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CONVOLUTION OPERATORS WITH HOMOGENEOUS SINGULAR MEASURES ON $\mathbb{R}^3$ OF POLYNOMIAL TYPE. THE REMAINDER CASE.

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Abstract

Let $\varphi(y_1, y_2) = y_2^l P(y_1, y_2)$ where P is a polynomial function of degree l such that $P(1,0) \neq 0$. Let μ_δ be the Borel measure on $\mathbb{R}^3$ defined by $\mu_\delta(E) = \int_{V_\delta} \chi_E(x, \varphi(x))\, dx$ where

$$V_\delta = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq 1, \text{and } |x_1| \leq \delta |x_2| \right\}$$

and let T_{μ_δ} be the convolution operator with the measure μ_δ. In this paper we explicitely describe the type set

$$E_{\mu_\delta} := \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in [0,1] \times [0,1] : \|T_{\mu_\delta}\|_{p,q} < \infty \right\},$$

for δ small enough.

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Contents

1. Introduction

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial function of degree $m \geq 2$ and let $D = \{y \in \mathbb{R}^2 : |y| \leq 1\}$. Let μ be the Borel measure on $\mathbb{R}^3$ given by

$$\mu(E) = \int_D \chi_E(y, \varphi(y))\, dy \tag{1.1}$$

and let T_μ be the operator defined, for $f \in S(\mathbb{R}^3)$, by $T_\mu f = \mu * f$. Let E_μ be the set of the pairs $\left(\frac{1}{p}, \frac{1}{q}\right) \in [0,1] \times [0,1]$ such that there exists a positive constant c satisfying $\|Tf\|_q \leq c\|f\|_p$ for all $f \in S(\mathbb{R}^3)$, where the L^p spaces are taken with respect to the Lebesgue measure on $\mathbb{R}^3$. For $\left(\frac{1}{p}, \frac{1}{q}\right) \in E_\mu$, T can be extended to a bounded operator, still denoted by T, from $L^p(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3)$.

Let $\varphi = \varphi_1^{e_1} \ldots \varphi_n^{e_n}$ be a decomposition of φ in irreducible factors with $\varphi_i \nmid \varphi_j$ for $i \neq j$. In [3] we could give a complete description of the set E_μ under the assumption that $e_i \neq \frac{m}{2}$ for each φ_i of degree 1. If $\det \varphi''(y)$ is not identically zero and if it vanishes somewhere on $\mathbb{R}^2 - \{0\}$, the set of the points y where $\det \varphi''(y)$ vanishes is a finite union of lines $L_1, \ldots, L_k$ through the origin. So, after a possibly linear change of variables, we localized the problem to the x axes and we studied the type set corresponding to measures μ_δ defined by

$$\mu_\delta(E) = \int_{V_\delta} \chi_E(y, \varphi(y))\, dy,$$

where $V_\delta = D \cap \{(y_1, y_2) \in \mathbb{R}^2 : |y_2| \leq \delta |y_1|\}$ and δ is small enough such that $\det \varphi''(y)$ only vanishes, on V_δ, along the x axes. The only case left was the one corresponding to functions φ of the form $\varphi(y_1, y_2) = y_2^l P(y_1, y_2)$ with $l = \frac{m}{2}$, P being a homogeneous polynomial function of degree l such that $P(1,0) \neq 0$.

In this paper we characterize E_{μ_δ} in this remainder case.

L^p improving properties of convolution operators with singular measures supported on hypersurfaces in $\mathbb{R}^n$ have been widely studied in [2], [5], [6]. In particular, in [5], the type set was studied under our actual hypothesis, but the endpoint problem was left open there. Our proof of the main result involves a biparametric family of dilations and will be based on a suitable adaptation of arguments due to M. Christ, developed in [1], where the author studied the type set associated to the two dimensional measure supported on the parabola.

Also, oscillatory integral estimates are involved. A very careful study of this kind of estimate can be found in [4] where the authors study the boundedness of maximal operators associated to mixed homogeneous hypersurfaces.

Throughout this paper c will denote a positive constant, not the same at each occurrence.

2. The Main Result

We assume $\varphi(y_1, y_2) = y_2^l P(y_1, y_2)$, where $l = \frac{m}{2}$ and P is a homogeneous polynomial function of degree l such that $P(1,0) \neq 0$. We take $\delta_1 > 0$ such that, for $y \in V_{\delta_1}$ such that $y_2 \neq 0$, $\det \varphi''(y) \neq 0$. Moreover, since $P(1,0) \neq 0$ we can assume that $P(y) \neq 0$ and $P_1(y) \neq 0$ for all $y \in V_{\delta_1}$. Now, if $\max_{V_{\delta_1}} |P_2(y_1, y_2)| \neq 0$, we choose $\delta < \min\left(\frac{l \min_{V_{\delta_1}} |P(y_1,y_2)|}{2 \max_{V_{\delta_1}} |P_2(y_1,y_2)|}, \delta_1 \right)$. In the other case we take $\delta = \delta_1$.

The main result we prove is the following.

Theorem 2.1. *Let $\varphi(y_1, y_2) = y_2^l P(y_1, y_2)$ where $l = \frac{m}{2}$ and P is a homogeneous polynomial function of degree l such that $P(1,0) \neq 0$ and $y_2 \nmid P(y_1, y_2)$. Let V_δ be defined as above and let E_{V_δ} be the corresponding type set. Then E_{V_δ} is the closed polygonal region with vertices $(0,0)$, $(1,1)$, $\left(\frac{2l+1}{2l+2}, \frac{2l-1}{2l+2}\right)$ and $\left(\frac{3}{2l+2}, \frac{1}{2l+2}\right)$.*

Standard arguments (see, for example Lemma 2 and Lemma 3 in [3]) imply the following result.

Lemma 2.2. *If $\left(\frac{1}{p}, \frac{1}{q}\right) \in E_{\mu_\delta}$ then $\frac{1}{q} \leq \frac{1}{p}$, $\frac{1}{q} \geq \frac{3}{p} - 2$ and $\frac{1}{q} \geq \frac{1}{p} - \frac{1}{l+1}$.*

So, since $\|T_{\mu_\delta}\|_{1,1} < \infty$, by duality arguments it only remains to prove that

$$\|T_{\mu_\delta}\|_{\frac{2l+2}{2l+1}, \frac{2l+2}{2l-1}} < \infty. \tag{2.1}$$

We set $Q_0 = \left[\frac{1}{4}, 2\right] \times \left[\frac{\delta}{64}, \frac{\delta}{8}\right]$. We take a truncation function $\theta \in C^\infty(\mathbb{R}^2)$, $\theta(y_1, y_2) \geq 0$, $\operatorname{supp} \theta \subset Q_0$ and $\theta(y_1, y_2) = 1$ on $\left[\frac{1}{2}, 1\right] \times \left[\frac{\delta}{32}, \frac{\delta}{16}\right]$. We define, for $\varepsilon,\gamma>0$, the biparametric family of dilations on $\mathbb{R}^2$ and $\mathbb{R}^3$ given by $(\varepsilon,\gamma)\circ(y_1,y_2)=(\varepsilon y_1,\gamma y_2)$ and $(\varepsilon,\gamma)\circ(y_1,y_2,y_3)=\left(\varepsilon y_1,\gamma y_2,\varepsilon^l\gamma^l y_3\right)$ repectively. Also, for $j,k\geq 0$, we set $Q_{j,k}=\left(2^{-j},2^{-k}\right)\circ Q_0$.

For $f\in S\left(\mathbb{R}^3\right)$, we define

$$T_{j,k}f\left(x_1,x_2,x_3\right) = \int f\left(x_1-y_1,x_2-y_2,x_3-\varphi\left(y_1,y_2\right)\right)\theta\left(2^j y_1,2^k y_2\right)dy_1dy_2 \tag{2.2}$$

so for $f\geq 0$,

$$T_{\mu_{\frac{\delta}{8}}}f\leq c\sum_{0\leq j\leq k}T_{j,k}f. \tag{2.3}$$

To study $\sum_{0\leq j\leq k}T_{j,k}f$, we will adapt the argument developed by M. Christ (see [1]) to the setting of biparametric dilations. First of all, we prove the following

Proposition 2.3. *There exists a positive constant $c>0$ such that for $0\leq j\leq k$,*

$$\left\|T_{j,k}\right\|_{\frac{2l+2}{2l+1},\frac{2l+2}{2l-1}}\leq c.$$

Proof.

$$\begin{aligned}
&T_{j,k}f\left(x_1,x_2,x_3\right)\\
&=\int f\left(x_1-y_1,x_2-y_2,x_3-\varphi\left(y_1,y_2\right)\right)\theta\left(2^j y_1,2^k y_2\right)dy_1dy_2\\
&=2^{-(j+k)}\int f\left(x_1-2^{-j}y_1,x_2-2^{-k}y_2,x_3-\varphi\left(2^{-j}y_1,2^{-k}y_2\right)\right)\theta\left(y_1,y_2\right)dy_1dy_2\\
&=2^{-(j+k)}T^{(j-k)}f_{j,k}\left(2^j x_1,2^k x_2,2^{(j+k)l}x_3\right),
\end{aligned}$$

where we denote

$$T^{(j)}f\left(x_1,x_2,x_3\right)=\int f\left(x_1-y_1,x_2-y_2,x_3-y_2^lP\left(y_1,2^jy_2\right)\right)\theta\left(y_1,y_2\right)dy_1dy_2$$

and

$$f_{j,k}\left(x_1,x_2,x_3\right)=f\left(\left(2^{-j},2^{-k}\right)\circ\left(x_1,x_2,x_3\right)\right).$$

So

$$\left\|T_{j,k}f\left(x_1,x_2,x_3\right)\right\|_q=2^{(j+k)\left(\frac{1+l}{p}-\frac{1+l}{q}-1\right)}\left\|T^{(j-k)}\right\|_{p,q}\left\|f\right\|_p. \tag{2.4}$$

Now,

$$\det\left(y_2^lP\left(y_1,2^{j-k}y_2\right)\right)''=2^{(2-2l)(j-k)}\det\left(\varphi\right)''\left(y_1,2^{j-k}y_2\right).$$

so as in the proof of Lemma 4 in [3] we obtain that there exists $c>0$ such that $\left\|T^{(j-k)}\right\|_{\frac{2l+2}{2l+1},\frac{2l+2}{2l-1}}\le c$ for $0\le j\le k$, and the proposition follows. □

We take $0\le j\le k$, and denote by $\mu_{j,k}$ and $\mu^{(j)}$ the measures associated to $T_{j,k}$ and $T^{(j)}$ respectively. For $\xi=\left(\xi_1,\xi_2,\xi_3\right)$,

$$\widehat{\mu^{(j-k)}}\left(\xi\right)=\int e^{-i\left(\xi_1y_1+\xi_2y_2+\xi_3y_2^lP\left(y_1,2^{j-k}y_2\right)\right)}\theta\left(y_1,y_2\right)dy_1dy_2.$$

If for some ξ on the unit sphere, $\Omega_\xi^{(j-k)}\left(y_1,y_2\right)=\xi_1y_1+\xi_2y_2+\xi_3y_2^lP\left(y_1,2^{j-k}y_2\right)$ has a critical point belonging to the $\operatorname{supp}\theta$, then

$$\xi_1+\xi_3y_2^lP_1\left(y_1,2^{j-k}y_2\right)=0$$

and

$$\xi_2 + \xi_3 \left(2^{j-k} y_2^l P_2 \left(y_1, 2^{j-k} y_2\right) + l y_2^{l-1} P \left(y_1, 2^{j-k} y_2\right)\right) = 0,$$

but then, since $P_1(y) \neq 0$ for $y \in V_{\delta_1}$, from the first equation we obtain that there exist constants $a, b \in \mathbb{Z}$ with $a < b$ such $2^a |\xi_3| \leq |\xi_1| \leq 2^b |\xi_3|$, and, from the second one and the choice of δ we obtain constants $c, d \in \mathbb{Z}^2$ with $c < d$ such that $2^c |\xi_3| \leq |\xi_2| \leq 2^d |\xi_3|$. So ξ belongs to the cone

$$C_0 = \left\{\xi \in \mathbb{R}^3 : 2^a |\xi_3| < |\xi_1| < 2^b |\xi_3|, \ 2^c |\xi_3| < |\xi_2| < 2^d |\xi_3|\right\}.$$

Lemma 2.4. *Suppose C_0 is as above. Then the family of cones $\left\{\left(2^j, 2^k\right) \circ C_0\right\}_{j,k \in \mathbb{Z}}$ has finite overlapping (i.e., $\#\left\{(j,k) \in \mathbb{Z}^2 : C_0 \cap \left(\left(2^j, 2^k\right) \circ C_0\right) \neq \emptyset\right\} < \infty$).*

Proof. We suppose $\xi \in C_0$ and $\left(2^j, 2^k\right) \circ \xi \in C_0$, then

$$2^a |\xi_3| < |\xi_1| < 2^b |\xi_3|, \qquad 2^c |\xi_3| < |\xi_2| < 2^d |\xi_3|$$

and

$$2^{(j+k)l+a} |\xi_3| < 2^j |\xi_1| < 2^{(j+k)l+b} |\xi_3|,$$
$$2^{(j+k)l+c} |\xi_3| < 2^k |\xi_2| < 2^{(j+k)l+d} |\xi_3|$$

so

$$2^j |\xi_1| < 2^{(j+k)l+b} |\xi_3| < 2^{(j+k)l+b-a} |\xi_1|$$

and

$$2^b |\xi_3| > |\xi_1| > 2^{-j} 2^{(j+k)l+a} |\xi_3|,$$

so

$$a - b - kl < j(l-1) < b - a - kl,$$

analogously we obtain

$$c-d-jl<k\left(l-1\right)<d-c-jl,$$

thus

$$\frac{\left(c-d\right)\left(l-1\right)+\left(a-b\right)l}{l^{2}-\left(l-1\right)^{2}}<k<\frac{\left(d-c\right)\left(l-1\right)+\left(b-a\right)l}{l^{2}-\left(l-1\right)^{2}}$$

and so

$$\frac{a-b}{l-1}-l\frac{\left(d-c\right)\left(l-1\right)+\left(b-a\right)l}{\left(l^{2}-\left(l-1\right)^{2}\right)\left(l-1\right)} \\ <j<\frac{\left(b-a\right)}{l-1}+l\frac{\left(d-c\right)\left(l-1\right)+\left(b-a\right)l}{\left(l^{2}-\left(l-1\right)^{2}\right)\left(l-1\right)}.$$

□

We define $m_0\left(\xi\right)=n\left(\xi_1,\xi_3\right)r\left(\xi_2,\xi_3\right)$ where n and r belong to $C^{\infty}\left(R^2-\{0\}\right)$, are homogeneous of degree zero with respect to the isotropic dilations,

$$\operatorname{supp} n\subset\left\{\left(\xi_1,\xi_3\right):2^{a-1}\left|\xi_3\right|<\left|\xi_1\right|<2^{b+1}\left|\xi_3\right|\right\}$$

$n\geq 0$ and $n\equiv 1$ on $\left\{\left(\xi_1,\xi_3\right):2^{a}\left|\xi_3\right|<\left|\xi_1\right|<2^{b}\left|\xi_3\right|\right\}$,

$$\operatorname{supp} r\subset\left\{\left(\xi_2,\xi_3\right):2^{c-1}\left|\xi_3\right|<\left|\xi_2\right|<2^{d+1}\left|\xi_3\right|\right\},$$

$r \geq 0$ and $r \equiv 1$ on $\left\{(\xi_2, \xi_3) : 2^c |\xi_3| < |\xi_2| < 2^d |\xi_3|\right\}$, so m_0 is homogeneous of degree zero with respect to the isotropic dilations, it belongs to C^∞ on each octant of $\mathbb{R}^3$, $m_0 \geq 0$, $m_0 \equiv 1$ on C_0 and

$$\begin{aligned}
&\operatorname{supp} m_0 \subset \widetilde{C_0} \\
&\quad = \left\{\xi \in \mathbb{R}^3 : 2^{a-1} |\xi_3| < |\xi_1| < 2^{b+1} |\xi_3|,\ 2^{c-1} |\xi_3| < |\xi_2| < 2^{d+1} |\xi_3|\right\}.
\end{aligned}$$

For $(j,k) \in \mathbb{Z}^2$, we define $m_{j,k}(\xi) = m_0\left(\left(2^{-j}, 2^{-k}\right) \circ \xi\right)$ and $\mathfrak{Q}_{j,k}$ the operator with multiplier $m_{j,k}$. If ξ belongs to an open octant of $\mathbb{R}^3$ then ξ belongs to $\left(2^j, 2^k\right) \circ C_0$ for some $(j,k) \in \mathbb{Z}^2$ (indeed $2^{-k} \sim \frac{|\xi_1|}{|\xi_3|}$ and $2^{-j} \sim \frac{|\xi_2|}{|\xi_3|}$) and from the previous lemma, it belongs to a finite number of them (independent of ξ). So $\sum_{(j,k)\in\mathbb{Z}^2} m_{j,k}(\xi) \leq c$. Now it is easy to check that, for $1 < p < \infty$, there exists $A_p > 0$ such that for $f \in L^2 \cap L^p$ and any choice of $\varepsilon_{j,k} = \pm 1$,

$$\left\| \sum_{(j,k)\in\mathbb{Z}^2} \varepsilon_{j,k} \mathfrak{Q}_{j,k} f \right\|_p \leq A_p \|f\|_p . \tag{2.5}$$

Indeed, we now show that

$$m(\xi) = \sum_{(j,k)\in\mathbb{Z}^2} \varepsilon_{j,k} m_{j,k}(\xi)$$

satisfies the hypothesis of the Marcinkiewicz Theorem, as stated in Theorem 6' in [7].

We have just observed that

$$|m(\xi)| \leq \sum_{(j,k)\in\mathbb{Z}^2} m_{j,k}(\xi) \leq c.$$

Now we want to estimate $\left|\frac{\partial}{\partial\xi_1} m(\xi)\right|$. We recall that $\frac{\partial}{\partial\xi_1} m_0$ is homogeneous of degree -1. We pick ξ in an open octant. In a small neighborhood of ξ only finitely many $(j,k) \in \mathbb{Z}^2$ (independent of ξ) are involved. For each one of them,

$$\begin{aligned}\frac{\partial}{\partial\xi_1} m_{j,k}(\xi) &= 2^{-j} \frac{\partial}{\partial\xi_1} m_0 \left(2^{-j}\xi_1, 2^{-k}\xi_2, 2^{-(j+k)l}\xi_3\right) \\ &\leq c2^{-j} \left|2^{-j}\xi_1, 2^{-k}\xi_2, 2^{-(j+k)l}\xi_3\right|^{-1} \leq c2^{-j} \left|2^{-j}\xi_1\right|^{-1},\end{aligned}$$

so

$$\sup_{\xi_2,\xi_3} \int_{2^s}^{2^{s+1}} \left|\frac{\partial}{\partial\xi_1} m(\xi)\right| d\xi_1 \leq c,$$

and in a similar way (using the homogeneity of the derivatives of $m_{j,k}$) we obtain that for each $0 < k \leq 3$,

$$\sup_{\xi_{k+1},\dots,\xi_3} \int_\rho \left|\frac{\partial^k}{\partial\xi_1 \dots \partial\xi_k} m(\xi)\right| d\xi_1 \leq c,$$

as ρ ranges over dyadic rectangles of $\mathbb{R}^k$ and that this inequality holds for every one of the six pemutations of the variables ξ_1, ξ_2, ξ_3.

We now define $h(\xi) \in C^\infty(\mathbb{R}^3)$, $h \geq 0$, $h \equiv 1$ on the unit ball of $\mathbb{R}^3$, $h_{j,k}(\xi) = h\left(\left(2^{-j}, 2^{-k}\right) \circ \xi\right)$ and $R_{j,k}$ the operators with multipliers $h_{j,k}$.

Lemma 2.5. *There exists a constant $C > 0$, independent of K, such that*

$$\left\| \sum_{0 \le j \le k \le K} T_{j,k} R_{j,k} \right\|_{\frac{2l+2}{2l+1}, \frac{2l+2}{2l-1}} \le C.$$

Proof. Let $K_{j,k}$ be the kernel of $T_{j,k}R_{j,k}$. A computation shows that,

$$K_{j,k}(x) = 2^{(j+k)l} \left(\mu^{(j-k)} * h^{\wedge\vee} \right) \left(\left(2^j, 2^k \right) \circ x \right).$$

Thus

$$\sum_{0 \le j \le k \le K} \left| K_{j,k}(\xi) \right| \le \sum_{0 \le j \le k} 2^{(j+k)l} \left| G^{(j,k)} \left(\left(2^j, 2^k \right) \circ \xi \right) \right|$$

with $G^{(j,k)}$ defined by $\left(G^{(j,k)} \right)^{\wedge} = \left(\mu^{(j-k)} \right)^{\wedge} h$. Since $j - k \le 0$, as in Lemma 7 in [3] we obtain that $\left(G^{(j,k)} \right)^{\wedge} \in S\left(\mathbb{R}^3 \right)$ with each seminorm bounded on j, k, it follows that the same holds for $G^{(j,k)}$. Now

$$\sum_{0 \le j \le k} 2^{(j+k)l} \left| G^{(j,k)} \left(\left(2^j, 2^k \right) \circ \xi \right) \right| \le \sum_{j,k,h \ge 0} 2^{ja+ka+ha} \left| G^{(j,k,h)} \left(2^j \xi_1, 2^k \xi_2, 2^h \xi_3 \right) \right|$$

with $a = \frac{l}{l+1}$, $G^{(j,k,h)} = G^{(j,k)}$ for $h = l(j+k)$ and $G^{(j,k,h)} = 0$ otherwise. It is well known that from the uniform boundedness properties of $G^{(j,k,h)}$ it follows that

$$\sum_{j,k,h \ge 0} 2^{ja+ka+ha} \left| G^{(j,k,h)} \left(2^j \xi_1, 2^k \xi_2, 2^h \xi_3 \right) \right| \le \frac{c}{|\xi_1|^a \, |\xi_2|^a \, |\xi_3|^a},$$

so

$$\sum_{0\leq j\leq k\leq K}\left|K_{j,k}\left(\xi\right)\right|\leq\frac{c}{\left|\xi_1\right|^{\frac{l}{l+1}}\left|\xi_2\right|^{\frac{l}{l+1}}\left|\xi_3\right|^{\frac{l}{l+1}}},$$

so $\sum_{0\leq j\leq k\leq K}T_{j,k}R_{j,k}$ convolves $L^p\left(\mathbb{R}^3\right)$ into $L^q\left(\mathbb{R}^3\right)$ for $\frac{1}{q}=\frac{1}{p}-\frac{1}{l+1}$ with bounds independent of K. □

Lemma 2.6. *There exists a constant $C>0$, independent of K, such that*

$$\left\|\sum_{1\leq j\leq k\leq K}T_{j,k}\left(I-P_{j,k}\right)\left(I-\mathfrak{Q}_{j,k}\right)\right\|_{\frac{2l+2}{2l+1},\frac{2l+2}{2l-1}}\leq C.$$

Proof. The kernel $H_{j,k}$ of

$$\sum_{1\leq j\leq k\leq K}T_{j,k}\left(I-P_{j,k}\right)\left(I-\mathfrak{Q}_{j,k}\right)$$

satisfies

$$\sum_{1\leq j\leq k\leq K}\left|H_{j,k}\left(\xi\right)\right|\leq\sum_{0\leq j\leq k}2^{(j+k)l}\left|g^{(j,k)}\left(\left(2^j,2^k\right)\circ\xi\right)\right|$$

with $g^{(j,k)}$ defined by $\left(g^{(j,k)}\right)^{\wedge}=\left(\mu^{(j-k)}\right)^{\wedge}\left(1-h\right)\left(1-m_0\right)$.

Observe that, from Lemma 7 in [3], we have $\left(\mu^{(j-k)}\right)^{\wedge}\left(1-h\right)\left(1-m_0\right)\in S(\mathbb{R}^3)$ with each seminorm bounded on j,k. From this fact the proof follows as in the previous lemma. □

Proof of the theorem. We have just observed that it is enough to prove (2.1). Since we can suppose $f \geq 0$, by (2.3), we need only check that there exists $C > 0$, independent of K such that

$$\left\| \sum_{0 \leq j \leq k \leq K} T_{j,k} \right\|_{\frac{2l+2}{2l+1}, \frac{2l+2}{2l-1}} \leq C,$$

where $T_{j,k}$ are defined by (2.2). For a constant $c_0 > 0$, we define $\mathfrak{Q}'_{j,k} = \sum_{|i-j| \leq c_0} \mathfrak{Q}_{i,k}$. So $\mathfrak{Q}'_{j,k}$ have the same properties as $\mathfrak{Q}_{j,k}$ and $\mathfrak{Q}'_{j,k} \circ \mathfrak{Q}_{j,k} = \mathfrak{Q}_{j,k}$ thus we have that (2.5) holds for $\mathfrak{Q}'_{j,k}$. Then, for $1 < p < \infty$ and

$$F = \{f_{j,k}\}_{j,k \geq 0} \in L^p\left(l^2\right), \qquad \left\| \sum_{j,k \geq 0} \mathfrak{Q}'_{j,k} f_{j,k} \right\|_p \leq c_p \left\| F \right\|_{L^p(l^2)}.$$

We decompose

$$\begin{aligned} \sum_{0 \leq j \leq k \leq K} & T_{j,k} f \\ &= \sum_{0 \leq j \leq k \leq K} T_{j,k} \left(I - P_{j,k}\right) \left(I - \mathfrak{Q}'_{j,k}\right) f + \sum_{0 \leq j \leq k \leq K} T_{j,k} P_{j,k} f \\ &\qquad + \sum_{0 \leq j \leq k \leq K} T_{j,k} \mathfrak{Q}'_{j,k} \left(I - P_{j,k}\right) f. \end{aligned}$$

Now, proceeding as in [1], the theorem follows from Proposition 2.3, Lemmas 2.5 and 2.6 and the remarks in [8, p. 85] concerning the multiparameter maximal function. □

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