Journal of Integer Sequences, Vol. 10 (2007),
Article 07.4.7

A Simple Symmetry Generating Operads Related to Rooted Planar m-ary Trees and Polygonal Numbers

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Abstract

The aim of this paper is to further explore an idea from J.-L. Loday and extend some of his results. We impose a natural and simple symmetry on a unit action over the most general quadratic relation which can be written. This leads us to two families of binary, quadratic and regular operads whose free objects are computed, as much as possible, as well as their duals in the sense of Ginzburg and Kapranov. Roughly speaking, free objects found here are in relation to rooted planar m-ary trees, triangular numbers and more generally m-tetrahedral numbers, homogeneous polynomials on m commutative indeterminates over a field K and polygonal numbers. Involutive connected $\mathcal{P}$-Hopf algebras are constructed.

To J.-L. Loday on his 60$^{\text{th}}$ birthday

1 Introduction

Let K be a null characteristic field. In the sequel, if S is a set, then KS or $K[S]$ will be the free K-vector space spanned by S.

We pursue our ideas from our work [4] to view discrete structures through operad theory. These ideas started in studying a work of J.-L. Loday [6]. In this paper, J.-L. Loday uses rooted planar binary trees to construct the free object in the category of dendriform algebras

and consequently rooted planar binary trees were reconstructed from the simple one, $\curlyvee$, and two operations. A question comes to mind:
Is it possible to view more discrete structures as generated by a simple object and operations? An advantage of this point of view is to use the power of algebraic tools. For instance, it is proposed in [4] formal deformations of dendriform algebras. Operations reconstructing planar binary trees from $\curlyvee$ can be deformed and at each order of the deformation, new types of algebras are proposed whose free objects have to be discovered. As we are focusing on K-vector spaces equipped with several operations, the natural framework to be considered is K-linear operad theory. We suppose the reader to be familiar with it, otherwise see [3, 5] for instance. Our setting will be regular, binary and quadratic operads.

To answer partly our question, we proceed in a way different from [4]. From an action of a unit, which will be defined below, on the most general quadratic relation written with k binary operations, we will exhibit a class of operad ${}^k\mathcal{P}$ whose duals, in the sense of Ginzburg and Kapranov [3], can be studied. We show in Section 3, that these operads, so called the $k-$*Gonal* operads are related to polygonal numbers. We recover that any polygonal numbers can be reconstructed from the most simple one, that is 1, and k operations. We will focus on the case $k = 3$ in Section 4 and 5. In this case, ${}^3\mathcal{P}$ can be described and renamed in 3-*Dend*, the operad of the so called 3-dendriform algebras since on one generator, the free object is related to rooted planar ternary trees. To recover the rooted planar m-ary trees cases, we add quadratic relations to 3-*Dend* in Section 4 and get the operads m-*Dend*. We show their free objects on one generator are related to rooted planar m-ary trees. Duals of operads m-*Dend* are computed, giving birth to operads m-*Tetra*, related to tetrahedral numbers in dimension $m-1$ and homogeneous polynomials in m commutative indeterminates. Homologies are proposed. In Section 5, we restrict our attention to the case 3-*Dend* and propose an homology of 3-dendriform trialgebras. We also provide proofs of results obtained in Section 4 for this particular case. We sum up our results in Section 6.

Let us start with some technicalities on operads.

2 Some technicalities on operads

Let V be a K-vector space and $\mathcal{P}$ be a binary regular quadratic operads. The *free $\mathcal{P}$-algebra* $\mathcal{P}(V)$ on V is by definition a $\mathcal{P}$-algebra equipped with a linear map $i: V \to \mathcal{P}(V)$ which satisfies the following universal property: for any linear map $f: V \to A$, where A is a $\mathcal{P}$-algebra, there exists a unique $\mathcal{P}$-algebra morphism $\phi: \mathcal{P}(V) \to A$ such that $\phi \circ i = f$. Since our $\mathcal{P}$-algebras are regular, the free $\mathcal{P}$-algebra over a $K-$vector space V is of the form: $\mathcal{P}(V) := \bigoplus_{n\geq 1} \mathcal{P}(n) \otimes_{K[S_n]} V^{\otimes n}$, where $\mathcal{P}(n) := \mathcal{P}_n \otimes K[S_n]$ is the K-vector spaces spanned by all possible n-ary operations. In particular, the free $\mathcal{P}$-algebra on one generator is $\mathcal{P}(K) = \bigoplus_{n\geq 1} \mathcal{P}_n$. The generating function of the regular operad $\mathcal{P}$, or its Poincaré series, is given by: $f^{\mathcal{P}}(x) := \sum (-1)^n \dim \mathcal{P}_n\, x^n$. Below, we will indicate the sequence $(\dim \mathcal{P}_n)_{n\geq 1}$. By a *unit action* [7], we mean the choice of two linear applications: $\upsilon: \mathcal{P}(2) \to \mathcal{P}(1)$ and $\varpi: \mathcal{P}(2) \to \mathcal{P}(1)$, giving sense, when possible, to $x \diamond 1$ and $1 \diamond x$, for all operations $\diamond \in \mathcal{P}(2)$ and for all x in the $\mathcal{P}$-algebra A, *i.e.*, $x \diamond 1 = \upsilon(\diamond)(x)$ and $1 \diamond x = \varpi(\diamond)(x)$. If $\mathcal{P}(2)$ contains an associative operation, say $\star$, then we require that $x \star 1 := x := 1 \star x$, i.e., $\upsilon(\star) := Id := \varpi(\star)$.

We also set $T(V) := K \oplus \bigoplus_{n>0} V^{\otimes n}$.

3 Symmetry on a unit action and polygonal numbers

3.1 The ${}^k\mathcal{P}$ operads

Let $k > 1$ be an integer. We consider k binary operations $\bullet_i$, $1 \leq i \leq k$, over a K-vector space V, supposed to be related by quadratic and regular relations. We suppose the existence of a unit element, denoted by 1, which acts as follows. First we rename $\bullet_1$ as $\succ$ and $\bullet_n$ as $\prec$. Second, set for all $x \in V$, $1 \prec x = 0 = x \succ 1$, $x \prec 1 = x = 1 \succ x$ and for all $2 \leq i \leq k-1$ set $x \bullet_i 1 = 0 = 1 \bullet_i x$. Observe that the action of the unit is invariant under the transformation,

$$x \prec y \longmapsto y \succ x, \qquad x \succ y \longmapsto y \prec x, \qquad x \bullet_i y \longmapsto y \bullet_i x, \tag{1}$$

for all $2 \leq i \leq k-1$. We now write down the most general equation relating operations $\bullet_i$, $1 \leq i \leq k$ to their braces, that is:

$$\sum_{i,j=1}^{k} \lambda_{ij}\ (x \bullet_i y) \bullet_j z = \sum_{i,j=1}^{k} \lambda'_{ij}\ x \bullet_i (y \bullet_j z),$$

where the λ_{ij} and λ'_{ij} are scalars of K. Setting $x = 1$, then $y = 1$, then $z = 1$ in the previous equation and applying our choice of the unit action lead to the following system of relations. For all $x, y, z \in V$, for all $2 \leq i \leq k-1$, we get:

$$(x \prec y) \prec z = x \prec (y \star z), \qquad (x \prec y) \bullet_i z = x \bullet_i (y \succ z)$$

$$(x \succ y) \prec z = x \succ (y \prec z), \qquad (x \succ y) \bullet_i z = x \succ (y \bullet_i z)$$

$$(x \star y) \succ z = x \succ (y \succ z), \qquad (x \bullet_i y) \prec z = x \bullet_i (y \prec z),$$

where $x \star y := x \prec y + x \succ y$. Observe that $\star$ is associative and that this system of $3(k-1)$ relations is invariant under the transformation (1). Binary, regular and quadratic operads denoted by ${}^k\mathcal{P}$ can be naturally associated with each system of relations. The coefficients of their Poincaré series starts with $1, k, 2k^2 - 3(k-1), \ldots$, that is for $k = 2$, we get $1, 2, 5, \ldots$ which is the beginning of the Catalan numbers counting for instance the number of planar rooted binary trees on p internal vertices, with $p > 0$. The operad ${}^2\mathcal{P}$ is the operad *Dend* of dendriform algebras introduced by J.-L. Loday [6]. For $k = 3$, we get $1, 3, 12, \ldots$ which is the beginning of the sequence counting the number of planar rooted ternary trees on p internal vertices, with $p > 0$. For $k = 4$, we get $1, 4, 23, \ldots$ which is the beginning of the sequence counting the number of non-crossing connected graphs [2] on $p+1$ vertices, with $p > 0$ and so on.

3.2 Hopf algebra structures on ${}^k\mathcal{P}$

Let T be a ${}^k\mathcal{P}$-algebra. Define new operations by:

$$x \prec' y := y \succ x; \ x \succ' y := y \prec x; \ x \bullet'_i y := y \bullet_{n+1-i} x,$$

for all $2 \leq i \leq k-1$. Then, the K-vector space T equipped with these operations is a new ${}^k\mathcal{P}$-algebra, denoted by T^{op}, called the opposite ${}^k\mathcal{P}$-algebra. A ${}^k\mathcal{P}$-algebra is said to be commutative if $T^{op} = T$. We get the following results.

Theorem 3.1. *Let V be a K-vector space. For all $k > 1$, there exists a connected $\mathcal{P}$-Hopf algebra structure respectively on ${}^k\mathcal{P}(V)$, the free ${}^k\mathcal{P}$-algebra over V and on ${}^k\mathcal{P}_{com}(V)$, the free commutative ${}^k\mathcal{P}$-algebra over V.*

Proof: Observe that the choice of the unit action is such that $x \prec' 1 := 1 \succ x := x$ and $x \succ' 1 := 1 \prec x := x$ and thus in agreement with the opposite structure. Apply now the result in [7]. $\square$

3.3 Reconstructing polygonal numbers from 1: The operad k-gonal

The free objects associated with the operads ${}^k\mathcal{P}$ defined in that section seems to be difficult to construct systematically (except for the case $n = 2$, see [6] and the case $n = 3$, see next Section). However, free objects associated with their operadic duals can be computed systematically. In the literature (see N.J.A. Sloane Online Encyclopedia of Integers for instance) the n^{th} k-gonal number is defined by:

$$g_k(n) := n + (k-2)\frac{n(n-1)}{2}.$$

Up to the author's knowedge, serious results on polygonal numbers started with a Fermat's theorem in 1638. He pretended to have proved that any positive integers can always be written as a sum of at most k k-gonal numbers. Gauss proved the case of triangular numbers summarising his result by the formula $E\Upsilon PHKA = \triangle + \triangle + \triangle$ (1796). Euler left important results on the Fermat's assertion which were used by Lagrange to prove the square case, result also found independently by Jacobi (1772). Finally Cauchy proved the whole assertion (1813). We now define k-gonal algebras.

Definition 3.2. Fix an integer $k > 2$. A k-gonal algebra G_k is a K-vector space equipped with k binary operations $\vdash, \dashv, (\perp_i)_{2 \leq i \leq k-1} : G_k^{\otimes 2} \to G_k$ obeying the following system of quadratic relations for all $2 \leq i, j \leq k-1$ and all $x, y, z \in G_k$,

$$\begin{cases} (x \dashv y) \dashv z = x \dashv (y \dashv z), & (x \dashv y) \perp_i z = x \perp_i (y \vdash z), \\ (x \dashv y) \dashv z = x \dashv (y \vdash z), & (x \vdash y) \perp_i z = x \vdash (y \perp_i z), \\ (x \vdash y) \dashv z = x \vdash (y \dashv z), & (x \perp_i y) \dashv z = x \perp_i (y \dashv z), \\ (x \dashv y) \vdash z = x \vdash (y \vdash z), & (x \perp_i y) \perp_j z = 0 = x \perp_i (y \perp_j z), \\ (x \vdash y) \vdash z = x \vdash (y \vdash z), & (x \perp_i y) \vdash z = 0 = x \dashv (y \perp_i z). \end{cases} \tag{2}$$

The associated category (resp. operad) is denoted by k-**Gonal**, (resp. $k - Gonal$).

As expected, there are $2(k-2)^2 + 5(k-1) = 2k^2 - 3(k-1)$ relations. The functorial diagram between involved categories holds,

$$
\begin{array}{ccccccc}
\mathbf{As} \to & \mathbf{Dias} \to & 3-\mathbf{Gonal} := \mathbf{Triang} \to & \mathbf{4\text{-}Gonal} \to \ldots \\
\searrow & \downarrow & \swarrow & \swarrow \\
 & \mathbf{Leib} & &
\end{array}
$$

where **Leib** is the category of Leibniz algebras (a generalisation of Lie algebras [6]). Commutative k-gonal algebras can be also defined. Introduce new operations by: $x \vdash' y := y \dashv x$; $x \dashv' y := y \vdash x$; $x \perp_i' y := y \perp_{m+1-i} x$, for all $2 \leq i \leq m-1$. Then, the K-vector space G_k equipped with these operations is a new k-gonal algebra, denoted by G_k^{op}, called the opposite k-gonal algebra. A k-gonal is said to be commutative if $T^{op} = T$. For any k-gonal algebra G_k, let $as(G_k)$ be the quotient of G_k by the ideal generated by the elements $x \vdash y - x \dashv y$ and $x \perp_i y$, for all $2 \leq i \leq m-1$ and $x, y \in T$. Observe that $as(G_k)$ is an associative algebra and that the functor $as.(-)$:k-**Gonal**$\to$ **As** is left adjoint to the functor inc : **As** $\to$ k-Gonal. As expected, we get:

Theorem 3.3. *Fix $k > 2$. The operad k-Gonal is dual in the sense of [3] to the operad ${}^k\mathcal{P}$, that is k-Gonal$={}^k\mathcal{P}^!$ and thus ${}^k\mathcal{P}$=k-Gonal$^!$.*

Proof: Straightforward. □

To state the next theorem, we introduce for all $1 \leq p \leq k-3$, the linear maps,

$$\psi_1^p : \bigoplus_{n>0} T^{\otimes n} \to \underbrace{V \oplus \ldots \oplus V}_{k-3}, \; v_1 \otimes \ldots \otimes v_n \mapsto 0 \oplus \ldots \oplus 0 \oplus \underbrace{v_1}_{p^{th}\ position} \oplus 0 \oplus \ldots \oplus 0,$$

and $\psi_2 : \bigoplus_{n>0} T^{\otimes n} \to T(V)$ defined by $v_1 \otimes \ldots \otimes v_n \mapsto v_2 \otimes \ldots \otimes v_n$ for $n > 1$ and by $v \mapsto 1$. Denote by $\psi : (K \oplus V) \otimes T(V) \to K$ and by $\Psi : (K \oplus V) \otimes T(V) \otimes (K \oplus V) \otimes T(V) \to K$ the canonical projections.

Theorem 3.4. *Fix an integer $k > 2$. Let V be a K-vector space. Define on*

$$k - Gonal(V) := T(V) \otimes [V \otimes (K \oplus \underbrace{V \oplus \ldots \oplus V}_{k-3}) \otimes T(V)] \otimes T(V),$$

the following binary operations where the $\omega_i \in V^{\otimes p_i}$, $v, v' \in V$ and w, w' belong to either V or K:

$$
\begin{aligned}
\omega_1 \otimes [v \otimes w \otimes \omega_2] \otimes \omega_3 \dashv \omega_1' \otimes [v' \otimes w' \otimes \omega_2'] \otimes \omega_3' &= \psi(w', \omega_2')\omega_1 \otimes [v \otimes w \otimes \omega_2] \otimes \omega_3 \omega_1' v' \omega_3', \\
\omega_1 \otimes [v \otimes w \otimes \omega_2] \otimes \omega_3 \vdash \omega_1' \otimes [v' \otimes w' \otimes \omega_2'] \otimes \omega_3' &= \psi(w, \omega_2)\omega_1 v \omega_3 \omega_1' \otimes [v' \otimes w' \otimes \omega_2'] \otimes \omega_3', \\
\omega_1 \otimes [v \otimes w \otimes \omega_2] \otimes \omega_3 \perp_2 \omega_1' \otimes [v' \otimes w' \otimes \omega_2'] \otimes \omega_3' &= \Psi(w, \omega_2, w', \omega_2')\omega_1 \otimes [v \otimes \omega_3 \omega_1' v'] \otimes \omega_3', \\
\omega_1 \otimes [v \otimes w \otimes \omega_2] \otimes \omega_3 \perp_i \omega_1' \otimes [v' \otimes w' \otimes \omega_2'] \otimes \omega_3' &= \Psi(w, \omega_2, w', \omega_2')\omega_1 \otimes [v \otimes \psi_1^{i-2}(\omega_3 \omega_1' v') \otimes \psi_2(\omega_3 \omega_1' v')] \otimes \omega_3',
\end{aligned}
$$

for $3 \leq i \leq k-1$. Then, $k - Gonal(V)$ is the free k-gonal algebra over V. Therefore, the Poincaré series of the operad $k - Gonal$ is,

$$f_{k-gonal}(x) = \sum_{n>0} (-1)^n g_k(n) x^n = \frac{(k-3)x^2 - x}{(x+1)^3}.$$

Proof: The proof that $k - Gonal(V)$ is a k-gonal algebra does not present any difficulties and is left to the reader. The map $i : V \hookrightarrow k - Gonal(V)$ is the composite:

$$V \simeq K \otimes [V \otimes K \otimes K] \otimes K \hookrightarrow \quad k - Gonal(V).$$

Let $f : V \longrightarrow G_k$ be a linear map, where G_k is a k-gonal algebra. We construct its universal extension ϕ as follows. First on monomials from k-$Gonal(V)$, then extended by K-linearity. Therefore, define $\phi : k$-$Gonal(V) \longrightarrow G_k$ by:

$$\begin{aligned}\phi(v_{-r} \otimes \ldots \otimes v_{-1} \otimes \quad [\quad & v_0 \otimes \psi_1^p(w) \otimes u_1 \otimes \ldots \otimes u_s] \otimes v_1 \otimes \ldots \otimes v_q) := \\ & f(v_{-r}) \vdash \ldots \vdash f(v_{-1}) \vdash [f(v_0) \perp_{p+2} (f(w) \vdash f(u_1) \vdash \ldots \vdash f(u_s))] \dashv f(v_1) \dashv \ldots \dashv f(v_q), \\ \phi(v_{-r} \otimes \ldots \otimes v_{-1} \otimes \quad [\quad & v_0 \otimes u_1 \otimes \ldots \otimes u_s] \otimes v_1 \otimes \ldots \otimes v_q) := \\ & f(v_{-r}) \vdash \ldots \vdash f(v_{-1}) \vdash [f(v_0) \perp_2 (f(u_1) \vdash \ldots \vdash f(u_s))] \dashv f(v_1) \dashv \ldots \dashv f(v_q),\end{aligned}$$

The map ϕ is a morphism of k-gonal algebra, hence the unicity of such a map since it has to coincide with f on V. □

Remark: Observe that k-gonal numbers can be reconstructed from k operations and 1 played here by $1 \otimes [v \otimes 1 \otimes 1] \otimes 1$ if we consider a vector space V spanned by say v, *i.e.*, $V := Kv$.

4 On planar m-ary trees and tetrahedral numbers

Let us focus on the case $k = 3$. It appears that ${}^3\mathcal{P}$ can be described with the help of planar rooted ternary trees. This leads us to generalise axioms of ${}^3\mathcal{P}$ to recover the case of planar rooted m-ary trees, m-ary trees for short. We first deal with the general case before focusing on ${}^3\mathcal{P}$.

4.1 The operad $m - Dend$

Definition 4.1. Fix an integer $m > 1$. A K-vector space T is a m-dendriform algebra if it is equipped with m binary operations $\prec, \succ, \bullet_2, \ldots, \bullet_{m-1} : T^{\otimes 2} \longrightarrow T$ verifying for all $x, y, z \in T$, and for all $2 \leq i \leq m-1$, the $\frac{m(m+1)}{2}$ axioms.

$$(x \prec y) \prec z = x \prec (y \star z), \qquad (x \prec y) \bullet_i z = x \bullet_i (y \succ z)$$

$$(x \succ y) \prec z = x \succ (y \prec z), \qquad (x \succ y) \bullet_i z = x \succ (y \bullet_i z)$$

$$(x \star y) \succ z = x \succ (y \succ z), \qquad (x \bullet_i y) \prec z = x \bullet_i (y \prec z),$$

where $x \star y := x \prec y + x \succ y$ and,

$$(x \bullet_i y) \bullet_j z = x \bullet_i (y \bullet_j z),$$

for all $2 \leq i < j \leq m-1$. A m-dendriform algebra is said to be involutive if it is equipped with a linear involutive map $\dagger : T \longrightarrow T$ acting as follows, $(x \prec y)^\dagger = y^\dagger \succ x^\dagger$, $(x \succ y)^\dagger = y^\dagger \prec x^\dagger$, $(x \bullet_i y)^\dagger = y^\dagger \bullet_{n+1-i} x^\dagger$, for $2 \leq i \leq k-1$. We extend then $\dagger$ on $T^{\otimes 2}$ by the following formula, $(x \otimes y)^\dagger = x^\dagger \otimes y^\dagger$. New operations can be defined by: $x \prec' y := y \succ x$; $x \succ' y := y \prec x$; $x \bullet'_i y := y \bullet_{m+1-i} x$, for all $2 \leq i \leq m-1$. The K-vector space T, equipped with these operations is a new m-dendriform algebra, denoted by T^{op}, the opposite m-dendriform algebra. A m-dendriform algebra is said to be commutative if $T^{op} = T$. Recall the case $m = 2$ has been studied in [6].

Therefore, each $m > 1$ gives birth to a regular, binary and quadratic operad denoted by $m - Dend$, and a category denoted by m-**Dend**. The functorial diagram between categories holds:

$$\begin{array}{ccccccc} 2-\mathbf{Dend} \longrightarrow & & 3-\mathbf{Dend} \longrightarrow \ldots & & \longrightarrow m-\mathbf{Dend} & & \ldots \longrightarrow \\ & + \searrow & \downarrow + & \swarrow + & & & \\ & & \mathbf{As} & & & & \end{array}$$

We recall some general facts about m-ary trees. Their names comes from the fact that each father vertex has exactly m children. The degree of such a tree will be the number of internal vertices. By $\overset{m}{\Psi}_n$, we mean the set of m-ary trees of degree n. It is known that the cardinal of $\overset{m}{\Psi}_n$ is $\frac{(mn)!}{n!(n(m-1)+1)!}$. Each m-ary tree can be decomposed in a unique way *via* the so called grafting operation defined for all $p_1, \ldots, p_m \in \mathbb{N}$ as follows,

$$\vee : \overset{m}{\Psi}_{p_1} \times \ldots \times \overset{m}{\Psi}_{p_m} \longrightarrow \overset{m}{\Psi}_{\sum_{i=1}^m p_i + 1},$$

$$(t_1, \ldots, t_m) \longmapsto t_1 \vee \ldots \vee t_m,$$

the last symbol meaning that the roots of t_i are glued together and that a new root is created. The tree $\overset{m}{\Psi}$ represents by definition the tree $| \vee \ldots \vee |$ m-times and is called a m-corolla. For instance, $\overset{3}{\Psi} := \Psi$, $\overset{4}{\Psi} := \Psi$, and so forth. There exists also an involution still denoted by $\dagger : \overset{m}{\Psi}_n \mapsto \overset{m}{\Psi}_n$ for all n, defined recursively by $t^\dagger = t_m^\dagger \vee \ldots \vee t_1^\dagger$ if $t = t_1 \vee \ldots \vee t_m$. Pictorially, this is the mirror image *via* the central axis of the tree. Over the K-vector space $K[\overset{m}{\Psi}_\infty] := K \mid \oplus \widehat{K[\overset{m}{\Psi}_\infty]}$, where $\widehat{K[\overset{m}{\Psi}_\infty]} := \bigoplus_{n>0} K\overset{m}{\Psi}_n$, we introduce recursively the following binary operations first on the trees. They are naturally extended by bilinearity to the whole $\widehat{K[\overset{m}{\Psi}_\infty]}$. Let $p, q > 0$ and set for any $t = t_1 \vee \ldots \vee t_m \in \overset{m}{\Psi}_p$ and $r = r_1 \vee \ldots \vee r_m \in \overset{m}{\Psi}_q$,

$$\begin{aligned} t \succ r &= (t \star r_1) \vee \ldots \vee r_m, & (3) \\ t \prec r &= t_1 \vee \ldots \vee (t_m \star r), & (4) \\ t \bullet_2 r &= r_1 \vee (r_2 \vee \ldots \vee r_m \star t_1 \vee t_2) \vee \ldots \vee t_m, & (5) \\ t \bullet_3 r &= r_1 \vee r_2 \vee (r_3 \vee \ldots \vee r_m \star t_1 \vee t_2 \vee t_3) \vee \ldots \vee t_m, & (6) \\ \vdots &= \vdots \qquad \vdots & (7) \\ t \bullet_{m-1} r &= r_1 \vee \ldots \vee (r_{m-1} \vee r_m \star t_1 \vee \ldots \vee t_{m-1}) \vee t_m, & (8) \\ | \prec t &= 0 = t \succ |, \qquad t \prec | = t = | \succ t, & (9) \\ | \bullet_i t &= 0 = t \bullet_i |, & (10) \end{aligned}$$

where as usual $x \star y = x \prec y + x \succ y$ is by construction associative. The symbols $1 \prec 1$, $1 \succ 1$ and $1 \bullet_i 1$ are not defined. Observe that $| \star t = t = t \star |$ and that our three operations respect the natural grading of $\widehat{K[\overset{m}{\Psi}_\infty]}$ since $\bullet_i, \succ, \prec: K\overset{m}{\Psi}_p \otimes K\overset{m}{\Psi}_q \longrightarrow K\overset{m}{\Psi}_{p+q}$. As expected, we get the following results whose proofs are similar of those described for $m = 3$, see next section.

Theorem 4.2. *Fix an integer $m \geq 2$.*

- *Equipped with these m binary operations, $\widehat{K[\overset{m}{\Upsilon}_\infty]}$ is the free m-dendriform algebra generated by $\overset{m}{\Upsilon}$ which is involutive when equipped with the involution $\dagger$.*
- *There exists a structure of involutive and connected $\mathcal{P}$-Hopf algebra on $K[\overset{m}{\Upsilon}_\infty]$.*
- *Let V be a K-vector space. The following operations turn $\bigoplus_{n>0} K[\overset{m}{\Upsilon}_n] \otimes V^{\otimes n}$ into a m-dendriform algebra:*

$$\begin{aligned} t \otimes \omega \prec t' \otimes \omega' &:= t \prec t' \otimes \omega\omega', \\ t \otimes \omega \succ t' \otimes \omega' &:= t \succ t' \otimes \omega\omega', \\ t \otimes \omega \bullet_i t' \otimes \omega' &:= t \bullet_i t' \otimes \omega\omega'. \end{aligned}$$

Therefore, the unique m-dendriform algebra map $m-Dend(V) \longrightarrow \bigoplus_{n>0} K[\overset{m}{\Upsilon}_n] \otimes V^{\otimes n}$, which sends the generator $v \in V$ to $\overset{m}{\Upsilon} \otimes v$ is an isomorphism.

4.2 The operad m-*Tetra*

To compute the dual in the sense of Ginzburg and Kapranov of $m - Dend$, we will need the definition of the tetrahedral numbers of dimension k, where here $k := m-1$. The n^{th} tetrahedral number of dimension k is by definition the number $t^{[n]}_{[k]} := \frac{n(n+1)\dots(n+k-1)}{k!}$ (see N.J.A. Sloane Online Encyclopedia of Integers for instance). For instance, for $k=3$, we got the triangular numbers $1, 3, 6, 10, 15, \dots$. Instead of dimension 2 (triangles are drawn in a plane), consider dimension 3. Triangles become tetraedrons, hence the tetrahedral numbers. They are $1, 4, 10, 20, \dots$ and they will be present in the definition of the Poincaré series of the operad *Tetra* defined below and so on. We now define these types of algebras.

Definition 4.3. A m-tetrahedral algebra T is a K-vector space equipped with m binary operations $\vdash, \dashv, \perp_2, \dots, \perp_{m-1}: T^{\otimes 2} \longrightarrow T$ verifying for all $x, y, z \in T$, for all $2 \leq i \leq m-1$, the following $\frac{m(3m-1)}{2}$ axioms.

$$\begin{cases} (x \dashv y) \dashv z = x \dashv (y \dashv z), & (x \dashv y) \perp_i z = x \perp_i (y \vdash z), \\ (x \dashv y) \dashv z = x \dashv (y \vdash z), & (x \vdash y) \perp_i z = x \vdash (y \perp_i z), \\ (x \vdash y) \dashv z = x \vdash (y \dashv z), & (x \perp_i y) \dashv z = x \perp_i (y \dashv z), \\ (x \dashv y) \vdash z = x \vdash (y \vdash z), & (x \perp_i y) \perp_i z = 0 = x \perp_i (y \perp_i z), \\ (x \vdash y) \vdash z = x \vdash (y \vdash z), & (x \perp_i y) \vdash z = 0 = x \dashv (y \perp_i z), \end{cases} \tag{11}$$

and also for all $2 \leq i < j \leq m-1$,

$$\begin{cases} (x \perp_i y) \perp_j z = x \perp_i (y \perp_j z), \\ (x \perp_j y) \perp_i z = 0 = x \perp_j (y \perp_i z). \end{cases} \tag{12}$$

Any associative algebra $(A, \cdot)$ is a m-tetrahedral algebra by setting $\vdash = \cdot = \dashv$ and $\perp_i = 0$ for all $2 \leq i \leq m-1$. Axioms above give birth to regular, binary and quadratic operads denoted by m-*Tetra* and categories denoted by m-**Tetra**. For convenience, we set 3-*Tetra*:= *Triang*, 3-**Tetra**:=**Triang** and 4-*Tetra*:=*Tetra* and 4-**Tetra**:=**Tetra**. The following functorial diagram holds:

$$\begin{array}{ccccccc} \mathbf{As} \to & \mathbf{Dias} \to & \mathbf{Triang} \to & \mathbf{Tetra} \to \ldots \\ \searrow & \downarrow & \swarrow & \swarrow \\ & \mathbf{Leib} & & \end{array}$$

Commutative m-tetrahedral algebras are left to the reader. For any m-tetrahedral algebra T, let $as(T)$ be the quotient of T by the ideal generated by the elements $x \vdash y - x \dashv y$ and $x \perp_i y$, for all $2 \leq i \leq m-1$ and $x, y \in T$. Observe that $as(T)$ is an associative algebra and that the functor $as.(-)$:m-**Tetra**$\rightarrow$ **As** is left adjoint to the functor inc : **As** $\rightarrow$ m-**Tetra**. As expected, we get the following result.

Theorem 4.4. *For each $m \geq 2$, the operad m-Tetra is dual in the sense of [3] to the operad m-Dend, that is m-Tetra=m-Dend$^!$ and m-Dend=m-Tetra$^!$.*

Proof: Straightforward. $\square$

Let V be a K-vector space. Let the linear map, $\Psi : \underbrace{T(V) \otimes \ldots \otimes T(V)}_{m-1 \ times} \longrightarrow K$, and the linear map, $\psi : \underbrace{T(V) \otimes \ldots \otimes T(V)}_{m-2 \ times} \longrightarrow K$, be the canonical projections.

Theorem 4.5. *Let V be a K-vector space. Consider the K-vector space,*

$$m - Tetra(V) := T(V) \otimes [V \otimes \underbrace{T(V) \otimes \ldots \otimes T(V)}_{m-2 \ times}] \otimes T(V).$$

Then, equipped with operations, $\perp_i$ $2 \leq i \leq m-1$, defined by,

$$\begin{array}{rcl} \omega_1 \otimes [v \otimes \omega_2 \otimes \ldots \otimes \omega_{m-1}] & \otimes & \omega_m \perp_2 \omega'_1 \otimes [v' \otimes \omega'_2 \otimes \ldots \otimes \omega'_{m-1}] \otimes \omega'_m \\ & = & \Psi(\omega_2, \ldots, \omega_{m-1}, \omega'_2)\omega_1 \otimes [v \otimes \omega_m\omega'_1 v' \otimes \omega'_3 \otimes \ldots \otimes \omega'_{m-1}] \otimes \omega'_m, \\ \omega_1 \otimes [v \otimes \omega_2 \otimes \ldots \otimes \omega_{m-1}] & \otimes & \omega_m \perp_3 \omega'_1 \otimes [v' \otimes \omega'_2 \otimes \ldots \otimes \omega'_{m-1}] \otimes \omega'_m \\ & = & \Psi(\omega_3, \ldots, \omega_{m-1}, \omega'_2, \omega'_3)\omega_1 \otimes [v \otimes \omega_2 \otimes \omega_m\omega'_1 v' \otimes \omega'_4 \otimes \ldots \otimes \omega'_{m-1}] \otimes \omega'_m, \\ \omega_1 \otimes [v \otimes \omega_2 \otimes \ldots \otimes \omega_{m-1}] & \otimes & \omega_m \perp_4 \omega'_1 \otimes [v' \otimes \omega'_2 \otimes \ldots \otimes \omega'_{m-1}] \otimes \omega'_m \\ & = & \Psi(\omega_4, \ldots, \omega_{m-1}, \omega'_2, \omega'_3, \omega'_4)\omega_1 \otimes [v \otimes \omega_2 \otimes \omega_3 \otimes \omega_m\omega'_1 v' \otimes \omega'_5 \otimes \ldots \otimes \omega'_{m-1}] \otimes \omega'_m \\ \vdots & = & \vdots \qquad \qquad \qquad \vdots \\ \omega_1 \otimes [v \otimes \omega_2 \otimes \ldots \otimes \omega_{m-1}] & \otimes & \omega_m \perp_{m-1} \omega'_1 \otimes [v' \otimes \omega'_2 \otimes \ldots \otimes \omega'_{m-1}] \otimes \omega'_m \\ & = & \Psi(\omega_{m-1}, \omega'_2, \ldots, \omega'_{m-1})\omega_1 \otimes [v \otimes \omega_2 \otimes \ldots \otimes \omega_{m-2} \otimes \omega_m\omega'_1 v'] \otimes \omega'_m, \end{array}$$

and,

$$\begin{array}{rcl} \omega_1 \otimes [v \otimes \omega_2 \otimes \ldots \otimes \omega_{m-1}] \otimes \omega_m & \dashv & \omega'_1 \otimes [v' \otimes \omega'_2 \otimes \ldots \otimes \omega'_{m-1}] \otimes \omega'_m \\ & = & \psi(\omega'_2, \ldots, \omega'_{m-1})\omega_1 \otimes [v \otimes \omega_2 \otimes \ldots \otimes \omega_{m-1}] \otimes \omega_m\omega'_1 v'\omega'_m, \\ \omega_1 \otimes [v \otimes \omega_2 \otimes \ldots \otimes \omega_{m-1}] \otimes \omega_m & \vdash & \omega'_1 \otimes [v' \otimes \omega'_2 \otimes \ldots \otimes \omega'_{m-1}] \otimes \omega'_m \\ & = & \psi(\omega_2, \ldots, \omega_{m-1})\omega_1 v\omega_m\omega'_1 \otimes [v' \otimes \omega'_2 \otimes \ldots \otimes \omega'_{m-1}] \otimes \omega'_m. \end{array}$$

$m-Tetra(V)$ is the free m-tetrahedral algebra on V. Consequently, the Poincaré series associated with the operad $m-Tetra$ is,

$$f_{m-tetra}(x) := \sum_{n\geq 1}(-1)^n t^{[n]}_{[m-1]} x^n = \frac{-x}{(1+x)^m}.$$

Remark: Observe that the brackets $[\ldots]$ can be dropped and are here only to recall the symmetry in the definitions of operations $\vdash$ and $\dashv$.

Proof: The proof will be only sketched to avoid tedious computations. Therefore, checking axioms of m-tetrahedral algebras is left to the reader. The map $i : V \hookrightarrow m-Tetra(V)$ is the composite:

$$V \simeq K \otimes [V \otimes K \otimes \ldots \otimes K] \otimes K \hookrightarrow \quad m-Tetra(V).$$

Let $f : V \longrightarrow T$ be a linear map, where T is a m-tetrahedral algebra. We construct its universal extension ϕ as follows. First on monomials from m-*Tetra*(V), then extended by K-linearity. Therefore, define $\phi : m\text{-}Tetra(V) \longrightarrow T$ by:

$$\begin{array}{lll} \phi(X) := f(v_{-p}) \vdash \ldots \vdash f(v_{-1}) \vdash [f(v_0) & \perp_2 & (f({}^2w_1) \vdash \ldots \vdash f({}^2w_{k_2})) \\ & \perp_3 & (f({}^3w_1) \vdash \ldots \vdash f({}^3w_{k_3})) \\ & \vdots & \\ & \perp_{m-1} & (f({}^{(m-1)}w_1) \vdash \ldots \vdash f({}^{(m-1)}w_{k_{m-1}}))] \dashv f(v_1) \dashv \ldots \dashv f(v_q), \end{array}$$

if,

$$\begin{array}{lll} X := v_{-p} \vdash \ldots \vdash v_{-1} \vdash [v_0 & \perp_2 & ({}^2w_1 \vdash \ldots \vdash^2 w_{k_2}) \\ & \perp_3 & ({}^3w_1 \vdash \ldots \vdash^3 w_{k_3}) \\ & \vdots & \\ & \perp_{m-1} & ({}^{(m-1)}w_1 \vdash \ldots \vdash^{(m-1)} w_{k_{m-1}})] \dashv v_1 \dashv \ldots \dashv v_q, \end{array}$$

where the v_j and the ${}^iw_j$ belong to V and where it is understood that the non-existence of a line $\perp_i \ldots$, $2 \leq i \leq m-1$, in the definition of X entails the vanishing of the line $\perp_i \ldots$ in the construction of ϕ. According to dimonoid calculus rules [6] and the group of equations (12) of Definition 4.3, the writings at the right hand side of the equalities do have a meaning. The proof that ϕ is a morphism of m-tetrahedral algebras is tedious but not difficult and is left to the reader. The unicity of such a morphism is now straightforward since it has to coincide on V with f. The last claim concerning the Poincaré series associated with the operad $m-Tetra$ is easy to compute. □

Remark: Observe that tetrahedral numbers in dimension $m-1$ can be reconstructed from m operations and 1 played here by $1 \otimes [v \otimes 1 \otimes \ldots \otimes 1] \otimes 1$ if we consider a vector space V spanned by say v, *i.e.*, $V := Kv$.

4.3 Relations with homogeneous polynomials

Set $\chi := 1 \otimes [v \otimes 1 \otimes \ldots \otimes 1] \otimes 1$. Let $\mathbb{P}^n Hmg[X_0, X_1, \ldots, X_{m-1}]$ be the K-vector space of homogeneous polynomials of degree $n-1$ over the commutative indeterminates $X_0, X_1, \ldots, X_{m-1}$.

Observe that as a K-vector, the dimension of $\mathbb{P}^n Hmg[X_0, X_1, \ldots, X_{m-1}]$ is $t^{[n]}_{[m-1]}$. Consider the bijection,

$$v^{\otimes p_1} \otimes [v \otimes v^{\otimes p_2} \otimes \ldots \otimes v^{\otimes p_m}] \otimes 1 \longmapsto \quad X_0^{p_1} X_1^{p_2} \ldots X_{m-1}^{p_m}.$$

Set $\mathbb{P}^\infty Hmg[X_0, X_1, \ldots, X_{m-1}] := \bigoplus_{n=2}^{\infty} \mathbb{P}^n Hmg[X_0, X_1, \ldots, X_{m-1}]$. This K-vector space inherits a m-tetrahedral algebra structure *via* the bijection constructed above. Still denote by χ the image of χ under this map. We get that, $K\chi \oplus \mathbb{P}^\infty Hmg[X_0, X_1, \ldots, X_{m-1}]$, is the free m-tetrahedral algebra generated by χ. Consequently, following [7], $\mathbb{P}^\infty Hmg[X_0, X_1, \ldots, X_{m-1}]$ inherits an operadic arithmetic, that is, a K-left-linear map (called usually the multiplication),

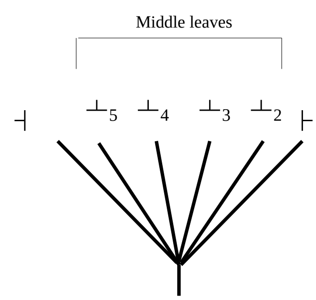
$$\circledast : \mathbb{P}^\infty Hmg[X_0, X_1, \ldots, X_{m-1}]^{\otimes 2} \to \mathbb{P}^\infty Hmg[X_0, X_1, \ldots, X_{m-1}],$$

distributive to the left as regards operations $\vdash$, $\dashv$, $\perp_i$ and consisting to replace the generator χ in the code describing the left object by the right one. For instance, $((\chi \perp_i \chi) \dashv \chi) \circledast z := (z \perp_i z) \dashv z$, where z is a homogeneous polynomial of degree say d. The combinatorial object underlying the free m-tetrahedral algebra on one generator is no longer linear combinations of planar m-ary trees but homogeneous polynomials over m commutative indeterminates, thus related to projective algebraic hypersurfaces in the projective space $\mathbb{P}^{m-1}(K)$.

4.4 (Co)Homology of m-tetrahedral algebras

The aim of this subsection is to propose an homology of m-tetrahedral algebras. We proceed as follows.

For any $t \in \overset{m}{\Psi}_n$, label its leaves from left to right starting from 0 to $(m-1)n$. Start with the leave 0, and begin to count from the place you reached. Every m leaves, place the operation $\perp_{m+1-i}$, $2 \leq i \leq m-1$, (resp. $\vdash$), (resp. $\dashv$), if the leave points in the same direction that the $(i-1)^{th}$ (resp. the $(m-1)^{th}$) (resp. 0^{th}) leave of $\overset{m}{\Psi}$. Here is an example of how the operations assignement depends on the leaves of here the corolla $\overset{6}{\Psi}$.



By convention the last leave of a given m-tree remains unassigned. Proceeding that way, a tree from $\overset{m}{\Psi}_{n+1}$ will give m binary operations, we label 1 to n from left to right. We have defined for $1 \leq j \leq n$, a map,

$$\circ_j : \overset{m}{\Psi}_{n+1} \longrightarrow \{\perp_i, \vdash, \dashv\},$$

assigning to each j the corresponding operation by the process just described. The image will be denoted by $\circ_j^t$. Denote by $\widetilde{\overset{m}{\Psi}}_n$ the set of trees of $\overset{m}{\Psi}_n$ whose leaves have been colored by the operations $\perp_i, \vdash, \dashv$ just as explained. We get a bijection $tilde : \overset{m}{\Upsilon}_n \longrightarrow \widetilde{\overset{m}{\Upsilon}}_n$. For any $1 \leq j \leq n$, define the face map, $d_j : K\overset{m}{\Upsilon}_n \longrightarrow K\overset{m}{\Upsilon}_{n-1}$, on $\overset{m}{\Upsilon}_n$ first and extended by linearity then to be the composite $tilde^{-1} \ \circ del_j \ \circ \ tilde$, where the linear map,

$$del_j : K\widetilde{\overset{m}{\Upsilon}}_n \longrightarrow K\widetilde{\overset{m}{\Upsilon}}_{n-1},$$

assigns to a tree t, the tree t' obtained from t as follows. Localise the leave colored by the operation labelled by j. Remove the offspring of its father vertex if the m children are leaves, or if all middle leaves are not colored. Otherwise, the result is zero. Let T be a m-tetrahedral algebra over K. Define the module of n-chains by $C\overset{m}{\Upsilon}_n(T) := K[\overset{m}{\Upsilon}_n] \otimes T^{\otimes n}$. Define a linear map $d : C\overset{m}{\Upsilon}_n(T) \longrightarrow C\overset{m}{\Upsilon}_{n-1}(T)$ by the following formula,

$$d(t; v_1, \ldots, v_n) := \sum_{j=1}^{n-1} (-1)^{j+1}(d_j(t); v_1, \ldots, v_{j-1}, v_j \circ_j^t v_{j+1}, \ldots, v_n) := \sum_{j=1}^{n-1} (-1)^{j+1} d_j(t; v_1, \ldots, v_n),$$

with a slight abuse of notation, and where $t \in \overset{m}{\Upsilon}_n$, $v_j \in T$.

Proposition 4.6. *The face maps $d_l : C\overset{m}{\Upsilon}_n(T) \longrightarrow C\overset{m}{\Upsilon}_{n-1}(T)$ satisfy the simplicial relations $d_k d_l = d_{l-1} d_k$ for any $1 \leq k < l \leq n-1$. Therefore $d \circ d = 0$ and so $(C\overset{m}{\Upsilon}_*(T), d)$ is a chain-complex.*

Proof: We give a sketch of this idendity for the lowest dimension, that is,

$$d_1 d_2 = d_1 d_1 : C\overset{m}{\Upsilon}_3(T) \to C\overset{m}{\Upsilon}_2(T).$$

Consider the 5 planar rooted trees on three internal vertices. Remove each offspring of two children by m-children. We obtain 5 m-trees of $\overset{m}{\Upsilon}_3$ which will give dialgebra constraints. Consider now the family of m-trees of the form,

$$((|\vee\overset{m}{\Upsilon}\vee\ldots\vee|)\vee|\vee\ldots\vee|),\ ((|\vee|\vee\overset{m}{\Upsilon}\vee\ldots\vee|)\vee|\vee\ldots\vee|), \ldots, ((|\vee\ldots\vee|\vee\overset{m}{\Upsilon}\vee|)\vee|\vee\ldots\vee|).$$

They will give equalities $(x \perp_{m+1-i} y) \vdash z = 0$ for $i \geq 2$. Apply formally the involution $\dagger$ (compatible with the opposite structure) on these trees to obtain the equalities $x \dashv (y \perp_i z) = 0$. Consider now the family of m-trees of the form,

$$(|\vee(\overset{m}{\Upsilon}\vee\ldots\vee|)\vee|\vee\ldots\vee|),\ (|\vee(|\vee\overset{m}{\Upsilon}\vee\ldots\vee|)\vee|\vee\ldots\vee|), \ldots, (|\vee(|\vee\ldots\vee|\vee\overset{m}{\Upsilon})\vee|\vee\ldots\vee|).$$

We obtain equations $(x \perp_i y) \perp_2 z = 0$, for $i \geq 2$ for the first $m-2$ trees. The last ones gives $x \perp_2 (y \perp_2 z) = 0$. The $(m-1)^{th}$ one gives $(x \dashv y) \perp_2 z = x \perp_2 (y \vdash z)$. Applying the involution $\dagger$ will give the constraints $x \perp_{m-1} (y \perp_{m+1-i} z) = 0$, $x \perp_{m-1} (y \perp_{m-1} z) = 0$ and $(x \dashv y) \perp_{m-1} z = x \perp_{m-1} (y \vdash z)$. Similarly the family,

$$(|\vee|\vee(\overset{m}{\Upsilon}\vee\ldots\vee|)\vee|\vee\ldots\vee|),\ (|\vee|\vee(|\vee\overset{m}{\Upsilon}\vee\ldots\vee|)\vee|\vee\ldots\vee|), \ldots,$$

$$\ldots(|\vee|\vee(|\vee\ldots\vee|\vee\overset{m}{\Upsilon})\vee|\vee\ldots\vee|),$$

will give equations $(x \perp_i y) \perp_3 z = 0$, for $i \geq 3$ for the first $m-3$ trees. The two last one give $x \perp_3 (y \perp_2 z) = 0$ and $x \perp_3 (y \perp_3 z) = 0$. The $(m-2)^{th}$ one gives $(x \dashv y) \perp_3 z = x \perp_3 (y \vdash z)$ and so forth. Observe now that the family,

$$(| \vee \ldots | \vee \overset{m}{\Upsilon} \vee | \vee \ldots | \vee \overset{m}{\Upsilon} \vee | \vee \ldots \vee |),$$

will give the constraints $(x \perp_i y) \perp_j z = x \perp_i (y \perp_j z)$ for $2 \leq i < j \leq m-1$ and also $(x \perp_i y) \dashv z = x \perp_i (y \dashv z)$ and $(x \vdash y) \perp_i z = x \vdash (y \perp_i z)$ for $2 \leq i \leq m-1$ as expected. The general case still splits into two cases. If $j = i+1$, then the proof follows from the low dimension cases and from axioms of m-tetrahedral algebras. The case $j > i+1$ is straightforward. $\square$

We get a chain-complex,

$$C\overset{m}{\Upsilon}_*(T): \quad \ldots \rightarrow K[\overset{m}{\Upsilon}_n] \otimes T^{\otimes n} \rightarrow \ldots \rightarrow K[\overset{m}{\Upsilon}_3] \otimes T^{\otimes n} \rightarrow K[\overset{m}{\Upsilon}_2] \otimes T^{\otimes n} \xrightarrow{\perp_i, \vdash, \dashv} T.$$

This allows us to define the homology of a m-tetrahedral algebra T as the homology of the chain-complex $C\overset{m}{\Upsilon}_*(T)$, that is $H\overset{m}{\Upsilon}_n(T) := H_n(C\overset{m}{\Upsilon}_*(T), d)$, $n > 0$. The cohomology of T is by definition $H\overset{m}{\Upsilon}^n(T) := H^n(Hom(C\overset{m}{\Upsilon}_*(T), K))$, $n > 0$. For the free m-tetrahedral algebra over the K-vector space V, we get $H\overset{m}{\Upsilon}_1(m - Tetra(V)) \simeq V$. We conjecture that for $n > 1$, $H\overset{m}{\Upsilon}_n(m - Tetra(V)) = 0$, that is the operad $m - Tetra$ is Koszul.

4.5 The Pascal triangle

We now summarise all the integer sequences we got by gathering them inside the Pascal triangle. Here is the beginning of this famous triangle. We have chosen the south-east direction (the south-west could be another one) and indicate by an arrow the begining of the coefficients of Poincaré series of the operads involved in this paper and the functors associated with the corresponding categories.

$$\begin{array}{cccccccccccccccc}
 & & & & & & & & 1 & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
 & \swarrow & & \searrow & & \searrow & & \searrow & & \searrow & & \searrow & & \searrow & & \searrow \\
As & \overset{as.}{\leftarrow} & \ldots \hookleftarrow & 7-Tetra & \hookleftarrow & 6-Tetra & \hookleftarrow & 5-Tetra & \hookleftarrow & Tetra & \hookleftarrow & Triang & \hookleftarrow & Dias & \hookleftarrow & As
\end{array}$$

Similarly, one could also construct a triangle dual to Pascal's by representing the generalised Catalan numbers, for instance, as follows. Recall that the operad of associative algebras is self-dual.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & 5 & 3 & & 1 & & & \\
 & & 1 & & 4 & 12 & 14 & 12 & 4 & & 1 & & \\
1 & & & 5 & 22 & 55 & 42 & 55 & 22 & 5 & & 1 & \\
\swarrow & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & & \searrow \\
As & \overset{\pm}{\leftarrow} & \ldots & 5-Dend \hookleftarrow & 4-Dend \hookleftarrow & 3-Dend \hookleftarrow & 2-Dend & \hookrightarrow 3-Dend & \hookrightarrow 4-Dend & \hookrightarrow 5-Dend & \ldots & \overset{\pm}{\rightarrow} & As
\end{array}$$

5 An operad over rooted planar ternary trees and triangular numbers

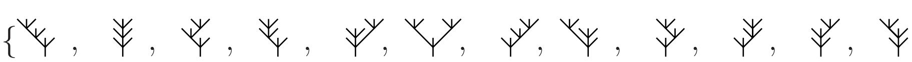
For $m = 3$, one can also propose an homology of 3-dendriform algebras. Before entering this subject, we propose other results and prove some theorems of the previous section in this context.

5.1 A focus on 3-*Dend*

Recall a K-vector space T is a 3-dendriform algebra if it is equipped with 3 binary operations $\prec, \succ, \cdot : T^{\otimes 2} \longrightarrow T$ verifying for all $x, y, z \in T$,

1. $(x \prec y) \prec z = x \prec (y \star z)$, 4. $(x \prec y) \bullet z = x \bullet (y \succ z)$
2. $(x \succ y) \prec z = x \succ (y \prec z)$, 5. $(x \succ y) \bullet z = x \succ (y \bullet z)$
3. $(x \star y) \succ z = x \succ (y \succ z)$, 6. $(x \bullet y) \prec z = x \bullet (y \prec z)$,

where $x \star y := x \prec y + x \succ y$.
Remark: Pay attention, this structure is different from tridendriform axioms obtained in [8] but is similar to the one obtained by F. Chapoton, the difference lying in an extra axiom, *i.e.,* the associativity of $\bullet$ required in [1].
For more information about ternary trees, we refer to the extended literature. In small dimensions, one gets: $\Psi_0 := \{|\}, \Psi_1 := \{\Psi\}$, $\Psi_2 := \{$  $\}$,

$\Psi_3 := \{$  $\}$

and so on, where Ψ_n denotes the set of ternary trees of degree n. Recall each tree of Ψ_n, $n > 0$, can be decomposed in a unique way *via* the so called grafting operation defined for all $p, q, r \in \mathbb{N}$ as follows,

$$\vee : \Psi_p \times \Psi_q \times \Psi_r \longrightarrow \Psi_{p+q+r+1},$$
$$(t_1, t_2, t_3) \longmapsto t_1 \vee t_2 \vee t_3.$$

Over the K-vector space $K[\Psi_\infty] := K \mid \oplus K[\hat{\Psi}_\infty]$, where $K[\hat{\Psi}_\infty] := \bigoplus_{n>0} K\Psi_n$, recall the recursive definitions we wrote down in Subsection 4.1.

Proposition 5.1. *Equipped with these three binary operations, $K[\hat{\Psi}_\infty]$ is an involutive 3-dendriform algebra generated by Ψ.*

Proof: We proceed by induction on the degree of trees. Observe that the involution $\dagger$ on Ψ_2 acts as expected since $(\Psi \prec \Psi)^\dagger = \Psi \succ \Psi$ and so on. By induction, one checks that $\dagger$ is an involutive map. For instance, if $r = r_1 \vee r_2 \vee r_3$ and t are trees, then $(r \prec t)^\dagger = (r_1 \vee r_2 \vee (r_3 \star t))^\dagger = (r_3 \star t)^\dagger \vee r_2^\dagger \vee r_1^\dagger$ by definition and $(r_3 \star t)^\dagger \vee r_2^\dagger \vee r_1^\dagger = (t^\dagger \star r_3^\dagger) \vee r_2^\dagger \vee r_1^\dagger$ by induction. Therefore, $(r \prec t)^\dagger = r^\dagger \succ t^\dagger$. Let us check the 6 axioms. Let r, s, t be ternary trees. We get: $(r \prec s) \prec t = r_1 \vee r_2 \vee (r_3 \star s) \star t = r_1 \vee r_2 \vee r_3 \star (s \star t)$ by induction, therefore Axiom 1 holds and Axiom 3 as well *via* involution on Axiom 1. Axiom 2 is straightforward. Axiom 4 leads to: $(r \prec s) \bullet t = r_1 \vee r_2 \vee (r_3 \star s) \bullet t = r_1 \vee (r_2 \vee ((r_3 \star s) \star t_1) \vee t_2) \vee t_3 = r_1 \vee (r_2 \vee (r_3 \star (s \star t_1)) \vee t_2) \vee t_3$

by induction, hence Axiom 4 holds. Axiom 5 and 6 are straightforward. Therefore, $K[\hat{\Psi}_\infty]$ is an involutive 3-dendriform algebra. We introduce the middle map $m : \Psi_p \longrightarrow \Psi_{p+1}$ for all $p \in \mathbb{N}$, such that $t \mapsto | \vee t \vee |$. We now prove that $K[\hat{\Psi}_\infty]$ is generated by Ψ by induction on the degree of trees. Indeed, the result holds in small dimension (checked by hand up to $p = 3$). Moreover we have for a tree t,

$$\begin{aligned} t := t_1 \vee t_2 \vee t_3 &= \Psi \quad \text{if } \ t_1 = t_2 = t_3 = |, \\ &= \Psi \prec t_3 \ \text{ if } \ t_1 = t_2 = |, \ t_3 \neq |, \\ &= t_1 \succ \Psi \ \text{ if } \ t_1 \neq |, \ t_2 = | = t_3, \\ &= t_1 \succ (m((t_2)_1) \bullet ((t_2)_2 \succ m((t_2)_3))) \prec t_3 \ \text{ otherwise.} \end{aligned}$$

□

Proposition 5.2. *The unique 3-dendriform algebra map* 3-*Dend*$(K) \longrightarrow K[\hat{\Psi}_\infty] := \bigoplus_{n>0} K[\Psi_n]$ *sending the generator x of* 3-*Dend*(K) *to Ψ is an isomorphism.*

Proof: We will check that $(K[\hat{\Psi}_\infty], \prec, \succ, \bullet)$ verifies the universal condition to be the free 3-dendriform algebra on one generator. Let T be a 3-dendriform algebra and let $a \in T$. By induction, we construct a linear map $\alpha : K[\hat{\Psi}_\infty] \longrightarrow T$ on its values on ternary trees as follows. Let $t = t_1 \vee t_2 \vee t_3 \in \Psi_p$ and set:

$$\begin{aligned} \alpha(t_1 \vee t_2 \vee t_3) &= a \quad \text{if } \ t_1 = t_2 = t_3 = |, \\ &= a \prec \alpha(t_3) \ \text{ if } \ t_1 = t_2 = |, \ t_3 \neq |, \\ &= \alpha(t_1) \succ a \ \text{ if } \ t_1 \neq |, \ t_2 = | = t_3, \\ &= \alpha(t_1) \succ (\alpha(m((t_2)_1)) \bullet (\alpha((t_2)_2) \succ \alpha(m((t_2)_3)))) \prec \alpha(t_3) \ \text{ otherwise.} \end{aligned}$$

The map α is unique since $K[\hat{\Psi}_\infty]$ is generated by Ψ and that $\alpha(\Psi) = a$. It is a morphism of 3-dendriform algebras as one can show by induction on the degree of trees. For instance,

$$\begin{aligned} \alpha(r \prec t) &:= \alpha(r_1) \succ (\alpha(m((r_2)_1)) \bullet (\alpha((r_2)_2) \succ \alpha(m((r_2)_3)))) \prec \alpha(r_3 \star t) \\ &= \alpha(r_1) \succ (\alpha(m((r_2)_1)) \bullet (\alpha((r_2)_2) \succ \alpha(m((r_2)_3)))) \prec (\alpha(r_3) \star \alpha(t)) \ \text{ by induction} \\ &= (\alpha(r_1) \succ (\alpha(m((r_2)_1)) \bullet (\alpha((r_2)_2) \succ \alpha(m((r_2)_3)))) \prec \alpha(r_3)) \prec \alpha(t) \ \text{ by Axiom 1} \\ &:= \alpha(r) \prec \alpha(t). \end{aligned}$$

The relation $\alpha(r \bullet t) = \alpha(r) \bullet \alpha(t)$ follows from the following general equality: Let $a, b, c, a', b', c' \in T$, where T is any 3-dendriform algebra. Then,

$$(a \succ b \prec c) \bullet (a' \succ b' \prec c') = a \succ (b \bullet ((c \star a') \succ b') \prec c',$$

holds. Therefore, $(K[\hat{\Psi}_\infty], \prec, \succ, \bullet)$ is the free dendriform algebra on one generator. □

Theorem 5.3 (Free 3-dendriform algebra)**.** *Let V be a K-vector space. The unique 3-dendriform algebra map* 3-*Dend*$(V) \longrightarrow \bigoplus_{n>0} K[\Psi_n] \otimes V^{\otimes n}$, *which sends the generator $v \in V$ to $\Psi \otimes v$ is an isomorphism.*

Proof: Define on $\bigoplus_{n>0} K[\curlyvee_n] \otimes V^{\otimes n}$ the following 3-dendriform algebra structure:

$$\begin{aligned} t \otimes \omega \prec t' \otimes \omega' &:= t \prec t' \otimes \omega\omega', \\ t \otimes \omega \succ t' \otimes \omega' &:= t \succ t' \otimes \omega\omega', \\ t \otimes \omega \bullet t' \otimes \omega' &:= t \bullet t' \otimes \omega\omega'. \end{aligned}$$

Since the relations defining 3-dendriform algebras are regular, the free 3-dendriform algebra over V is then determined by the free 3-dendriform algebra on one generator, hence 3-$Dend(V) := \bigoplus_{n>0}$ 3-$Dend(K)_n \otimes V^{\otimes n} = \bigoplus_{n>0} K[\curlyvee_n] \otimes V^{\otimes n}$ by Proposition 5.2. $\square$

5.2 (Co)Homology of 3-dendriform algebras

We now show the existence of a chain complex of Hochschild type for any 3-dendriform algebra and define a (co)homology theory for this category. Let T be a 3-dendriform algebra. For all $n \in \mathbb{N}$, define $X_n := \{(k,j); 0 \leq k \leq j \leq n\}$ and the module of n-chains of T as $C_n^{3-Dend}(T) := K(X_n) \otimes T^{\otimes n}$. Introduce the differential operator $d := \sum_{1 \leq i \leq n-1} (-1)^{i+1} d_i$, where $d_i : C_n^{3-Dend}(T) \longrightarrow C_{n-1}^{3-Dend}(T)$, $1 \leq i \leq n-1$, are the face operators and act on X_n as follows:
Step 1: Set $d_i(k,j) := (\tilde{d}_i(k), \tilde{d}_i(j))$ where $\tilde{d}_i : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n-1\}$ is such that $\tilde{d}_i(r) := r-1$ if $i \leq r$ and $\tilde{d}_i(r) := r$ if $i \geq r+1$. The face maps are extended linearly to maps $d_i : K[X_n] \longrightarrow K[X_{n-1}]$.
Step 2: Introduce now the symbol:

$$\circ_i^{(k,j)} := \begin{cases} \bullet & if \quad i-1 \in \{k,j\};\ i \in \{k,j\}, \\ \succ & if \quad i-1 \notin \{k,j\};\ i \in \{k,j\}, \\ \prec & if \quad i-1 \in \{k,j\};\ i \notin \{k,j\}, \\ \star & if \quad i-1 \notin \{k,j\};\ i \notin \{k,j\}. \end{cases}$$

We now explicit the action of the face maps $d_i : C_n^{3-Dend}(T) \longrightarrow C_{n-1}^{3-Dend}(T)$, $1 \leq i \leq n-1$ by $d_i((k,j); x_1 \otimes \ldots \otimes x_n) := (d_i(k,j); x_1 \otimes \ldots \otimes x_{i-1} \otimes x_i \circ_i^{(k,j)} x_{i+1} \otimes \ldots \otimes x_n)$.

Proposition 5.4. *The face maps d_i, $1 \leq i \leq n-1$ satisfy the simplicial relations $d_i d_j = d_{j-1} d_i$ for $i < j$. Therefore $(C_*^{3-Dend}(T), d)$ is a chain-complex.*

Proof: We prove first that $d_1 d_2 = d_1 d_1$ on $C_3^{3-Dend}(T)$. Let $x, y, z \in T$.

$$\begin{cases} d_1 d_2((0,0); x \otimes y \otimes z) = d_1((0,0); x \otimes y \star z) = ((0,0); x \prec (y \star z)), \\ d_1 d_1((0,0); x \otimes y \otimes z) = d_1((0,0); x \prec y \otimes z) = ((0,0); (x \prec y) \prec z), \end{cases}$$

hence the equality *via* Axiom 1.

$$\begin{cases} d_1 d_2((1,1); x \otimes y \otimes z) = d_1((1,1); x \otimes y \prec z) = ((0,0); x \succ (y \prec z)), \\ d_1 d_1((1,1); x \otimes y \otimes z) = d_1((0,0); x \succ y \otimes z) = ((0,0); (x \succ y) \prec z), \end{cases}$$

hence the equality *via* Axiom 2.

$$\begin{cases} d_1d_2((2,2); x\otimes y\otimes z) = d_1((1,1); x\otimes y \succ z) = ((0,0); x\succ (y\succ z)), \\ d_1d_1((2,2); x\otimes y\otimes z) = d_1((1,1); x\star y\otimes z) = ((0,0); (x\star y)\succ z), \end{cases}$$

hence the equality *via* Axiom 3.

$$\begin{cases} d_1d_2((0,2); x\otimes y\otimes z) = d_1((0,1); x\otimes y \succ z) = ((0,0); x\bullet (y\succ z)), \\ d_1d_1((0,2); x\otimes y\otimes z) = d_1((0,1); x\prec y\otimes z) = ((0,0); (x\prec y)\bullet z), \end{cases}$$

hence the equality *via* Axiom 4.

$$\begin{cases} d_1d_2((1,2); x\otimes y\otimes z) = d_1((1,1); x\otimes y \bullet z) = ((0,0); x\succ (y\bullet z)), \\ d_1d_1((1,2); x\otimes y\otimes z) = d_1((0,1); x\succ y\otimes z) = ((0,0); (x\succ y)\bullet z), \end{cases}$$

hence the equality *via* Axiom 5.

$$\begin{cases} d_1d_2((0,1); x\otimes y\otimes z) = d_1((0,1); x\otimes y \prec z) = ((0,0); x\bullet (y\prec z)), \\ d_1d_1((0,1); x\otimes y\otimes z) = d_1((0,0); x\bullet y\otimes z) = ((0,0); (x\bullet y)\prec z), \end{cases}$$

hence the equality *via* Axiom 6. The sequel of the proof splits into two cases. The case $j > i+1$ is straightforward and the case $j = i+1$ depends on the computations above and the fact that $\star$ is associative. □

5.3 The operad *Triang*

We obtained a (co)homology of 3-dendriform algebras by studying the dual, in the sense of Ginzburg and Kapranov, of the operad 3-*Dend*. Recall a triangular algebra T is a K-vector space equipped with three binary operations $\perp, \vdash, \dashv : T^{\otimes 2} \longrightarrow T$ verifying for all $x, y, z \in T$, the following 12 axioms.

$$\begin{cases} 1.\ (x\dashv y)\dashv z = x\dashv (y\dashv z), & 6.\ (x\dashv y)\perp z = x\perp (y\vdash z), \\ 2.\ (x\dashv y)\dashv z = x\dashv (y\vdash z), & 7.\ (x\vdash y)\perp z = x\vdash (y\perp z), \\ 3.\ (x\vdash y)\dashv z = x\vdash (y\dashv z), & 8.\ (x\perp y)\dashv z = x\perp (y\dashv z), \\ 4.\ (x\dashv y)\vdash z = x\vdash (y\vdash z), & 9.\ (x\perp y)\perp z = 0 \overset{10.}{=} x\perp (y\perp z), \\ 5.\ (x\vdash y)\vdash z = x\vdash (y\vdash z), & 11.\ (x\perp y)\vdash z = 0 \overset{12.}{=} x\dashv (y\perp z). \end{cases} \tag{13}$$

Example 5.5. Any associative algebra $(A, \cdot)$ is a triangular algebra by setting $\vdash = \cdot = \dashv$ and $\perp = 0$. In the non-graded setting, let (A, d) be a differential associative algebra, that is $d(ab) = d(a)b + ad(b)$ and $d^2 = 0$. Set $a\dashv b := ad(b)$, $a\vdash b := d(a)b$ and $a\perp b := d(a)d(b)$. Then, $(A, \vdash, \dashv, \perp)$ turns to be a triangular algebra.

Theorem 5.6. *The operad Triang is dual in the sense of [3] to the operad 3-Dend, that is Triang=3-Dend$^!$ and 3-Dend=Triang$^!$.*

Proof: We compute the dual of 3-*Dend*. Since our operads are regular, we know that the action of the symmetric group can be forgotten. We consider then only $\mathcal{P}_n$. The K-vector space generating operations is 3-$Dend_2 := K \prec \ \oplus K \succ \oplus K\bullet$. Set $OP := \{\prec, \succ, \bullet\}$. The K-vector space made out of three variables is $K[OP \times OP] \oplus K[OP \times OP]$. Its dimension is 18. The operad 3-*Dend.* is completely determined by some subspace $R \subset K[OP \times OP] \oplus K[OP \times OP]$. Denote by $(\circ_1)\circ_2$ (resp., $\circ_1(\circ_2)$), $\circ_i \in OP$, the basis vector of the first (resp., the second) summand $K[OP \times OP]$. Observe that R is spanned by 6 vectors of the form $(\circ_1) \circ_2 - \circ_1 (\circ_2)$ obtained from axioms of 3-dendriform algebras. Identify the dual of $K[OP]$ with itself by identifying a basis vector with its dual. According to [3], the dual operad 3-*Dend.* is then completely determined by $R^\perp \subset K[OP \times OP] \oplus K[OP \times OP]$, where $R^\perp$ is the orthogonal space of R under the quadratic form $\begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}$. Identify now $\prec$ with $\dashv$, $\succ$ with $\vdash$ and $\bullet$ with $\perp$. The K-vector space $R^\perp$ becomes the space $R^!$ spanned by the 12 vectors obtained from axioms of triangular algebras. For instance, consider the vector $(\dashv) \dashv - \dashv (\dashv)$ of $R^!$, identified with $(\prec) \prec - \prec (\prec)$ and observe that for instance $\langle\, (\prec) \prec - \prec (\prec); (\prec) \prec - \prec (\star)\, \rangle = 1 - 1 = 0$ and so on. $\square$

5.4 The free triangular algebra: Explicit proofs

Let V be a K-vector space. A tensor $v_1 \otimes \ldots \otimes v_p$ will be denoted sometimes, for commodity by $v_1, \ldots, v_p$ or by $v_1 \ldots v_p$, when no confusion is possible. The algebraic object to consider is $V \otimes T(V)^{\otimes 3}$. For esthetic reasons, we will work with an isomorphism copy written in an unusual way as:

$$\triangle(V) := \begin{array}{ccccc} & & T(V) & & \\ & & \otimes & & \\ T(V) & \otimes & V & \otimes & T(V) \end{array}$$

Let $\psi : T(V) \longrightarrow K$ and $\Psi : T(V)^{\otimes 2} \longrightarrow K$ be the canonical projections. Define now three binary operations $\vdash, \dashv, \perp$: $\triangle^{\otimes 2}(V) \longrightarrow \triangle(V)$ as follows,

$$\begin{array}{ccccc} & & H & & \\ & & \otimes & & \\ L & \otimes & v & \otimes & R \end{array} \vdash \begin{array}{ccccc} & & H' & & \\ & & \otimes & & \\ L' & \otimes & v' & \otimes & R' \end{array} = \begin{array}{ccccc} & & \psi(H)H' & & \\ & & \otimes & & \\ LvRL' & \otimes & v' & \otimes & R', \end{array}$$

$$\begin{array}{ccccc} & & H & & \\ & & \otimes & & \\ L & \otimes & v & \otimes & R \end{array} \dashv \begin{array}{ccccc} & & H' & & \\ & & \otimes & & \\ L' & \otimes & v' & \otimes & R' \end{array} = \begin{array}{ccccc} & & H\psi(H') & & \\ & & \otimes & & \\ L & \otimes & v & \otimes & RL'vR', \end{array}$$

and

$$\begin{array}{ccccc} & & H & & \\ & & \otimes & & \\ L & \otimes & v & \otimes & R \end{array} \perp \begin{array}{ccccc} & & H' & & \\ & & \otimes & & \\ L' & \otimes & v' & \otimes & R' \end{array} = \begin{array}{ccccc} & & \Psi(H,H')RL'v' & & \\ & & \otimes & & \\ L & \otimes & v & \otimes & R', \end{array}$$

for any $R \in V^{\otimes p_1}$, $L \in V^{\otimes p_2}$, $R \in V^{\otimes p_3}$, $R' \in V^{\otimes p_4}$, $L' \in V^{\otimes p_5}$, $H' \in V^{\otimes p_6}$ and any $v, v' \in V$ and extended by bilinearity then.

Theorem 5.7. *Let V be a K-vector space. The K-vector space $\triangle(V)$ equipped with the three operations just defined is the free triangular algebra on V.*

Proof: Checking axioms of triangular algebras for $\triangle(V)$ is left to the reader. The map $i : V \hookrightarrow \triangle(V)$ is the composite:

$$V \simeq \begin{array}{ccccc} & & K & & \\ & & \otimes & & \\ K & \otimes & V & \otimes & K \end{array} \hookrightarrow \quad \triangle(V).$$

Let $f : V \longrightarrow T$ be a linear map, where T is a triangular algebra. We construct ϕ as follows. First on monomials from $\triangle(V)$, second we extend it by K-linearity. Therefore, define its universal extension $\phi : \triangle(V) \longrightarrow T$ by:

$$\phi(X) := f(v_{-p}) \vdash \ldots \vdash f(v_{-1}) \vdash [f(v_0) \perp (f(w_1) \vdash \ldots \vdash f(w_k))] \dashv f(v_1) \dashv \ldots \dashv f(v_q),$$

if,

$$X := \begin{array}{ccccc} & & w_1 \otimes \ldots \otimes w_k & & \\ & & \otimes & & \\ v_{-p} \otimes \ldots \otimes v_{-1} & \otimes & v_0 & \otimes & v_1 \otimes \ldots \otimes v_q \end{array}$$

and obviously by,

$$\phi(X) := f(v_{-p}) \vdash \ldots \vdash f(v_{-1}) \vdash f(v_0) \dashv f(v_1) \dashv \ldots \dashv f(v_q),$$

if,

$$X := \begin{array}{ccccc} & & 1 & & \\ & & \otimes & & \\ v_{-p} \otimes \ldots \otimes v_{-1} & \otimes & v_0 & \otimes & v_1 \otimes \ldots \otimes v_q. \end{array}$$

According to dimonoid calculus rules [6], the writings at the right hand sides do have a meaning. Observe that the bracket $[\ldots]$ can be dropped and is just used here to recall the symmetry shape of $\triangle(V)$. We prove now, that so defined, ϕ is a morphism of triangular algebras. Let us start with the binary operation $\perp$. We replace, with a slight abuse of notation, tensors by capital letters so as to ease proofs. On the one hand,

$$\begin{aligned} A \quad &:= \quad \phi\left(\begin{array}{ccccc} & & H & & \\ & & \otimes & & \\ L & \otimes & v & \otimes & R \end{array}\right) \perp \phi\left(\begin{array}{ccccc} & & H' & & \\ & & \otimes & & \\ L' & \otimes & v' & \otimes & R' \end{array}\right) := \\ &= \quad (f(L) \vdash [f(v) \perp f(H)] \dashv f(R)) \perp (f(L') \vdash [f(v') \perp f(H')] \dashv f(R')). \end{aligned}$$

Set $z := f(L') \vdash [f(v') \perp f(H')] \dashv f(R')$, $y = [f(v) \perp f(H)] \dashv f(R)$, we get $A = (f(L) \vdash y) \perp z = f(L) \vdash (y \perp z)$ *via* Axiom 7. However, $y \perp z = ([f(v) \perp f(H)] \dashv f(R)) \perp z = (f(v) \perp [f(H) \dashv f(R)]) \perp z = 0$ *via* Axiom 8 first and Axiom 9 then. The case $H = 1$ and $H' \neq 1$ give the same result and is left to the reader (apply Axioms 8, 7 and 10). If $H = H' = 1$, then we get, $A := (f(L) \vdash f(v) \dashv f(R)) \perp (f(L') \vdash f(v') \dashv f(R'))$. Set $z := f(L') \vdash f(v') \dashv f(R')$, $y = f(v) \dashv f(R)$. Then, by applying Axiom 7 we get $(f(L) \vdash$

$y) \perp z = f(L) \vdash (y \perp z)$. However, $y \perp z := (f(v) \dashv f(R)) \perp z = f(v) \perp (f(R) \vdash z)$ by Axiom 6. Moreover, $f(R) \vdash z := f(R) \vdash ((f(L') \vdash f(v')) \dashv f(R')) = (f(R) \vdash (f(L') \vdash f(v'))) \dashv f(R')$ *via* Axiom 3, which is equal to $(f(R) \vdash f(L') \vdash f(v')) \dashv f(R')$ *via* Axiom 5. Set $x := f(R) \vdash f(L') \vdash f(v')$, we have $f(v) \perp (x \dashv f(R')) = (f(v) \perp x) \dashv f(R')$ *via* Axiom 8. Summarising our computations, we find,

$$A := f(L) \vdash (f(v) \perp (f(R) \vdash f(L') \vdash f(v'))) \dashv f(R').$$

On the other hand,

$$\begin{aligned} B &:= \phi\left(\begin{array}{ccccc} & & H & & \\ & & \otimes & & \\ L & \otimes & v & \otimes & R \end{array} \perp \begin{array}{ccccc} & & H' & & \\ & & \otimes & & \\ L' & \otimes & v' & \otimes & R' \end{array} \right) := \\ &= f(L) \vdash [f(v) \perp (f(R) \vdash (f(L') \vdash f(v'))] \dashv f(R'). \end{aligned}$$

if $H = H' = 1$ and vanishes otherwise. Hence $A = B$.
Concerning the binary operation $\vdash$, we get on the one hand:

$$\begin{aligned} B &:= \phi\left(\begin{array}{ccccc} & & H & & \\ & & \otimes & & \\ L & \otimes & v & \otimes & R \end{array} \vdash \begin{array}{ccccc} & & H' & & \\ & & \otimes & & \\ L' & \otimes & v' & \otimes & R' \end{array} \right) := \\ &= f(L) \vdash f(v) \vdash f(R) \vdash f(L') \vdash [f(v') \perp f(H')] \dashv f(R'), \end{aligned}$$

if $H = 1$ and vanishes otherwise. On the other hand,

$$\begin{aligned} A &:= \phi\left(\begin{array}{ccccc} & & H & & \\ & & \otimes & & \\ L & \otimes & v & \otimes & R \end{array} \right) \vdash \phi\left(\begin{array}{ccccc} & & H' & & \\ & & \otimes & & \\ L' & \otimes & v' & \otimes & R' \end{array} \right) := \\ &= (f(L) \vdash [f(v) \perp f(H)] \dashv f(R)) \vdash (f(L') \vdash [f(v') \perp f(H')] \dashv f(R')). \end{aligned}$$

Setting $z := f(L') \vdash [f(v') \perp f(H')] \dashv f(R')$, we get by applying successively Axioms 3,4,5 and 11,

$$\begin{aligned} A &= ((f(L) \vdash [f(v) \perp f(H)]) \dashv f(R)) \vdash z = (f(L) \vdash [f(v) \perp f(H)] \vdash f(R)) \vdash z \\ &= (f(L) \vdash ([f(v) \perp f(H)] \vdash f(R))) \vdash z = 0. \end{aligned}$$

If $H = 1$, then it is easy to check the required equality. Therefore, $A = B$. Proceeding the same way for the binary operation $\dashv$, shows that ϕ is a morphism of triangular algebras. Consequently, ϕ so constructed is unique since it has to coincide with f on V. This completes the proof. $\square$

5.5 On the construction of the homology of 3-dendriform algebras

Set $V := Kv$ the vector space spanned by v and

$$\chi := \begin{array}{ccccc} & & 1 & & \\ & & \otimes & & \\ 1 & \otimes & v & \otimes & 1. \end{array}$$

Let us describe in more detail the free triangular algebra on the generator χ. For all $n > 0$, introduce the set $Tab(n) := \{\frac{p_3}{p_1 \quad 1 \quad p_2};\quad p_1 + p_2 + p_3 = n - 1\}$. Check that the cardinality of $Tab(n)$ is $\frac{n(n+1)}{2}$. We can turn the K-vector space $Tab_\infty := \bigoplus_{n=1}^\infty KTab(n)$ into a triangular algebra by using the following bijection,

$$\begin{array}{ccccc} & & v^{\otimes p_3} & & \\ & & \otimes & & \\ v^{\otimes p_1} & \otimes & v & \otimes & v^{\otimes p_2} \end{array} \longmapsto \frac{p_3}{p_1 \quad 1 \quad p_2},$$

where by convention $v^{\otimes 0} = 1$. Consequently, $Tab_\infty := \bigoplus_{n=1}^\infty KTab(n)$ is also the free triangular algebra on the generator still denoted by $\chi = \frac{0}{0 \quad 1 \quad 0}$.

We now relate these results to the construction of the homology of 3-dendriform algebras. Recall that for all $n \in \mathbb{N}$, we defined $X_n := \{(k, j); 0 \leq k \leq j \leq n\}$. The map $\eta : Tab(n) \longrightarrow X_n$ defined for all n by,

$$\frac{p_3}{p_1 \quad 1 \quad p_2} \longmapsto \begin{cases} (p_1, p_3) \text{ if } p_1 \leq p_3, \\ (p_1, n - 1 - p_3) \text{ if } p_1 > p_3, \end{cases}$$

is clearly a bijection and explain why we constructed the chain-complex of 3-dendriform algebras as we did, according to Ginzburg and Kapranov's results [3].

6 Conclusions and open questions

In the following array, we sum up our results just coming from a simple symmetry on an action of the unit on the most general quadratic relation we can write.

Operads	Poincaré series Numbers	Dual Operads	Poincaré series Numbers
Dend (Loday)	1,2,5,... Catalan numbers	Dias	1,2,3,4,... Natural numbers
m−Dend	Generalized Catalan numbers	m−Tetra	m−tetrahedral numbers
^{k}P	Combinatorial objects to be discovered.	k−Gonal	k−gonal numbers

We conjecture that all these operads are Koszul. Indeed, if an operad $\mathcal{P}$, with dual $\mathcal{P}^!$, is Koszul then their Poincaré series are related by $f_{\mathcal{P}} \circ f_{\mathcal{P}^!} = id$. A hint is to observe that this is the case for our operads.

What are the combinatorial objects behind operads ${}^k\mathcal{P}$, for $k > 4$? For instance, the inverse of the Poincaré series of the operad 5-*Tetra* is a series whose coefficients begin with $1, 5, 38, 347, 3507, 37788, 425490, \ldots$. Observe that operads could be also used indirectely to enumerate discrete structures. Unfortunately, for the time being no known sequence seems to have this starting.

We have also proposed an operadic point of view to recover m-ary trees from the most simple one and operations and also polygonal numbers from 1, played by χ, and operations.

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2000 *Mathematics Subject Classification*: Primary 05C99; Secondary 16W30, 17A30, 17A50, 17A42, 18D50, 55P99.

Keywords: Free m-dendriform algebras, involutive $\mathcal{P}$-Hopf algebras, rooted planar m-ary trees, homogeneous polynomials, duality, operads, k-gonal numbers.

(Concerned with sequences A000217, A000292, A000326, A001764, A002293, A002294, A113206, and A113207.)

Received July 26 2006; revised version received April 25 2007. Published in *Journal of Integer Sequences*, May 4 2007.

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