Journal of Integer Sequences, Vol. 10 (2007),
Article 07.5.7

Some Remarks On the Equation $F_n = kF_m$ In Fibonacci Numbers

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Abstract

Let $\{F_n\}_{n=1}^{\infty} = \{1, 1, 2, 3, \ldots\}$ be the sequence of Fibonacci numbers. In this paper we give some sufficient conditions on a natural number k such that the equation $F_n = kF_m$ is solvable with respect to the unknowns n and m. We also show that for $k > 1$ the equation $F_n = kF_m$ has at most one solution (n, m).

1 Preliminaries

Let F_n be the nth Fibonacci number, i.e.,

$$F_1 = F_2 = 1 \ , \ F_{n+2} = F_n + F_{n+1}, \ \forall n \in \mathbb{N}.$$

It is known that these numbers have the following properties :

(1) $F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$;
(2) $\gcd(F_m, F_n) = F_{\gcd(m,n)}$;
(3) if $m|n$, then $F_m|F_n$;
(4) if $F_m|F_n$ and $m > 2$, then $m|n$.

Now, put

$$\begin{aligned} \mathcal{P} &= \{k \in \mathbb{N} \ : \ \exists m, n \in \mathbb{N}, F_n = kF_m\}, \\ \mathcal{Q} &= \{k \in \mathbb{N} \ : \ \nexists m, n \in \mathbb{N}, F_n = kF_m\}. \end{aligned}$$

A simple computations show that the natural numbers which satisfy in $\mathcal{P}$, less than 100, are as follows:

$$1, 2, 3, 4, 5, 7, 8, 11, 13, 17, 18, 21, 29, 34, 47, 48, 55, 72, 76, 89.$$

By definition of $\mathcal{P}$ and the properties (3) and (4), for each $k \in \mathcal{P}$ there exist $m, n \in \mathbb{N}$ such that $k = \frac{F_{mn}}{F_n}$. However, it seems that the elements of $\mathcal{Q}$ do not have any special form.

Using a theorem of R. D. Carmichael [2], it can be shown that the product of Fibonacci numbers and their quotients belong to $\mathcal{Q}$ except for some cases (see Theorem 3.10).

In this paper, we use elementary methods to prove our claim. In section 3, we obtain some more properties of $\mathcal{P}$. For example, we show that for every element $k(> 1)$ of $\mathcal{P}$, the equation $F_n = kF_m$ has a unique solution (n, m). Moreover, we give a necessary and sufficient condition for which the product of two elements of $\mathcal{P}$ is again in $\mathcal{P}$.

2 The Main Theorem

In this section, we introduce some elements k in $\mathcal{Q}$, so that for each fixed $n \in \mathbb{N}$,

$$k = F_{a_1} F_{a_2} \cdots F_{a_n}$$

belongs to $\mathcal{Q}$, for all natural numbers $a_1, \ldots, a_n$ but a finite number.

In order to prove the above claim, we need the following elementary properties of Fibonacci numbers.

Lemma 2.1. *For all $a, b, c, a_1, a_2, \ldots, a_n \in \mathbb{N}$, the following conditions hold*

a) $F_{a+b-1} = F_a F_b + F_{a-1} F_{b-1}$;

b) $F_{a+b-2} = F_a F_b - F_{a-2} F_{b-2}$;

c) $F_{a+b+c-3} = F_a F_b F_c + F_{a-1} F_{b-1} F_{c-1} - F_{a-2} F_{b-2} F_{c-2}$;

d) if $n \geq 3$, *then* $F_{a_1+\cdots+a_n-n} \geq F_{a_1} F_{a_2} \cdots F_{a_n}$.

Proof. Parts (a) and (b) are easily verified.

(c) Using (1), we obtain

$$\begin{aligned} F_{a+b+c-3} &= F_{a-1} F_{b+c-3} + F_a F_{b+c-2} \\ &= F_{a-1}(F_{b-2} F_{c-2} + F_{b-1} F_{c-1}) + F_a (F_b F_c - F_{b-2} F_{c-2}) \\ &= F_a F_b F_c + F_{a-1} F_{b-1} F_{c-1} - (F_a - F_{a-1}) F_{b-2} F_{c-2} \\ &= F_a F_b F_c + F_{a-1} F_{b-1} F_{c-1} - F_{a-2} F_{b-2} F_{c-2}. \end{aligned}$$

(d) We use induction on n. By part (c), the result holds for $n = 3$. Now assume it is true for $n \geq 3$. Clearly

$$\begin{aligned} F_{a_1+\cdots+a_{n+1}-(n+1)} &= F_{a_{n+1}-1} F_{a_1+\cdots+a_n-(n+1)} + F_{a_{n+1}} F_{a_1+\cdots+a_n-n} \\ &\geq F_{a_{n+1}} F_{a_1+\cdots+a_n-n} \\ &\geq F_{a_1} F_{a_2} \cdots F_{a_{n+1}}, \end{aligned}$$

which gives the assertion. □

Remark 1. *In Lemma 2.1(d), if $a_1 = \cdots = a_n = 1$, then $a_1 + \cdots + a_n - (n+1) = -1$ and by generalizing the recursive relation for negative numbers, we get $F_{-1} = F_1 - F_0 = 1$.*

Remark 2. *Note that all the formulas in Lemma 2.1 can be also deduced from Binet's formula*

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},$$

where

$$\alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2}.$$

Lemma 2.2. *Suppose m, n and k are any natural numbers with $k|n$, then*

$$\frac{F_{mn}}{F_n} \overset{F_k}{\equiv} mF_{n-1}^{m-1}.$$

Proof. We proceed by induction on m. Clearly, the result is true for $m = 1$. Assume it is true for m. Now, using (1) and (3), we have

$$\begin{aligned} \frac{F_{(m+1)n}}{F_n} &\overset{F_k}{\equiv} F_{n-1}\frac{F_{mn}}{F_n} + F_{mn+1} \\ &\overset{F_k}{\equiv} mF_{n-1}^{m} + F_{mn+1} \\ &\overset{F_k}{\equiv} mF_{n-1}^{m} + F_{n-1}F_{(m-1)n+1} + F_nF_{(m-1)n+2} \\ &\overset{F_k}{\equiv} mF_{n-1}^{m} + F_{n-1}F_{(m-1)n+1} \\ &\vdots \\ &\overset{F_k}{\equiv} mF_{n-1}^{m} + F_{n-1}^{m} \\ &\overset{F_k}{\equiv} (m+1)F_{n-1}^{m}. \end{aligned}$$

□

Lemma 2.3. *Let $a_1, \ldots, a_n, n \geq 3$ and $F_{a_1}F_{a_2}\cdots F_{a_n} = F_b$, then*

$$b + n \leq a_1 + \cdots + a_n \leq b + 2n - 2.$$

Proof. By Lemma 2.1, $F_b = F_{a_1}F_{a_2}\cdots F_{a_n} \leq F_{a_1+a_2+\cdots+a_n-n}$ and hence $b \leq a_1 + a_2 + \cdots + a_n - n$. This gives the left hand side of the inequality. By repeated application of Lemma 2.1 we have

$$\begin{aligned} F_b &= F_{a_1}F_{a_2}\cdots F_{a_n} \\ &\geq F_{a_1+a_2-2}F_{a_3}\cdots F_{a_n} \\ &\geq F_{a_1+a_2+a_3-4}F_{a_4}\cdots F_{a_n} \\ &\vdots \\ &\geq F_{a_1+\cdots+a_n-2(n-1)}, \end{aligned}$$

and so $b \geq a_1 + \cdots + a_n - 2(n-1)$, which completes the proof. □

Remark 3. *Note that using Binet's formula, for $n > 2$, one obtains*

$$(1-\beta^8)\alpha^n \leq \sqrt{5}F_n \leq (1+\beta^6)\alpha^n,$$

which implies the following inequalities

$$vn - u \leq a_1 + \cdots + a_n - b \leq un - v,$$

where

$$u = -\frac{\log((1-\beta^8)/\sqrt{5})}{\log\alpha} = 1.717\ldots$$

and

$$v = -\frac{\log((1+\beta^6)/\sqrt{5})}{\log\alpha} = 1.559\ldots.$$

One observes that the above inequalities are sharper than Lemma 2.3.

Definition. *A solution of the equation $F_{a_1}F_{a_2}\cdots F_{a_n} = F_b$ is said to be nontrivial, whenever $a_1, \ldots, a_n \geq 3$ or equivalently $F_{a_1}, \ldots, F_{a_n} > 1$.*

Lemma 2.4. *The equation $F_aF_b = F_c$ has no nontrivial solution, for any natural numbers a, b and c.*

Proof. We may assume $a \leq b$ and the triple (a, b, c) is a nontrivial solution of the equation, i.e., $a, b \geq 3$. Clearly, $F_b|F_c$ and hence $b|c$. Now put $c = kb$ which gives $k \geq 2$ and therefore $F_aF_b = F_{kb} \geq F_{2b} = F_b(F_{b-1} + F_{b+1}) > F_b^2 \geq F_aF_b$, which is impossible. □

We are now able to prove the main theorem of this section.

Theorem 2.5. *For each fixed $n \geq 2$, the equation $F_{a_1}F_{a_2}\cdots F_{a_n} = F_b$ has at most finitely many nontrivial solutions.*

Proof. By Lemma 2.4, the result follows for $n = 2$. Assume, $n \geq 3$ and let $(a_1, \ldots, a_n; b)$ be a nontrivial solution of the equation. Without loss of generality, we may assume $3 \leq a_1 \leq a_2 \leq \cdots \leq a_n$. Put $a_1 + \cdots + a_n = b + k$. Clearly, by Lemma 2.3 there are only finitely many natural numbers k, which can satisfy the latter equation. As $F_{a_n}|F_b$ and $a_n \geq 3$, we have $a_n|b$ and so $b = k'a_n$ for some $k' \in \mathbb{N}$. Similarly, $F_{a_{n-1}}|F_b = F_{k'a_n}$ and $a_{n-1} \geq 3$, which implies that $a_{n-1}|k'a_n$ and so $a_{n-1} = k''k'''$ with $k''|k'$ and $k'''|a_n$. Now since $F_{k'''}|F_{a_{n-1}}|\frac{F_{k'a_n}}{F_{a_n}}$, Lemma 2.2 implies that $F_{k'''}|k'$. By Lemma 2.3, there are only finitely many k, k', k'', k''' satisfying these equations. Thus there are only finitely many choices for a_{n-1} and consequently for $a_1, \ldots, a_{n-2}$. Finally, there are only finitely many choices for a_n and b satisfying the equation. □

Remark 4. *The above theorem shows that except finitely many cases if $k = F_{a_1}\cdots F_{a_n}$, where $a_1, \ldots, a_n \geq 3$ the equation $F_t = kF_s$ has no solution.*

3 Some More Results

In this section, we consider some more properties of the elements of $\mathcal{P}$ and $\mathcal{Q}$. For instant, it is shown that every element $k > 1$ of $\mathcal{P}$ satisfies a unique equation of the form $F_n = kF_m$.

Theorem 3.1. *The equation $F_aF_b = F_cF_d$ holds for natural numbers a, b, c, d if and only if $F_a = F_c$ and $F_b = F_d$, or $F_a = F_d$ and $F_b = F_c$.*

Proof. Clearly, if one the numbers a, b, c or d, (a, say), is less than 3 then $F_b = F_cF_d$ and Lemma 2.4 implies that either $F_c = F_a = 1$ and $F_b = F_d$, or $F_d = F_a = 1$ and $F_b = F_c$. Therefore, we assume that $a, b, c, d \geq 3$ and by symmetry we may assume that $3 \leq a \leq b, c, d$. Using Lemma 2.1, we have

$$F_{a+b-2} < F_aF_b = F_cF_d < F_{c+d-1},$$

which implies that $a + b - 2 < c + d - 1$ and hence $a + b \leq c + d$. Similarly $c + d \leq a + b$ and so $a + b = c + d$. By repeated application of Lemma 2.1, we obtain

$$\begin{array}{rrcl} & F_aF_b & = & F_cF_d \\ \Rightarrow & F_{a-1}F_{b-1} & = & F_{c-1}F_{d-1} \\ & & \vdots & \\ \Rightarrow & F_2F_{b-a+2} & = & F_{c-a+2}F_{d-a+2} \\ \Rightarrow & F_{b-a+2} & = & F_{c-a+2}F_{d-a+2}. \end{array}$$

Now by Lemma 2.4, $F_{c-a+2} = 1$ or $F_{d-a+2} = 1$, which implies that either $a = c$ and $b = d$, or $a = d$ and $b = c$. □

The following corollaries follow immediately.

Corollary 3.2. *Suppose $\frac{F_a}{F_b} = \frac{F_c}{F_d} \neq 1$, then $F_a = F_c$ and $F_b = F_d$.*

Corollary 3.3. *Every element $k > 1$ of $\mathcal{P}$ satisfies a unique equation of the form $F_n = kF_m$, for some natural numbers m and n.*

Corollary 3.4. *The least common multiple of two Fibonacci numbers is again a Fibonacci number if and only if one divides the other.*

Proof. Suppose $\mathrm{lcm}(F_m, F_n) = F_k$, for some natural numbers m and n. Then clearly

$$F_mF_n = \gcd(F_m, F_n)\mathrm{lcm}(F_m, F_n) = F_{\gcd(m,n)}F_k$$

and so $\gcd(F_m, F_n) = F_{\gcd(m,n)}$ is either F_m or F_n. Hence either $F_m|F_n$ or $F_n|F_m$. □

Theorem 3.5. *For any natural numbers a, b, c, d and e, the equation $F_aF_bF_c = F_dF_e$ has no nontrivial solution.*

Proof. Assume $(a, b, c; d, e)$ is a nontrivial solution of the equation $F_aF_bF_c = F_dF_e$. Hence $a, b, c, d, e \geq 3$. By Lemma 2.1, we have

$$F_{a+b+c-4} < F_aF_bF_c = F_dF_e < F_{d+e-1}$$

and

$$F_{d+e-2} < F_dF_e = F_aF_bF_c \leq F_{a+b+c-3},$$

which imply that $a+b+c = d+e+2$. Using Lemma 2.1 once more and noting the identity $a+b+c-3 = d+e-1$, we obtain

$$\begin{aligned} F_{d+e-4} &\leq F_{d-1}F_{e-1} \\ &= F_{a-1}F_{b-1}F_{c-1} - F_{a-2}F_{b-2}F_{c-2} \\ &< F_{a-1}F_{b-1}F_{c-1} \\ &\leq F_{a+b+c-6}. \end{aligned}$$

Thus $d+e+2 < a+b+c$, which is impossible. □

Theorem 3.6. *Let $(a, b, c; d, e, f)$ be a nontrivial solution of the equation $F_aF_bF_c = F_dF_eF_f$, then a, b, c are equal to d, e, f, in some order.*

Proof. Without loss of generality, we may assume that $a \leq d$, $3 \leq a \leq b \leq c$ and $3 \leq d \leq e \leq f$. If $a = d$, the result follows immediately by Theorem 3.1. Now assume that $a < d$. Using Lemma 2.1, we have

$$F_{a+b+c-4} < F_aF_bF_c = F_dF_eF_f \leq F_{d+e+f-3}$$

and

$$F_{d+e+f-4} < F_dF_eF_f = F_aF_bF_c \leq F_{a+b+c-3}.$$

Thus $a+b+c = d+e+f$, and so by Lemma 2.1 we obtain

$$\begin{aligned} F_{a-1}F_{b-1}F_{c-1} - F_{a-2}F_{b-2}F_{c-2} &= F_{d-1}F_{e-1}F_{f-1} - F_{d-2}F_{e-2}F_{f-2} \\ 2F_{a-2}F_{b-2}F_{c-2} - F_{a-3}F_{b-3}F_{c-3} &= 2F_{d-2}F_{e-2}F_{f-2} - F_{d-3}F_{e-3}F_{f-3} \\ &\vdots \end{aligned}$$

Hence for each $i \geq 1$

$$F_{i+1}F_{a-i}F_{b-i}F_{c-i} - F_iF_{a-i-1}F_{b-i-1}F_{c-i-1} = F_{i+1}F_{d-i}F_{e-i}F_{f-i} - F_iF_{d-i-1}F_{e-i-1}F_{f-i-1}.$$

By replacing i by a in the above equality, we obtain

$$0 \geq -F_aF_{b-a-1}F_{c-a-1} = F_{a+1}F_{d-a}F_{e-a}F_{f-a} - F_aF_{d-a-1}F_{e-a-1}F_{f-a-1} \geq 0.$$

Then

$$F_{a+1}F_{d-a}F_{e-a}F_{f-a} - F_aF_{d-a-1}F_{e-a-1}F_{f-a-1} = 0,$$

which is impossible, since otherwise we must have

$$F_{d-a}F_{e-a}F_{f-a} = F_{d-a-1}F_{e-a-1}F_{f-a-1} = 0,$$

which implies that $d = a$. □

The following corollary is an immediate consequence of the above theorem.

Corollary 3.7. *Let* $x = \frac{F_a}{F_b}, y = \frac{F_c}{F_d}$ *be in* $\mathcal{P}$*. Then* $xy \in \mathcal{P}$ *if and only if one of the following occurs*

i) $x = 1$;
ii) $y = 1$;
iii) $x = y = 2$;
iv) $F_a = F_d$, *or*
v) $F_b = F_c$.

Now we turn to the equation $F_{a_1} F_{a_2} \cdots F_{a_n} = F_b$. The special case when a_i's are equal follows easily from the following theorem. We are not aware of its proof so we prove it here (see [3]).

Theorem 3.8. *Let* p *be a prime and let* m *and* n *be natural numbers such that* $p \nmid m$ *and* $p^\alpha \| F_n$, *for* $\alpha > 0$*. Then*

i) $p^{\alpha+1} \| F_{nmp}$, *if* $(p, \alpha) \neq (2, 1)$;
ii) $p^{\alpha+2} \| F_{nmp}$, *if* $(p, \alpha) = (2, 1)$.

Proof. By the assumption and Lemma 2.2,

$$\frac{F_{nm}}{F_n} \overset{p}{\equiv} m F_{n-1}^{m-1}.$$

Thus if $p \nmid m$ then $p^\alpha \| F_{nm}$ and hence it is enough to show that $p^{\alpha+1} \| F_{np}$. By repeated applications of (1), we have

$$\begin{aligned}
\frac{F_{pn}}{F_n} &= F_{n-1} \frac{F_{(p-1)n}}{F_n} + F_{(p-1)n+1} \\
&= F_{n-1} \left(F_{n-1} \frac{F_{(p-2)n}}{F_n} + F_{(p-2)n+1} \right) + F_{(p-1)n+1} \\
&\vdots \\
&= F_{n-1}^{p-1} + F_{n-1}^{p-2} F_{n+1} + F_{n-1}^{p-3} F_{2n+1} + \cdots + F_{n-1} F_{(p-2)n+1} + F_{(p-1)n+1}.
\end{aligned}$$

Now, for each $k \in \mathbb{N}$

$$\begin{aligned}
F_{kn+1} &= F_n F_{(k-1)n} + F_{n+1} F_{(k-1)n+1} \\
&\overset{p^{2\alpha}}{\equiv} F_{n+1} F_{(k-1)n+1} \\
&\vdots \\
&\overset{p^{2\alpha}}{\equiv} F_{n+1}^k \\
&\overset{p^{2\alpha}}{\equiv} (F_n + F_{n-1})^k \\
&\overset{p^{2\alpha}}{\equiv} k F_n F_{n-1}^{k-1} + F_{n-1}^k.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{F_{pn}}{F_n} &\overset{p^{2\alpha}}{\equiv} F_{n-1}^{p-1} + F_{n-1}^{p-2}F_{n+1} + \cdots + F_{n-1}F_{(p-2)n+1} + F_{(p-1)n+1} \\
&\overset{p^{2\alpha}}{\equiv} F_{n-1}^{p-1} + F_{n-1}^{p-2}(F_n + F_{n-1}) + \cdots + F_{n-1}\big((p-2)F_nF_{n-1}^{p-3} + F_{n-1}^{p-1}\big) \\
&\quad + \big((p-1)F_nF_{n-1}^{p-2} + F_{n-1}^{p-1}\big) \\
&\overset{p^{2\alpha}}{\equiv} \frac{p(p-1)}{2}F_nF_{n-1}^{p-2} + pF_{n-1}^{p-1},
\end{aligned}$$

which implies that $p^{\alpha+1}\|F_{np}$ whenever $(p,\alpha) \neq (2,1)$. This proves (i).

Now, if $(p,\alpha) = (2,1)$ then F_n is even, $3|n$ and $\frac{n}{3}$ is odd. On the other hand, $8\|F_6$ and by the proof of part (i), $8\|F_{2n}$ which completes the proof of part (ii). □

Theorem 3.9. *For all $k > 1$, the equation $F_n = F_m^k$ has only the solutions $F_m = F_n = 1$, or $k = 3$, $m = 3$ and $n = 6$.*

Proof. Let $k > 1$, $n \geq m \geq 3$ and $F_n = F_m^k$. As $F_m|F_n$, we have $m|n$ and so $n = dm$, for some $d \in \mathbb{N}$. Also, by Lemma 2.2, $F_m|d$. Now, if p is a prime divisor of F_m such that $p^a\|F_m$, where $(p,a) \neq (2,1)$, then p is also a divisor of d and by Theorem 3.8, $p^{a+b}\|F_n$, where $p^b\|d$. On the other hand, $p^{ka}\|F_n$ and so $a + b = ka$, i.e., $b = (k-1)a$. Now, we have

$$k - 1 \geq d = p^bd' \geq p^b = p^{(k-1)a} \geq p^{k-1} \geq k,$$

which is impossible and hence $F_m = 2$. If $p > 3$ and p divides n, then $F_p|2^k$, which is also impossible. Hence $n = 2^s3^t$ and as $F_4, F_9 \nmid 2^k$, we must have $n = 6$. □

R. D. Carmichael [2] showed that if $n > 2$ and $n \neq 6, 12$ then F_n has a prime divisor p, which does not divide the Fibonacci numbers F_m, for all $1 \leq m < n$. Applying this result one can obtain the general solutions of the equation $F_{a_1} \cdots F_{a_m} = F_b$ and more generally the solutions of the equation $F_{a_1} \cdots F_{a_m} = F_{b_1} \cdots F_{b_n}$. For some applications of this beautiful theorem, see [1].

We say a solution of the equation $F_{a_1} \cdots F_{a_m} = F_{b_1} \cdots F_{b_n}$ is nontrivial, whenever $a_i, b_j \geq 3$ and $a_i \neq b_j$, for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Theorem 3.10. *i) The only nontrivial solutions of the equation $F_{a_1}F_{a_2} \cdots F_{a_n} = F_b$ with $n > 1$ and $a_1 \leq \cdots \leq a_n$ are*

$$(3,3,3;6)\ ,\ (3,4,4,6;12)\ ,\ (3,3,3,3,4,4;12)$$

ii) The only nontrivial solutions of the equation $F_{a_1} \cdots F_{a_m} = F_{b_1} \cdots F_{b_n}$ are

$$\begin{aligned}
(3,\ldots,3;6,\ldots,6) &\quad , \quad m = 3n \\
(\overbrace{3,\ldots,3}^{a},\overbrace{6,\ldots,6}^{b},4,\ldots,4;12,\ldots,12) &\quad , \quad a + 3b = 4n \\
(\overbrace{3,\ldots,3}^{a},4,\ldots,4;\overbrace{6,\ldots,6}^{b},12,\ldots,12) &\quad , \quad a = 3b + 4n \\
(\overbrace{6,\ldots,6}^{a},4,\ldots,4;\overbrace{3,\ldots,3}^{b},12,\ldots,12) &\quad , \quad 3a = b + 4n
\end{aligned}$$

Proof. The proofs of both parts follow easily from Carmichael's theorem. □

The following theorem is another consequence of Carmichael's theorem.

Theorem 3.11. *Suppose $p_1, p_2, \ldots, p_n$ are arbitrary distinct prime numbers. Then there are only finitely many n-tuples $(a_1, \ldots, a_n)$ of nonnegative integers such that $p_1^{a_1} \cdots p_n^{a_n} \in \mathcal{P}$.*

Proof. Assume $\{(a_{i_1}, \ldots, a_{i_n})\}_{i=1}^{\infty}$ is an infinite sequence of distinct n-tuples such that for each i the number $k_i = p_1^{i_1} \cdots p_n^{i_n}$ belongs to $\mathcal{P}$. Then there exist some natural numbers m_i and n_i such that $F_{n_i} = k_i F_{m_i}$. Without loss of generality, we may assume that $n_i \neq m_i$ and $n_i's$ are all distinct and greater than 12. Since there are infinitely many n-tuples, we may ignore the prime factors of the equations $F_{n_i} = k_i F_{m_i}$ so that we obtain an equation of type as in Theorem 3.10, which contradicts Theorem 3.10. □

Although we were able to obtain the general solutions of the equation $F_{a_1} \cdots F_{a_n} = F_b$ using Carmichael's theorem, an elementary proof may nevertheless be of interest.

4 Acknowledgment

The author would like to thank the referee for some useful suggestions and corrections.

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2000 *Mathematics Subject Classification*: Primary 11B39; Secondary 11B50, 11D99.
Keywords: Fibonacci numbers.

(Concerned with sequence A000045.)

Received March 28 2007; revised version received May 22 2007. Published in *Journal of Integer Sequences*, June 4 2007.

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