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A VARIANT OF JESSEN'S INEQUALITY OF MERCER'S TYPE FOR SUPERQUADRATIC FUNCTIONS

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ABSTRACT. A variant of Jessen's inequality for superquadratic functions is proved. This is a refinement of a variant of Jessen's inequality of Mercer's type for convex functions. The result is used to refine some comparison inequalities of Mercer's type between functional power means and between functional quasi-arithmetic means.

Key words and phrases: Isotonic linear functionals, Jessen's inequality, Superquadratic functions, Functional quasi-arithmetic and power means of Mercer's type.

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1. INTRODUCTION

Let E be a nonempty set and L be a linear class of real valued functions $f : E \to \mathbb{R}$ having the properties:

$L1$: $f, g \in L \Rightarrow (\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;

$L2$: $1 \in L$, i.e., if $f(t) = 1$ for $t \in E$, then $f \in L$.

An isotonic linear functional is a functional $A : L \to \mathbb{R}$ having the properties:

$A1$: $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for $f, g \in L$, $\alpha, \beta \in \mathbb{R}$ (A is linear);

$A2$: $f \in L$, $f(t) \geq 0$ on $E \Rightarrow A(f) \geq 0$ (A is isotonic).

The following result is Jessen's generalization of the well known Jensen's inequality for convex functions [10] (see also [12, p. 47]):

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Theorem A. *Let L satisfy properties $L1$, $L2$ on a nonempty set E, and let φ be a continuous convex function on an interval $I \subset \mathbb{R}$. If A is an isotonic linear functional on L with $A(1) = 1$, then for all $g \in L$ such that $\varphi(g) \in L$, we have $A(g) \in I$ and*

$$\varphi(A(g)) \leq A(\varphi(g)).$$

Similar to Jensen's inequality, Jessen's inequality has a converse [7] (see also [12, p. 98]):

Theorem B. *Let L satisfy properties $L1$, $L2$ on a nonempty set E, and let φ be a convex function on an interval $I = [m, M]$, $-\infty < m < M < \infty$. If A is an isotonic linear functional on L with $A(1) = 1$, then for all $g \in L$ such that $\varphi(g) \in L$ so that $m \leq g(t) \leq M$ for all $t \in E$, we have*

$$A(\varphi(g)) \leq \frac{M - A(g)}{M - m} \cdot \varphi(m) + \frac{A(g) - m}{M - m} \cdot \varphi(M).$$

Inspired by I.Gavrea's [9] result, which is a generalization of Mercer's variant of Jensen's inequality [11], recently, W.S. Cheung, A. Matković and J. Pečarić, [8] gave the following extension on a linear class L satisfying properties $L1$, $L2$.

Theorem C. *Let L satisfy properties $L1$, $L2$ on a nonempty set E, and let φ be a continuous convex function on an interval $I = [m, M]$, $-\infty < m < M < \infty$. If A is an isotonic linear functional on L with $A(1) = 1$, then for all $g \in L$ such that $\varphi(g), \varphi(m + M - g) \in L$ so that $m \leq g(t) \leq M$ for all $t \in E$, we have the following variant of Jessen's inequality*

$$\varphi(m + M - A(g)) \leq \varphi(m) + \varphi(M) - A(\varphi(g)). \tag{1.1}$$

In fact, to be more specific we have the following series of inequalities

$$\begin{aligned} &\varphi(m + M - A(g)) \\ &\qquad \leq A(\varphi(m + M - g)) \\ &\qquad \leq \frac{M - A(g)}{M - m} \cdot \varphi(M) + \frac{A(g) - m}{M - m} \cdot \varphi(m) \\ &\qquad \leq \varphi(m) + \varphi(M) - A(\varphi(g)). \end{aligned} \tag{1.2}$$

If the function φ is concave, inequalities (1.1) and (1.2) are reversed.

In this paper we give an analogous result for superquadratic function (see also different analogous results in [6]). We start with the following definition.

Definition A ([1, Definition 2.1])**.** *A function $\varphi : [0, \infty) \to \mathbb{R}$ is **superquadratic** provided that for all $x \geq 0$ there exists a constant $C(x) \in \mathbb{R}$ such that*

$$\varphi(y) - \varphi(x) - \varphi(|y - x|) \geq C(x)(y - x) \tag{1.3}$$

*for all $y \geq 0$. We say that f is **subquadratic** if $-f$ is a superquadratic function.*

For example, the function $\varphi(x) = x^p$ is superquadratic for $p \geq 2$ and subquadratic for $p \in (0, 2]$.

Theorem D ([1, Theorem 2.3])**.** *The inequality*

$$f\left(\int g d\mu\right) \leq \int \left(f(g(s)) - f\left(\left|g(s) - \int g d\mu\right|\right)\right) d\mu(s)$$

holds for all probability measures μ and all non-negative $\mu-$integrable functions g, if and only if f is superquadratic.

The following discrete version that follows from the above theorem is also used in the sequel.

Lemma A. *Suppose that f is superquadratic. Let $x_r \geq 0$, $1 \leq r \leq n$ and let $\bar{x} = \sum_{r=1}^{n} \lambda_r x_r$ where $\lambda_r \geq 0$ and $\sum_{r=1}^{n} \lambda_r = 1$. Then*

$$\sum_{r=1}^{n} \lambda_r f(x_r) \geq f(\bar{x}) + \sum_{r=1}^{n} \lambda_r f(|x_r - \bar{x}|).$$

In [3] and [4] the following converse of Jensen's inequality for superquadratic functions was proved.

Theorem E. *Let(Ω, A, μ) be a measurable space with $0 < \mu(r) < \infty$ and let $f : [0, \infty) \to \mathbb{R}$ be a superquadratic function. If $g : \Omega \to [m, M] \leq [0, \infty)$ is such that $g,\ f \circ g \in L_1(\mu)$, then we have*

$$\frac{1}{\mu(\Omega)} \int_{\Omega} f(g) d\mu \leq \frac{M - \bar{g}}{M - m} f(m) + \frac{\bar{g} - m}{M - m} f(M) - \frac{1}{\mu(\Omega)} \frac{1}{M - m} \int_{\Omega} \left((M - g) f(g - m) + (g - m) f(M - g) \right) d\mu,$$

for $\bar{g} = \frac{1}{\mu(\Omega)} \int_{\Omega} g d\mu$.

The discrete version of this theorem is:

Theorem F. *Let $f : [0, \infty) \to \mathbb{R}$ be a superquadratic function. Let $(x_1, \ldots, x_n)$ be an $n-$tuple in $[m, M]^n$ $(0 \leq m < M < \infty)$, and $(p_1, \ldots, p_n)$ be a non-negative $n-$tuple such that $P_n = \sum_{i=1}^{n} p_i > 0$. Denote $\bar{x} = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i$, then*

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \leq \frac{M - \bar{x}}{M - m} f(m) + \frac{\bar{x} - m}{M - m} f(M) - \frac{1}{P_n (M - m)} \sum_{i=1}^{n} p_i \left[(M - x_i) f(x_i - m) + (x_i - m) f(M - x_i) \right].$$

In Section 2 we give the main result of our paper which is an analogue of Theorem C for superquadratic functions. In Section 3 we use that result to derive some refinements of the inequalities obtained in [8] which involve functional power means of Mercer's type and functional quasi-arithmetic means of Mercer's type.

2. MAIN RESULTS

Theorem 2.1. *Let L satisfy properties $L1$, $L2$, on a nonempty set E, $\varphi : [0, \infty) \to \mathbb{R}$ be a continuous superquadratic function, and $0 \leq m < M < \infty$. Assume that A is an isotonic linear functional on L with $A(1) = 1$. If $g \in L$ is such that $m \leq g(t) \leq M$, for all $t \in E$, and such that $\varphi(g)$, $\varphi(m + M - g)$, $(M - g)\varphi(g - m)$, $(g - m)\varphi(M - g) \in L$, then we have*

$$\begin{aligned} &\varphi(m + M - A(g)) \\ &\leq \frac{A(g) - m}{M - m} \varphi(m) + \frac{M - A(g)}{M - m} \varphi(M) \\ &\qquad - \frac{1}{M - m} \left[(A(g) - m)\varphi(M - A(g)) + (M - A(g))\varphi(A(g) - m) \right] \end{aligned}$$

$$\begin{aligned}(2.1)\qquad &\leq \varphi(m)+\varphi(M)-A\left(\varphi(g)\right)\\ &\quad-\frac{1}{M-m}A\left((g-m)\varphi(M-g)+(M-g)\varphi(g-m)\right)\\ &\quad-\frac{1}{M-m}\left[(A(g)-m)\varphi(M-A(g))+(M-A(g))\varphi(A(g)-m)\right].\end{aligned}$$

If the function φ is subquadratic, then all the inequalities above are reversed.

Proof. From Lemma A for $n=2$, as well as from Theorem F, we get that for $0\leq m\leq t\leq M$,

$$\varphi(t)\leq\frac{M-t}{M-m}\varphi(m)+\frac{t-m}{M-m}\varphi(M)-\frac{M-t}{M-m}\varphi(t-m)-\frac{t-m}{M-m}\varphi(M-t). \tag{2.2}$$

Replacing t with $M+m-t$ in (2.2) it follows that

$$\begin{aligned}\varphi(M+m-t)&\leq\frac{t-m}{M-m}\varphi(m)+\frac{M-t}{M-m}\varphi(M)\\ &\qquad-\frac{t-m}{M-m}\varphi(M-t)-\frac{M-t}{M-m}\varphi(t-m)\\ &=\varphi(m)+\varphi(M)-\left[\frac{t-m}{M-m}\varphi(M)+\frac{M-t}{M-m}\varphi(m)\right]\\ &\qquad-\frac{t-m}{M-m}\varphi(M-t)-\frac{M-t}{M-m}\varphi(t-m).\end{aligned}$$

Since $m\leq g(t)\leq M$ for all $t\in E,$ it follows that $m\leq A(g)\leq M$ and we have

$$\begin{aligned}\varphi(m+M-A(g))&\leq\varphi(m)+\varphi(M)-\left[\frac{A(g)-m}{M-m}\varphi(M)+\frac{M-A(g)}{M-m}\varphi(m)\right]\\ &\qquad-\frac{A(g)-m}{M-m}\varphi(M-A(g))-\frac{M-A(g)}{M-m}\varphi(A(g)-m).\end{aligned}\tag{2.3}$$

On the other hand, since $m\leq g(t)\leq M$ for all $t\in E$ it follows that

$$\begin{aligned}\varphi(g(t))&\leq\frac{M-g(t)}{M-m}\varphi(m)+\frac{g(t)-m}{M-m}\varphi(M)\\ &\qquad-\frac{M-g(t)}{M-m}\varphi(g(t)-m)-\frac{g(t)-m}{M-m}\varphi(M-g(t)).\end{aligned}$$

Using functional calculus we have

$$\begin{aligned}A(\varphi(g))&\leq\frac{M-A(g)}{M-m}\varphi(m)+\frac{A(g)-m}{M-m}\varphi(M)-\frac{1}{M-m}A\left((M-g(t)\varphi(g(t)-m)\right)\\ &\qquad-\frac{1}{M-m}A\left((g(t)-m)\varphi(M-g(t))\right).\end{aligned}\tag{2.4}$$

Using inequalities (2.3) and (2.4), we obtain the desired inequality (2.1).

The last statement follows immediately from the fact that if φ is subquadratic then $-\varphi$ is a superquadratic function. □

Remark 1. If a function φ is superquadratic and nonnegative, then it is convex [1, Lema 2.2]. Hence, in this case inequality (2.1) is a refinement of inequality (1.1).

On the other hand, we can get one more inequality in (2.1) if we use a result of S. Banić and S. Varosǎnec [5] on Jessen's inequality for superquadratic functions:

Theorem 2.2 ([5, Theorem 8, Remark 1]). *Let L satisfy properties $L1$, $L2$, on a nonempty set E, and let $\varphi : [0,\infty) \to \mathbb{R}$ be a continuous superquadratic function. Assume that A is an isotonic linear functional on L with $A(1) = 1$. If $f \in L$ is nonnegative and such that $\varphi(f)$, $\varphi(|f - A(f)|) \in L$, then we have*

$$\varphi(A(f)) \leq A(\varphi(f)) - A(\varphi(|f - A(f)|)). \tag{2.5}$$

If the function φ is subquadratic, then the inequality above is reversed.

Using Theorem 2.2 and some basic properties of superquadratic functions we prove the next theorem.

Theorem 2.3. *Let L satisfy properties $L1$, $L2$, on a nonempty set E, and let $\varphi : [0,\infty) \to \mathbb{R}$ be a continuous superquadratic function, and let $0 \leq m < M < \infty$. Assume that A is an isotonic linear functional on L with $A(1) = 1$. If $g \in L$ is such that $m \leq g(t) \leq M$, for all $t \in E$, and such that $\varphi(g)$, $\varphi(m + M - g)$, $(M - g)\varphi(g - m)$, $(g - m)\varphi(M - g)$, $\varphi(|g - A(g)|) \in L$, then we have*

$$\begin{aligned}
&\varphi(m + M - A(g)) \\
&\leq A(\varphi(m + M - g)) - A(\varphi(|g - A(g)|)) &(2.6)\\
&\leq \frac{A(g) - m}{M - m}\varphi(m) + \frac{M - A(g)}{M - m}\varphi(M) &(2.7)\\
&\qquad - \frac{1}{M - m} A\left((g - m)\varphi(M - g) + (M - g)\varphi(g - m)\right) - A(\varphi(|g - A(g)|)) \\
&\leq \varphi(m) + \varphi(M) - A(\varphi(g)) &(2.8)\\
&\qquad - \frac{2}{M - m} A\left((g - m)\varphi(M - g) + (M - g)\varphi(g - m)\right) - A(\varphi(|g - A(g)|)).
\end{aligned}$$

If the function φ is subquadratic, then all the inequalities above are reversed.

Proof. Notice that $(m + M - g) \in L$. Since $m \leq g(t) \leq M$ for all $t \in E$, it follows that $m \leq m + M - g(t) \leq M$ for all $t \in E$. Applying (2.5) to the function $f = m + M - g$ we get

$$\begin{aligned}
&\varphi(A(m + M - g)) \\
&= \varphi(m + M - A(g)) \\
&\leq A(\varphi(m + M - g)) - A(\varphi(|m + M - g - A(m + M - g)|)) \\
&= A(\varphi(m + M - g)) - A(\varphi(|m + M - g - m - M + A(g)|)) \\
&= A(\varphi(m + M - g)) - A(\varphi(|g - A(g)|)),
\end{aligned}$$

which is the inequality (2.6).

From the discrete Jensen's inequality for superquadratic functions we get for all $m \leq x \leq M$,

$$\varphi(x) \leq \frac{M - x}{M - m}\varphi(m) + \frac{x - m}{M - m}\varphi(M) - \frac{M - x}{M - m}\varphi(x - m) - \frac{x - m}{M - m}\varphi(M - x). \tag{2.9}$$

Replacing x in (2.9) with $m + M - g(t) \in [m, M]$ for all $t \in E$, we have

$$\begin{aligned}
\varphi(m + M - g(t)) \leq{}& \frac{g(t) - m}{M - m}\varphi(m) + \frac{M - g(t)}{M - m}\varphi(M) \\
&- \frac{g(t) - m}{M - m}\varphi(M - g(t)) - \frac{M - g(t)}{M - m}\varphi(g(t) - m).
\end{aligned}$$

Since A is linear, isotonic and satisfies $A(1) = 1$, from the above inequality it follows that

$$A(\varphi(m+M-g)) \leq \frac{A(g)-m}{M-m}\varphi(m) + \frac{M-A(g)}{M-m}\varphi(M) - \frac{1}{M-m}A\left((g-m)\varphi(M-g) + (M-g)\varphi(g-m)\right). \tag{2.10}$$

Adding $-A(\varphi(|g-A(g)|))$ on both sides of (2.10) we get

$$A(\varphi(m+M-g)) - A(\varphi(|g-A(g)|)) \leq \frac{A(g)-m}{M-m}\varphi(m) + \frac{M-A(g)}{M-m}\varphi(M) - \frac{1}{M-m}A\left((g-m)\varphi(M-g) + (M-g)\varphi(g-m)\right) - A(\varphi(|g-A(g)|)), \tag{2.11}$$

which is the inequality (2.7).

The right hand side of (2.11) can be written as follows

$$\varphi(m) + \varphi(M) - \frac{M-A(g)}{M-m}\varphi(m) - \frac{A(g)-m}{M-m}\varphi(M) - \frac{1}{M-m}A\left((g-m)\varphi(M-g) + (M-g)\varphi(g-m)\right) - A(\varphi(|g-A(g)|)). \tag{2.12}$$

On the other hand, replacing x, in (2.9), with $g(t) \in [m, M]$, for all $t \in E$, we get

$$\varphi(g(t)) \leq \frac{M-g(t)}{M-m}\varphi(m) + \frac{g(t)-m}{M-m}\varphi(M) - \frac{M-g(t)}{M-m}\varphi(g(t)-m) - \frac{g(t)-m}{M-m}\varphi(M-g(t)). \tag{2.13}$$

Applying the functional A on (2.13) we have

$$A(\varphi(g)) \leq \frac{M-A(g)}{M-m}\varphi(m) + \frac{A(g)-m}{M-m}\varphi(M) - \frac{1}{M-m}A\left((M-g)\varphi(g-m) + (g-m)\varphi(M-g)\right), \tag{2.14}$$

The inequality (2.14) can be written as follows

$$-\frac{M-A(g)}{M-m}\varphi(m) - \frac{A(g)-m}{M-m}\varphi(M) \leq -A(\varphi(g)) - \frac{1}{M-m}A\left((g-m)\varphi(M-g) + (M-g)\varphi(g-m)\right).$$

Using (2.12) we get

$$\frac{A(g)-m}{M-m}\varphi(m) + \frac{M-A(g)}{M-m}\varphi(M) - \frac{1}{M-m}A\left((g-m)\varphi(M-g) + (M-g)\varphi(g-m)\right) - A(\varphi(|g-A(g)|))$$

$$\begin{aligned}
&\leq \varphi(m)+\varphi(M)-A(\varphi(g)) \\
&\qquad -\frac{1}{M-m}A\left((g-m)\varphi(M-g)+(M-g)\varphi(g-m)\right) \\
&\qquad -\frac{1}{M-m}A\left((g-m)\varphi(M-g)+(M-g)\varphi(g-m)\right)-A(\varphi(|g-A(g)|)) \\
&= \varphi(m)+\varphi(M)-A(\varphi(g)) \\
&\qquad -\frac{2}{M-m}A\left((g-m)\varphi(M-g)+(M-g)\varphi(g-m)\right)-A(\varphi(|g-A(g)|)).
\end{aligned}$$

Now, it follows that

$$\begin{aligned}
&\frac{A(g)-m}{M-m}\varphi(m)+\frac{M-A(g)}{M-m}\varphi(M) \\
&\qquad -\frac{1}{M-m}A\left((g-m)\varphi(M-g)+(M-g)\varphi(g-m)\right)-A(\varphi(|g-A(g)|)) \\
&\leq \varphi(m)+\varphi(M)-A(\varphi(g)) \\
&\qquad -\frac{2}{M-m}A\left((g-m)\varphi(M-g)+(M-g)\varphi(g-m)\right)-A(\varphi(|g-A(g)|)),
\end{aligned}$$

which is the inequality (2.8). □

3. APPLICATIONS

Throughout this section we suppose that:

(i) L is a linear class having properties $L1$, $L2$ on a nonempty set E.

(ii) A is an isotonic linear functional on L such that $A(1) = 1$.

(iii) $g \in L$ is a function of E to $[m, M]$ $(0 < m < M < \infty)$ such that all of the following expressions are well defined.

Let ψ be a continuous and strictly monotonic function on an interval $I = [m, M]$, $(0 < m < M < \infty)$.

For any $r \in \mathbb{R}$, a power mean of Mercer's type functional

$$Q(r,g) := \begin{cases} \left[m^r + M^r - A(g^r)\right]^{\frac{1}{r}}, & r \neq 0 \\ \dfrac{mM}{\exp\left(A(\log g)\right)}, & r = 0, \end{cases}$$

and a quasi-arithmetic mean functional of Mercer's type

$$\widetilde{M}_{\psi}(g,A) = \psi^{-1}\left(\psi(m)+\psi(M)-A(\psi(g))\right)$$

are defined in [8] and the following theorems are proved.

Theorem G. *If* $r, s \in \mathbb{R}$ *and* $r \leq s$*, then*

$$Q(r,g) \leq Q(s,g).$$

Theorem H.

(i) *If either* $\chi \circ \psi^{-1}$ *is convex and* χ *is strictly increasing, or* $\chi \circ \psi^{-1}$ *is concave and* χ *is strictly decreasing, then*

$$\widetilde{M}_{\psi}(g, A) \leq \widetilde{M}_{\chi}(g, A). \tag{3.1}$$

(ii) *If either* $\chi \circ \psi^{-1}$ *is concave and* χ *is strictly increasing, or* $\chi \circ \psi^{-1}$ *is convex and* χ *is strictly decreasing, then the inequality (3.1) is reversed.*

Applying the inequality (2.1) to the adequate superquadratic functions we shall give some refinements of the inequalities in Theorems G and H. To do this, we will define following functions.

$$\begin{aligned}\Diamond(m, M, r, s, g, A) = & \frac{1}{M^r - m^r} A\left((M^r - g^r)(g^r - m^r)^{\frac{s}{r}}\right) \\ & + \frac{1}{M^r - m^r} A\left((g^r - m^r)(M^r - g^r)^{\frac{s}{r}}\right) \\ & + \frac{1}{M^r - m^r}\left(A(g^r) - m^r\right)\left(M^r - A(g^r)\right)^{\frac{s}{r}} \\ & + \frac{1}{M^r - m^r}\left(M^r - A(g^r)\right)\left(A(g^r) - m^r\right)^{\frac{s}{r}}.\end{aligned}$$

and

$$\begin{aligned}&\Diamond(m, M, \psi, \chi, g, A) \\ &= \frac{1}{\psi(M) - \psi(m)} A\left((\psi(M) - \psi(g))\chi\left(\psi^{-1}\left(\psi(g) - \psi(m)\right)\right)\right) \\ &\quad + \frac{1}{\psi(M) - \psi(m)} A\left((\psi(g) - \psi(m))\chi\left(\psi^{-1}\left(\psi(M) - \psi(g)\right)\right)\right) \\ &\quad + \frac{1}{\psi(M) - \psi(m)}\left(A(\psi(g)) - \psi(m)\right)\chi\left(\psi^{-1}\left(\psi(M) - A(\psi(g))\right)\right) \\ &\quad + \frac{1}{\psi(M) - \psi(m)}\left(\psi(M) - A(\psi(g))\right)\chi\left(\psi^{-1}\left(A(\psi(g)) - \psi(m)\right)\right).\end{aligned}$$

Now, the following theorems are valid.

Theorem 3.1. *Let* $r, s \in \mathbb{R}$.

(i) *If* $0 < 2r \leq s$, *then*

$$Q(r, g) \leq \left[(Q(s, g))^s - \Diamond(m, M, r, s, g, A)\right]^{\frac{1}{s}}. \tag{3.2}$$

(ii) *If* $2r \leq s < 0$, *then for* $(Q(s, g))^s - \Diamond(M, m, r, s, g, A) > 0$

$$Q(r, g) \leq \left[(Q(s, g))^s - \Diamond(M, m, r, s, g, A)\right]^{\frac{1}{s}}, \tag{3.3}$$

where we used $\Diamond(M, m, r, s, g, A)$ *to denote the new function derived from the function* $\Diamond(m, M, r, s, g, A)$ *by changing the places of* m *and* M.

(iii) *If* $0 < s \leq 2r$, *then for* $(Q(s, g))^s - \Diamond(M, m, r, s, g, A) > 0$ *the reverse inequality (3.2) holds.*

(iv) *If* $s \leq 2r < 0$, *then the reversed inequality (3.3) holds.*

Proof.

(i) It is given that

$$0 < m \le g \le M < \infty.$$

Since $0 < 2r \le s$, it follows that

$$0 < m^r \le g^r \le M^r < \infty.$$

Applying Theorem 2.1, or more precisely inequality (2.1) to the superquadratic function $\varphi(t) = t^{\frac{s}{r}}$ (note that $\frac{s}{r} \ge 2$ here) and replacing g, m and M with g^r, m^r and M^r, respectively, we have

$$\begin{aligned} &[m^r + M^r - A(g^r)]^{\frac{s}{r}} \\ &\quad + \frac{1}{M^r - m^r}\left(A(g^r) - m^r\right)\left(M^r - A(g^r)\right)^{\frac{s}{r}} \\ &\quad + \frac{1}{M^r - m^r}\left(M^r - A(g^r)\right)\left(A(g^r) - m^r\right)^{\frac{s}{r}} \\ &\le m^s + M^s - A(g^s) \\ &\quad - \frac{1}{M^r - m^r} A\left((M^r - g^r)(g^r - m^r)^{\frac{s}{r}}\right) \\ &\quad - \frac{1}{M^r - m^r} A\left((g^r - m^r)(M^r - g^r)^{\frac{s}{r}}\right). \end{aligned}$$

i.e.

$$[Q(r,g)]^s \le [Q(s,g)]^s - \diamondsuit(m, M, r, s, g, A). \tag{3.4}$$

Raising both sides of (3.4) to the power $\frac{1}{s} > 0$, we get desired inequality (3.2).

(ii) In this case we have

$$0 < M^r \le g^r \le m^r < \infty.$$

Applying Theorem 2.1 or, more precisely, the reversed inequality (2.1) to the subquadratic function $\varphi(t) = t^{\frac{s}{r}}$ (note that now we have $0 < \frac{s}{r} \le 2$) and replacing g, m and M with g^r, m^r and M^r, respectively, we get

$$\begin{aligned} &[M^r + m^r - A(g^r)]^{\frac{s}{r}} \\ &\quad + \frac{1}{m^r - M^r}\left(A(g^r) - M^r\right)\left(m^r - A(g^r)\right)^{\frac{s}{r}} \\ &\quad + \frac{1}{m^r - M^r}\left(m^r - A(g^r)\right)\left(A(g^r) - M^r\right)^{\frac{s}{r}} \\ &\ge M^s + m^s - A(g^s) \\ &\quad - \frac{1}{m^r - M^r} A\left((m^r - g^r)(g^r - M^r)^{\frac{s}{r}}\right) \\ &\quad - \frac{1}{m^r - M^r} A\left((g^r - M^r)(m^r - g^r)^{\frac{s}{r}}\right). \end{aligned}$$

Since $2r \le s < 0$, raising both sides to the power $\frac{1}{s}$, it follows that

$$[M^r + m^r - A(g^r)]^{\frac{1}{r}} \le [M^s + m^s - A(g^s) - \diamondsuit(M, m, r, s, g, A)]^{\frac{1}{s}},$$

or

$$Q(r,g) \le [(Q(s,g))^s - \diamondsuit(M, m, r, s, g, A)]^{\frac{1}{s}}.$$

(iii) In this case we have $0 < \frac{s}{r} \le 2$. Since $0 < m^r \le g^r \le M^r < \infty$, we can apply Theorem 2.1, or more precisely, the reversed inequality (2.1) to the subquadratic function $\varphi(t) = t^{\frac{s}{r}}$. Replacing g, m and M with g^r, m^r and M^r, respectively, it follows that

$$\begin{aligned}
&[m^r + M^r - A(g^r)]^{\frac{s}{r}} \\
&\quad + \frac{1}{M^r - m^r}\left(A(g^r) - m^r\right)\left(M^r - A(g^r)\right)^{\frac{s}{r}} \\
&\quad + \frac{1}{M^r - m^r}\left(M^r - A(g^r)\right)\left(A(g^r) - m^r\right)^{\frac{s}{r}} \\
&\ge m^s + M^s - A(g^s) \\
&\quad - \frac{1}{M^r - m^r} A\left((M^r - g^r)(g^r - m^r)^{\frac{s}{r}}\right) \\
&\quad - \frac{1}{M^r - m^r} A\left((g^r - m^r)(M^r - g^r)^{\frac{s}{r}}\right),
\end{aligned}$$

i.e.

$$[Q(r,g)]^s \ge [Q(s,g)]^s - \diamondsuit(m,M,r,s,g,A). \tag{3.5}$$

Raising both sides of (3.5) to the power $\frac{1}{s} > 0$ we get

$$Q(r,g) \ge \left[(Q(s,g))^s - \diamondsuit(m,M,r,s,g,A)\right]^{\frac{1}{s}}.$$

(iv) Since $r < 0$, from $0 < m \le g \le M < \infty$ it follows that $0 < M^r \le g^r \le m^r < \infty$. Now, we are applying Theorem 2.1 to the superquadratic function $\varphi(t) = t^{\frac{s}{r}}$, because $\frac{s}{r} \ge 2$ here, and analogous to the previous theorem we get

$$[Q(r,g)]^s \le [Q(s,g)]^s - \diamondsuit(M,m,r,s,g,A).$$

Raising both sides to the power $\frac{1}{s} < 0$ it follows that

$$Q(r,g) \ge \left[(Q(s,g))^s - \diamondsuit(M,m,r,s,g,A)\right]^{\frac{1}{s}}.$$

□

Theorem 3.2. *Let $r, s \in \mathbb{R}$.*

(i) *If $0 < 2s \le r$, then*

$$Q(r,g) \ge \left[(Q(s,g))^r + \diamondsuit(m,M,s,r,g,A)\right]^{\frac{1}{r}}, \tag{3.6}$$

where we used $\diamondsuit(m,M,s,r,g,A)$ to denote the new function derived from the function $\diamondsuit(m,M,r,s,g,A)$ by changing the places of r and s.

(ii) *If $2s \le r < 0$, then*

$$Q(r,g) \le \left[(Q(s,g))^r + \diamondsuit(M,m,s,r,g,A)\right]^{\frac{1}{r}}. \tag{3.7}$$

(iii) *If $0 < r \le 2s$, then the reversed inequality (3.6) holds.*

(iv) *If $r \le 2s < 0$, then the reversed inequality (3.7) holds.*

Proof.

(i) Applying inequality (2.1) to the superquadratic function $\varphi(t) = t^{\frac{r}{s}}$ (note that $\frac{r}{s} \ge 2$ here) and replacing g, m and M with g^s, m^s and M^s, $(0 < m^s \le g^s \le M^s < \infty)$

respectively, we have

$$\begin{aligned}
&[m^s + M^s - A(g^s)]^{\frac{r}{s}} \\
&\quad + \frac{1}{M^s - m^s}\left(A(g^s) - m^s\right)\left(M^s - A(g^s)\right)^{\frac{r}{s}} \\
&\quad + \frac{1}{M^s - m^s}\left(M^s - A(g^s)\right)\left(A(g^s) - m^s\right)^{\frac{r}{s}} \\
&\geq m^r + M^r - A(g^r) \\
&\quad - \frac{1}{M^s - m^s} A\left((M^s - g^s)(g^s - m^s)^{\frac{r}{s}}\right) \\
&\quad - \frac{1}{M^s - m^s} A\left((g^s - m^s)(M^s - g^s)^{\frac{r}{s}}\right),
\end{aligned}$$

i.e.

$$[Q(s,g)]^r \leq [Q(r,g)]^r - \diamondsuit(m, M, s, r, g, A).$$

Raising both sides to the power $\frac{1}{r} > 0$, the inequality (3.6) follows.

(ii) Since $s < 0$, we have $0 < M^s \leq g^s \leq m^s < \infty$ so the function $\diamondsuit$ will be of the form $\diamondsuit(M, m, s, r, g, A)$. Since $0 < \frac{r}{s} \leq 2$, we will apply Theorem 2.1 to the subquadratic function $\varphi(t) = t^{\frac{r}{s}}$ and, as in previous case, it follows that

$$[Q(s,g)]^r + \diamondsuit(M, m, s, r, g, A) \geq [Q(r,g)]^r.$$

Raising both sides to the power $\frac{1}{r} < 0$, the inequality (3.7) follows.

(iii) Since $0 < \frac{r}{s} \leq 2$, we will apply Theorem 2.1 to the subquadratic function $\varphi(t) = t^{\frac{r}{s}}$ and then it follows that

$$[Q(s,g)]^r + \diamondsuit(m, M, s, r, g, A) \geq [Q(r,g)]^r.$$

Raising both sides to the power $\frac{1}{r} > 0$, we get

$$Q(r,g) \leq \left[(Q(s,g))^r + \diamondsuit(m, M, s, r, g, A)\right]^{\frac{1}{r}}.$$

(iv) Since $\frac{r}{s} \geq 2$, we will apply Theorem 2.1 to the superquadratic function $\varphi(t) = t^{\frac{r}{s}}$ and use the function $\diamondsuit(M, m, s, r, g, A)$ instead of $\diamondsuit(m, M, s, r, g, A)$. Then we get

$$[Q(s,g)]^r + \diamondsuit(M, m, s, r, g, A) \leq [Q(r,g)]^r.$$

Raising both sides to the power $\frac{1}{r} < 0$, it follows that

$$Q(r,g) \geq \left[(Q(s,g))^r + \diamondsuit(M, m, s, r, g, A)\right]^{\frac{1}{r}}.$$

□

Remark 2. Notice that some cases in the last theorems have common parts. In some of them we can establish double inequalities. For example, if $0 < r \leq 2s$ and $0 < s \leq 2r$, then for $(Q(s,g))^s - \diamondsuit(M, m, r, s, g, A) > 0$

$$\left[(Q(s,g))^r + \diamondsuit(m, M, s, r, g, A)\right]^{\frac{1}{r}} \geq Q(r,g) \geq \left[(Q(s,g))^s - \diamondsuit(m, M, r, s, g, A)\right]^{\frac{1}{s}}.$$

Theorem 3.3. *Let $\psi \in C([m,M])$ be strictly increasing and let $\chi \in C([m,M])$ be strictly monotonic functions.*

(i) *If either $\chi \circ \psi^{-1}$ is superquadratic and χ is strictly increasing, or $\chi \circ \psi^{-1}$ is subquadratic and χ is strictly decreasing, then*

$$\widetilde{M}_\psi(g, A) \leq \chi^{-1}\left(\chi\left(\widetilde{M}_\chi(g, A)\right) - \diamondsuit(m, M, \psi, \chi, g, A)\right), \tag{3.8}$$

(ii) *If either $\chi\circ\psi^{-1}$ is subquadratic and χ is strictly increasing or $\chi\circ\psi^{-1}$ is superquadratic and χ is strictly decreasing, then the inequality (3.8) is reversed.*

Proof. Suppose that $\chi\circ\psi^{-1}$ is superquadratic. Letting $\varphi=\chi\circ\psi^{-1}$ in Theorem 2.1 and replacing g, m and M with $\psi(g)$, $\psi(m)$ and $\psi(M)$ respectively, we have

$$\begin{aligned}
&\chi\left(\psi^{-1}\left(\psi(m)+\psi(M)-A(\psi(g))\right)\right)\\
&\quad+\frac{1}{\psi(M)-\psi(m)}\left((A(\psi(g))-\psi(m))\,\chi\left(\psi^{-1}\left(\psi(M)-A(\psi(g))\right)\right)\right)\\
&\quad+\frac{1}{\psi(M)-\psi(m)}\left((\psi(M)-A(\psi(g)))\,\chi\left(\psi^{-1}\left(A(\psi(g))-\psi(m)\right)\right)\right)\\
&\le \chi(m)+\chi(M)-A(\chi(g))\\
&\quad-\frac{1}{\psi(M)-\psi(m)}A\left((\psi(M)-\psi(m))\,\chi\left(\psi^{-1}\left(\psi(g)-\psi(m)\right)\right)\right)\\
&\quad-\frac{1}{\psi(M)-\psi(m)}A\left((\psi(g)-\psi(m))\,\chi\left(\psi^{-1}\left(\psi(M)-\psi(g)\right)\right)\right),
\end{aligned}$$

i.e.,

$$\begin{aligned}
\chi\left(\widetilde{M}_{\psi}(g,A)\right) &\le \chi(m)+\chi(M)-A(\chi(g))-\Diamond(m,M,\psi,\chi,g,A)\\
&\le \chi\circ\chi^{-1}\left(\chi(m)+\chi(M)-A(\chi(g))\right)-\Diamond(m,M,\psi,\chi,g,A)\\
&\le \chi\left(\widetilde{M}_{\chi}(g,A)\right)-\Diamond(m,M,\psi,\chi,g,A).
\end{aligned}\tag{3.9}$$

If χ is strictly increasing, then the inverse function χ^{-1} is also strictly increasing and inequality (3.9) implies the inequality (3.8). If χ is strictly decreasing, then the inverse function χ^{-1} is also strictly decreasing and in that case the reverse of (3.9) implies (3.8). Analogously, we get the reverse of (3.8) in the cases when $\chi\circ\psi^{-1}$ is superquadratic and χ is strictly decreasing, or $\chi\circ\psi^{-1}$ is subquadratic and χ is strictly increasing. □

Remark 3. If the function ψ in Theorem 3.3 is strictly decreasing, then the inequality (3.8) and its reversal also hold under the same assumptions, but with m and M interchanged.

Remark 4. Obviously, Theorem 3.1 and Theorem 3.2 follow from Theorem 3.3 and Remark 3 by choosing $\psi(t)=t^r$ and $\chi(t)=t^s$, or vice versa.

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