c_{n+1})\sum_{k=n+1}^{\infty} a_k}{2-2\sum_{k=2}^{n} a_k + (1-c_{n+1})\sum_{k=n+1}^{\infty} a_k}.$$

Now $|w(z)| \le 1$ if and only if

$$\sum_{k=2}^{n} a_k + c_{n+1}\sum_{k=n+1}^{\infty} a_k \le 1,$$

which is true by (6.3). This immediately leads to the assertion (6.2) of Theorem 6.1.

The estimate in (6.2) is sharp with the extremal function $f(z)$ given by (6.4). This completes the proof of Theorem 6.1. □

Letting $\alpha = 0$ in the above theorem, we have:

Corollary 6.2. *If $f(z) \in T^*(\beta)$, then*

$$\operatorname{Re}\frac{f(z)}{f_n(z)} \ge \frac{n}{(n+1-\beta)}, \qquad (z \in U)$$

and

$$\operatorname{Re}\frac{f_n(z)}{f(z)} \ge \frac{n+1-\beta}{(n+2-2\beta)}, \qquad (z \in U)\,.$$

The result is sharp for every n, with the extremal function

$$f(z) = z - \frac{1-\beta}{(n+1-\beta)} z^{n+1}. \tag{6.5}$$

We now turn to the ratios involving derivatives. The proof of Theorem 6.3 below follows the pattern of that in Theorem 6.1, and so the details may be omitted.

Theorem 6.3. *If $f(z) \in \overline{H}(\alpha,\beta)$, then*

$$\operatorname{Re}\frac{f'(z)}{f'_n(z)} \ge 1 - \frac{n+1}{c_{n+1}} \qquad (z \in U)\,, \tag{6.6}$$

and

$$\operatorname{Re}\left(\frac{f'_n(z)}{f'(z)}\right) \ge \frac{c_{n+1}}{n+1+c_{n+1}} \qquad (z \in U,\ n \in N)\,. \tag{6.7}$$

The estimates in (6.6) and (6.7) are sharp with the extremal function given by (6.4).

Letting $\alpha = 0$ in the above theorem, we have:

Corollary 6.4. *If $f(z) \in T^*(\beta)$, then*

$$\operatorname{Re}\frac{f'(z)}{f'_n(z)} \ge \frac{\beta n}{(n+1-\beta)}, \qquad (z \in U)\,,$$

and

$$\operatorname{Re}\frac{f'_n(z)}{f'(z)} \ge \frac{n+1-\beta}{n+(1-\beta)(n+2)}, \qquad (z \in U)\,.$$

The result is sharp for every n, with the extremal function given by (6.5).

REFERENCES

[1] A.W. GOODMAN, Univalent functions and analytic curves, *Proc. Amer. Math. Soc.*, **8**(3) (1957), 598–601.

[2] J.L. LI AND S. OWA, Sufficient conditions for starlikeness, *Indian J. Pure Appl. Math.*, **33** (2002), 313-318.

[3] J.E. LITTLEWOOD, On inequalities in the theory of functions, *Proc. London Math. Soc.*, **23** (1925), 481–519.

[4] T.H. MacGREGOR, The radius for starlike functions of order $\frac{1}{2}$, *Proc. Amer. Math. Soc.*, **14** (1963), 71–76.

[5] M. OBRADOVIC AND S.B. JOSHI, On certain classes of strongly starlike functions, *Taiwanese J. Math.*, **2**(3) (1998), 297–302.

[6] K.S. PADMANABHAN, On sufficient conditions for starlikeness, *Indian J. Pure Appl. Math.*, **32**(4) (2001), 543–550.

[7] B. PINCHUK, On starlike and convex functions of order a, *Duke Math. J.*, **35** (1968), 89–103.

[8] M.S. ROBERTSON, On the theory of univalent functions, *Ann. of Math.*, **37** (1936), 374–408.

[9] S. RUSCHEWEYH, Neighborhoods of univalent functions, *Proc. Amer. Math. Soc.*, **81**(4) (1981), 521–527.

[10] A. SCHILD, On starlike functions of order a, *Amer. J. Math.*, **87** (1965), 65–70.

[11] H. SILVERMAN, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, **51** (1975), 109–116.

[12] H. SILVERMAN, Partial sums of starlike and convex functions, *J. Math. Anal. Appl.*, **209** (1997), 221–227.

[13] S. SINGH AND S. GUPTA, First order differential subordinations and starlikeness of analytic maps in the unit disc, *Kyungpook Math. J.*, **45** (2005), 395–404.

[14] N. XU AND D. YANG, Some criteria for starlikeness and strongly starlikeness, *Bull. Korean Math. Soc.*, **42**(3) (2005), 579–590.