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New Ramanujan-Type Formulas and Quasi-Fibonacci Numbers of Order 7

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Abstract

We give applications of the quasi-Fibonacci numbers of order 7 and the so-called sine-Fibonacci numbers of order 7 and many other new kinds of recurrent sequences to the decompositions of some polynomials. We also present the characteristic equations, generating functions and some properties of all these sequences. Finally, some new Ramanujan-type formulas are generated.

1 Introduction

The scope of the paper is the generalization of the following decompositions of polynomials [2, 3, 4, 8, 9]:

$$\big(\mathbb{X}-2\sin(\tfrac{2\pi}{7})\big)\big(\mathbb{X}-2\sin(\tfrac{4\pi}{7})\big)\big(\mathbb{X}-2\sin(\tfrac{8\pi}{7})\big)=\mathbb{X}^3-\sqrt{7}\,\mathbb{X}^2+\sqrt{7}, \tag{1.1}$$

$$\big(\mathbb{X}-4\sin^2(\tfrac{2\pi}{7})\big)\big(\mathbb{X}-4\sin^2(\tfrac{4\pi}{7})\big)\big(\mathbb{X}-4\sin^2(\tfrac{8\pi}{7})\big)=\mathbb{X}^3-7\,\mathbb{X}^2+14\,\mathbb{X}-7, \tag{1.2}$$

$$\big(\mathbb{X}-8\sin^3(\tfrac{2\pi}{7})\big)\big(\mathbb{X}-8\sin^3(\tfrac{4\pi}{7})\big)\big(\mathbb{X}-8\sin^3(\tfrac{8\pi}{7})\big)=\mathbb{X}^3-4\sqrt{7}\,\mathbb{X}^2+21\,\mathbb{X}+7\sqrt{7}, \tag{1.3}$$

$$\big(\mathbb{X}-2\cos(\tfrac{2\pi}{7})\big)\big(\mathbb{X}-2\cos(\tfrac{4\pi}{7})\big)\big(\mathbb{X}-2\cos(\tfrac{8\pi}{7})\big)=\mathbb{X}^3+\mathbb{X}^2-2\,\mathbb{X}-1, \tag{1.4}$$

$$\big(\mathbb{X}-4\cos^2(\tfrac{2\pi}{7})\big)\big(\mathbb{X}-4\cos^2(\tfrac{4\pi}{7})\big)\big(\mathbb{X}-4\cos^2(\tfrac{8\pi}{7})\big)=\mathbb{X}^3-5\,\mathbb{X}^2+6\,\mathbb{X}-1, \tag{1.5}$$

$$\big(\mathbb{X}-8\cos^3(\tfrac{2\pi}{7})\big)\big(\mathbb{X}-8\cos^3(\tfrac{4\pi}{7})\big)\big(\mathbb{X}-8\cos^3(\tfrac{8\pi}{7})\big)=\mathbb{X}^3+4\,\mathbb{X}^2-11\,\mathbb{X}-1, \tag{1.6}$$

$$\begin{aligned}&\big(\mathbb{X}-8\sin(\tfrac{2\pi}{7})\,\cos^2(\tfrac{8\pi}{7})\big)\big(\mathbb{X}-8\sin(\tfrac{4\pi}{7})\,\cos^2(\tfrac{2\pi}{7})\big)\big(\mathbb{X}-8\sin(\tfrac{8\pi}{7})\,\cos^2(\tfrac{4\pi}{7})\big)=\\&\qquad=\mathbb{X}^3-3\sqrt{7}\,\mathbb{X}^2+14\,\mathbb{X}+\sqrt{7},\end{aligned} \tag{1.7}$$

etc.

The main incentive for generating the decompositions of these polynomials is provided by the properties of the so-called quasi-Fibonacci numbers of order 7, $A_n(\delta)$, $B_n(\delta)$ and $C_n(\delta)$, $n\in\mathbb{N}$, described in [10] by means of the relations

$$(1+\delta\,(\xi^k+\xi^{6k}))^n=A_n(\delta)+B_n(\delta)\,(\xi^k+\xi^{6k})+C_n(\delta)\,(\xi^{2k}+\xi^{5k}) \tag{1.8}$$

for $k=1,2,3$, where $\xi\in\mathbb{C}$ is a primitive root of unity of order 7 (i.e., $\xi^7=1$ and $\xi\neq1$), $\delta\in\mathbb{C}$, $\delta\neq0$. Besides, an essential rôle in the decompositions of polynomials discussed in the paper is played by related numbers ($\delta\in\mathbb{C}$, $n\in\mathbb{N}$):

$$\begin{aligned}\mathcal{A}_n(\delta)&:=3\,A_n(\delta)-B_n(\delta)-C_n(\delta)=\\&=\big(1+\delta\,(\xi+\xi^6)\big)^n+\big(1+\delta\,(\xi^2+\xi^5)\big)^n+\big(1+\delta\,(\xi^3+\xi^4)\big)^n\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}\mathcal{B}_n(\delta)&:=\frac{1}{2}\big((\mathcal{A}_n(\delta))^2-\mathcal{A}_{2n}(\delta)\big)=\\&=\Big(\big(1+2\,\delta\,\cos(\tfrac{2\pi}{7})\big)\big(1+2\,\delta\,\cos(\tfrac{4\pi}{7})\big)\Big)^n+\\&\quad+\Big(\big(1+2\,\delta\,\cos(\tfrac{2\pi}{7})\big)\big(1+2\,\delta\,\cos(\tfrac{8\pi}{7})\big)\Big)^n+\\&\quad+\Big(\big(1+2\,\delta\,\cos(\tfrac{4\pi}{7})\big)\big(1+2\,\delta\,\cos(\tfrac{8\pi}{7})\big)\Big)^n=\\&=3\,\big(A_n(\delta)\big)^2-2\,A_n(\delta)\,B_n(\delta)-2\,A_n(\delta)\,C_n(\delta)+\\&\quad+3\,B_n(\delta)\,C_n(\delta)-2\,\big(B_n(\delta)\big)^2-2\,\big(C_n(\delta)\big)^2=\\&=A_n(\delta)\,\Big(2\,\mathcal{A}_n(\delta)-3A_n(\delta)\Big)-2\,\Big(B_n(\delta)-C_n(\delta)\Big)^2-B_n(\delta)\,C_n(\delta).\end{aligned} \tag{1.10}$$

Furthermore, to simplify the formulas, we will write

$$\mathcal{A}_n=\mathcal{A}_n(1),\quad \mathcal{B}_n=\mathcal{B}_n(1),\quad A_n=A_n(1),\quad B_n=B_n(1)\quad\text{and}\quad C_n=C_n(1), \tag{1.11}$$

for every $n \in \mathbb{N}$. We note that the tables of values of these numbers can be found in the article [10].

2 Basic decompositions

Wituła et al. [10] determined the following two formulas:

$$\big(\mathbb{X}-\big(2\cos(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X}-\big(2\cos(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-\big(2\cos(\tfrac{8\pi}{7})\big)^n\big)= \\ =\mathbb{X}^3-\mathcal{B}_n\,\mathbb{X}^2+(-1)^n\,\mathcal{A}_n\,\mathbb{X}-1 \quad (2.1)$$

and

$$\big(\mathbb{X}-\big(1+2\,\delta\cos(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X}-\big(1+2\,\delta\cos(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-\big(1+2\,\delta\cos(\tfrac{8\pi}{7})\big)^n\big)= \\ =\mathbb{X}^3-\mathcal{A}_n(\delta)\,\mathbb{X}^2+\mathcal{B}_n(\delta)\,\mathbb{X}-\big(1-\delta-2\,\delta^2+\delta^3\big)^n. \quad (2.2)$$

From (2.2) three special formulas follow:

$$\big(\mathbb{X}-\big(1+\cos^2(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X}-\big(1+\cos^2(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-\big(1+\cos^2(\tfrac{8\pi}{7})\big)^n\big)= \\ =\mathbb{X}^3-\big(\tfrac{3}{2}\big)^n\mathcal{A}_n\big(\tfrac{1}{6}\big)\,\mathbb{X}^2+\big(\tfrac{3}{2}\big)^{2n}\mathcal{B}_n\big(\tfrac{1}{6}\big)\,\mathbb{X}-\big(\tfrac{13}{8}\big)^{2n}, \quad (2.3)$$

$$\big(\mathbb{X}-\big(2\sin(\tfrac{2\pi}{7})\big)^{2n}\big)\big(\mathbb{X}-\big(2\sin(\tfrac{4\pi}{7})\big)^{2n}\big)\big(\mathbb{X}-\big(2\sin(\tfrac{8\pi}{7})\big)^{2n}\big)= \\ =\mathbb{X}^3-2^n\,\mathcal{A}_n\big(-\tfrac{1}{2}\big)\,\mathbb{X}^2+2^{2n}\,\mathcal{B}_n\big(-\tfrac{1}{2}\big)\,\mathbb{X}-7^n \quad (2.4)$$

and

$$\big(\mathbb{X}-\big(2\cos(\tfrac{2\pi}{7})\big)^{2n}\big)\big(\mathbb{X}-\big(2\cos(\tfrac{4\pi}{7})\big)^{2n}\big)\big(\mathbb{X}-\big(2\cos(\tfrac{8\pi}{7})\big)^{2n}\big)= \\ =\mathbb{X}^3-2^n\,\mathcal{A}_n\big(\tfrac{1}{2}\big)\,\mathbb{X}^2+2^{2n}\,\mathcal{B}_n\big(\tfrac{1}{2}\big)\,\mathbb{X}-1. \quad (2.5)$$

Comparing formulas (2.1) and (2.5) two new identities are generated

$$\mathcal{A}_{2n}=2^{2n}\,\mathcal{B}_n\big(\tfrac{1}{2}\big) \quad \text{and} \quad \mathcal{B}_{2n}=2^n\,\mathcal{A}_n\big(\tfrac{1}{2}\big). \quad (2.6)$$

We also have the decomposition

$$\Big(\mathbb{X}-\big(\big(1+\delta\,(\xi+\xi^6)\big)\,\big(1+\delta\,(\xi^2+\xi^5)\big)\big)^n\Big)\,\Big(\mathbb{X}-\big(\big(1+\delta\,(\xi+\xi^6)\big)\,\big(1+\delta\,(\xi^3+\xi^4)\big)\big)^n\Big)\times \\ \times\Big(\mathbb{X}-\big(\big(1+\delta\,(\xi^2+\xi^5)\big)\,\big(1+\delta\,(\xi^3+\xi^4)\big)\big)^n\Big)= \\ =\Big(\mathbb{X}-\big(\big(1+2\,\delta\cos(\tfrac{2\pi}{7})\big)\,\big(1+2\,\delta\cos(\tfrac{4\pi}{7})\big)\big)^n\Big)\,\Big(\mathbb{X}-\big(\big(1+2\,\delta\cos(\tfrac{2\pi}{7})\big)\,\big(1+2\,\delta\cos(\tfrac{6\pi}{7})\big)\big)^n\Big)\times \\ \times\Big(\mathbb{X}-\big(\big(1+2\,\delta\cos(\tfrac{4\pi}{7})\big)\,\big(1+2\,\delta\cos(\tfrac{6\pi}{7})\big)\big)^n\Big)= \\ =\mathbb{X}^3-\mathcal{B}_n(\delta)\,\mathbb{X}^2+\big(1-\delta-2\,\delta^2+\delta^3\big)\,\mathcal{A}_n(\delta)\,\mathbb{X}-\big(1-\delta-2\,\delta^2+\delta^3\big)^{2n}= \\ :=r_n(\mathbb{X};\delta). \quad (2.7)$$

Now let us set

$$\Xi_n := \Xi_n(\delta,\varepsilon,\eta) = 2^n \left(\delta\, \cos^n\left(\tfrac{2\pi}{7}\right) + \varepsilon\, \cos^n\left(\tfrac{4\pi}{7}\right) + \eta\, \cos^n\left(\tfrac{8\pi}{7}\right)\right), \tag{2.8}$$

$$\Upsilon_n := \Upsilon_n(\delta,\varepsilon,\eta) = 2^n \left(\varepsilon\, \cos^n\left(\tfrac{2\pi}{7}\right) + \eta\, \cos^n\left(\tfrac{4\pi}{7}\right) + \delta\, \cos^n\left(\tfrac{8\pi}{7}\right)\right), \tag{2.9}$$

$$\Theta_n := \Theta_n(\delta,\varepsilon,\eta) = 2^n \left(\eta\, \cos^n\left(\tfrac{2\pi}{7}\right) + \delta\, \cos^n\left(\tfrac{4\pi}{7}\right) + \varepsilon\, \cos^n\left(\tfrac{8\pi}{7}\right)\right) \tag{2.10}$$

for any $\delta, \varepsilon, \eta \in \mathbb{C}$ and $n \in \mathbb{N}_0$.

Lemma 2.1 *The following general decomposition formula holds*

$$\begin{aligned}
&(\mathbb{X} - \Xi_n)(\mathbb{X} - \Upsilon_n)(\mathbb{X} - \Theta_n) = \\
&\quad = \mathbb{X}^3 - (\delta+\varepsilon+\eta)\,\mathcal{A}_n\,\mathbb{X}^2 + \tfrac{1}{2}\,(-1)^n\left((\delta+\varepsilon)^2 + (\delta+\eta)^2 + (\varepsilon+\eta)^2\right)\mathcal{A}_n\,\mathbb{X} - \\
&\quad - \Big[\delta\,\varepsilon\,\eta\,(\mathcal{B}_{3n}+3) + \delta^3 + \varepsilon^3 + \eta^3 + (-1)^n\left(\delta\,\varepsilon^2 + \varepsilon\,\eta^2 + \eta\,\delta^2\right)\mathcal{A}_n(-1) + \\
&\qquad + 2^{2n}\left(\delta^2\,\varepsilon + \varepsilon^2\,\eta + \eta^2\,\delta\right)\sum_{k=0}^{n}(-1)^k\binom{n}{k}\mathcal{A}_{2n-k}\left(\tfrac{1}{2}\right)\Big].
\end{aligned} \tag{2.11}$$

Below an illustrative example connected with Lemma 2.1 will be presented.

Examle 2.2 A. M. Yaglom and I. M. Yaglom [11] (see also [5, problem 230] and [6, problem 329]) considered the following polynomial:

$$w(x) = \binom{2m+1}{1}x^m - \binom{2m+1}{3}x^{m-1} + \binom{2m+1}{5}x^{m-2} - \ldots$$

and proved that it has the roots

$$x_k = \cot^2\left(\frac{k\,\pi}{2m+1}\right), \qquad k = 1, 2, \ldots, m.$$

In particular, for $m = 3$ taking into account that

$$\cot^2\left(\tfrac{\pi}{7}\right) = \cot^2\left(\tfrac{8\pi}{7}\right), \qquad \cot^2\left(\tfrac{3\pi}{7}\right) = \cot^2\left(\tfrac{4\pi}{7}\right)$$

we have the decomposition (see also formula (6.14) below):

$$\left(x - \cot^2\left(\tfrac{2\pi}{7}\right)\right)\left(x - \cot^2\left(\tfrac{4\pi}{7}\right)\right)\left(x - \cot^2\left(\tfrac{8\pi}{7}\right)\right) = x^3 - 5\,x^2 + 3\,x - \tfrac{1}{7}$$

and as a corollary – the decomposition

$$\left(x + 7\,\cot^2\left(\tfrac{2\pi}{7}\right)\right)\left(x + 7\,\cot^2\left(\tfrac{4\pi}{7}\right)\right)\left(x + 7\,\cot^2\left(\tfrac{8\pi}{7}\right)\right) = x^3 + 35\,x^2 + 147\,x + 49. \tag{2.12}$$

According to (2.8) for $n = 1$, let us try to find a linear combination

$$-7\,\cot^2\left(\tfrac{2\pi}{7}\right) = 2\left(\delta\,\cos\left(\tfrac{2\pi}{7}\right) + \varepsilon\,\cos\left(\tfrac{4\pi}{7}\right) + \eta\,\cos\left(\tfrac{8\pi}{7}\right)\right)$$

or, the same,

$$-\,7\,\cos^2\left(\tfrac{2\pi}{7}\right)=2\left(\delta\,\cos\left(\tfrac{2\pi}{7}\right)+\varepsilon\,\cos\left(\tfrac{4\pi}{7}\right)+\eta\,\cos\left(\tfrac{8\pi}{7}\right)\right)-$$
$$-\,\delta\,\cos^3\left(\tfrac{2\pi}{7}\right)-\varepsilon\,\cos\left(\tfrac{4\pi}{7}\right)\,\cos^2\left(\tfrac{2\pi}{7}\right)-\eta\,\cos\left(\tfrac{8\pi}{7}\right)\,\cos^2\left(\tfrac{2\pi}{7}\right).$$

Decreasing powers and taking into account the identity

$$\cos\left(\tfrac{2\pi}{7}\right)+\cos\left(\tfrac{4\pi}{7}\right)+\cos\left(\tfrac{8\pi}{7}\right)=-\frac{1}{2}$$

we find

$$\big(2\,\delta+\varepsilon-3\,\eta\big)\cos\left(\tfrac{2\pi}{7}\right)+\big(\delta+3\,\varepsilon-3\,\eta+7\big)\cos\left(\tfrac{4\pi}{7}\right)=-\frac{\delta}{2}+\frac{\varepsilon}{2}+\eta-7.$$

Thus, for finding δ, ε, η we have the linear system (see Corollary 2.5 in [10]):

$$\begin{cases} 2\,\delta+\varepsilon-3\,\eta=0\\ \delta+3\,\varepsilon-3\,\eta=-7\\ \delta-\varepsilon-2\,\eta=-14 \end{cases}$$

with the solution

$$\delta=17,\qquad \varepsilon=5,\qquad \eta=13.$$

Hence,

$$-7\,\cot^2\left(\tfrac{2\pi}{7}\right)=2\left(17\,\cos\left(\tfrac{2\pi}{7}\right)+5\,\cos\left(\tfrac{4\pi}{7}\right)+13\,\cos\left(\tfrac{8\pi}{7}\right)\right).$$

Analogously, we obtain

$$-7\,\cot^2\left(\tfrac{4\pi}{7}\right)=2\left(13\,\cos\left(\tfrac{2\pi}{7}\right)+17\,\cos\left(\tfrac{4\pi}{7}\right)+5\,\cos\left(\tfrac{8\pi}{7}\right)\right),$$
$$-7\,\cot^2\left(\tfrac{8\pi}{7}\right)=2\left(5\,\cos\left(\tfrac{2\pi}{7}\right)+13\,\cos\left(\tfrac{4\pi}{7}\right)+17\,\cos\left(\tfrac{8\pi}{7}\right)\right).$$

Thus the decomposition (2.12) corresponds to Lemma 2.1 with

$$\Xi_1(17,5,13),\quad \Upsilon_1(17,5,13),\quad \Theta_1(17,5,13).$$

Remark 2.3 The identity (2.12) was found earlier by Shevelev [4].

Remark 2.4 The formula (2.11), in some cases which are subject of our interest, especially when coefficients δ, ε, η are the corresponding values of trigonometric functions, becomes rather complicated. Thus, in the next two sections we attempt to designate the relevant coefficients of decomposition (2.11), including the recurrent coefficients, on the grounds of new sequences that are easier to analyze.

3 The first group of special cases of (2.11)

Let us set

$$a_n = 2^{2n+1}\left[\sin(\tfrac{2\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^{2n} + \sin(\tfrac{4\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^{2n} + \sin(\tfrac{8\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^{2n}\right], \tag{3.1}$$

$$b_n = 2^{2n+1}\left[\sin(\tfrac{4\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^{2n} + \sin(\tfrac{8\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^{2n} + \sin(\tfrac{2\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^{2n}\right], \tag{3.2}$$

$$c_n = 2^{2n+1}\left[\sin(\tfrac{8\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^{2n} + \sin(\tfrac{2\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^{2n} + \sin(\tfrac{4\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^{2n}\right], \tag{3.3}$$

for $n = 0, 1, 2, \dots$

Lemma 3.1 *The following recurrence relations hold:*

$$\begin{cases} a_{n+1} = 2\,a_n + b_n, \\ b_{n+1} = a_n + 2\,b_n - c_n, \\ c_{n+1} = c_n - b_n, \end{cases} \tag{3.4}$$

for $n = 0, 1, 2, \dots$ and $a_0 = b_0 = c_0 = \sqrt{7}$. Moreover, elements of each sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ satisfy the recurrence equation

$$x_{n+2} - 5\,x_{n+1} + 6\,x_n - x_{n-1} = 0, \qquad n = 0, 1, 2, \dots \tag{3.5}$$

(in view of decomposition (1.5) an appropriate characteristic polynomial is compatible with the definition of numbers a_n, b_n and c_n).

The first twelve values of numbers $a_n^* = a_n/\sqrt{7}$, $b_n^* = b_n/\sqrt{7}$ and $c_n^* = c_n/\sqrt{7}$ are presented in Table 1.

Now, let us set

$$\alpha_n = 2^{2n}\left[\sin(\tfrac{2\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^{2n-1} + \sin(\tfrac{4\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^{2n-1} + \sin(\tfrac{8\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^{2n-1}\right], \tag{3.6}$$

$$\beta_n = 2^{2n}\left[\sin(\tfrac{4\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^{2n-1} + \sin(\tfrac{8\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^{2n-1} + \sin(\tfrac{2\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^{2n-1}\right], \tag{3.7}$$

$$\gamma_n = 2^{2n}\left[\sin(\tfrac{8\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^{2n-1} + \sin(\tfrac{2\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^{2n-1} + \sin(\tfrac{4\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^{2n-1}\right], \tag{3.8}$$

for $n = 1, 2, \dots$

Lemma 3.2 *We have*

$$\alpha_1 = 0, \qquad \beta_1 = -2\sqrt{7}, \qquad \textit{and} \qquad \gamma_1 = \sqrt{7}.$$

The elements of sequences $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ satisfy the system of recurrence relations (3.4) and, selectively, recurrence relation (3.5).

The first twelve values of numbers $\alpha_n^* = \alpha_n/\sqrt{7}$, $\beta_n^* = \beta_n/\sqrt{7}$ and $\gamma_n^* = \gamma_n/\sqrt{7}$ are presented in Table 1.

Remark 3.3 There exists a simple relationships between numbers α_n, β_n, γ_n, $n \in \mathbb{N}$, and a_n, b_n, c_n, $n \in \mathbb{N}$. We have

$$\alpha_n \equiv c_n, \qquad \beta_n \equiv -a_{n-1} - b_{n-1}, \qquad \gamma_n \equiv a_{n-1}.$$

These relations are easily derived from the definitions of respective numbers, for example

$$\begin{aligned}
\beta_n = 2^{2(n-1)+1}\Big[&2\,\sin(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^{2(n-1)} + \\
&+ 2\,\sin(\tfrac{8\pi}{7})\,\cos(\tfrac{2\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^{2(n-1)} + 2\,\sin(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^{2(n-1)}\Big] = \\
= 2^{2(n-1)+1}\Big[&-\big(\sin(\tfrac{2\pi}{7}) + \sin(\tfrac{4\pi}{7})\big)\big(\cos(\tfrac{8\pi}{7})\big)^{2(n-1)} + \\
&+ \big(-\sin(\tfrac{4\pi}{7}) - \sin(\tfrac{8\pi}{7})\big)\big(\cos(\tfrac{2\pi}{7})\big)^{2(n-1)} - \big(\sin(\tfrac{2\pi}{7}) + \sin(\tfrac{8\pi}{7})\big)\big(\cos(\tfrac{4\pi}{7})\big)^{2(n-1)}\Big] = \\
= -a_{n-1} - b_{n-1}.&
\end{aligned}$$

Let us also set

$$f_n = 2^{n+1}\Big[\cos(\tfrac{2\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^n + \cos(\tfrac{4\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^n + \cos(\tfrac{8\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^n\Big], \tag{3.9}$$

$$g_n = 2^{n+1}\Big[\cos(\tfrac{8\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^n + \cos(\tfrac{2\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^n + \cos(\tfrac{4\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^n\Big], \tag{3.10}$$

$$h_n = 2^{n+1}\Big[\big(\cos(\tfrac{2\pi}{7})\big)^{n+1} + \big(\cos(\tfrac{4\pi}{7})\big)^{n+1} + \big(\cos(\tfrac{8\pi}{7})\big)^{n+1}\Big], \tag{3.11}$$

for $n = 0, 1, 2, \ldots$

Lemma 3.4 *We have*

$$f_0 = g_0 = h_0 = -1, \qquad \textit{and} \qquad h_1 = 5$$

and

$$\begin{cases} f_{n+1} = f_n + g_n, & n \geq 0, \\ g_{n+1} = f_n + h_n, & n \geq 0, \\ h_{n+1} = g_n + 2\,h_{n-1}, & n \geq 1. \end{cases} \tag{3.12}$$

The elements of sequences $\{f_n\}_{n=0}^{\infty}$, $\{g_n\}_{n=0}^{\infty}$ and $\{h_n\}_{n=0}^{\infty}$ satisfy the following recurrence relation (see formula (1.4)):

$$x_{n+3} + x_{n+2} - 2\,x_{n+1} - x_n = 0, \qquad n = 0, 1, \ldots \tag{3.13}$$

Proof: By (3.12) we obtain

$$g_n = f_{n+1} - f_n, \tag{3.14}$$

$$h_n = g_{n+1} - f_n = f_{n+2} - f_{n+1} - f_n, \tag{3.15}$$

and, finally, the following identity

$$f_{n+3} - f_{n+2} - f_{n+1} = 2\left(f_{n+1} - f_n - f_{n-1}\right) + f_{n+1} - f_n,$$

i.e.,

$$f_{n+3} - f_{n+2} - 4\, f_{n+1} + 3\, f_n + 2\, f_{n-1} = 0. \tag{3.16}$$

But we also have the following decomposition of respective characteristic polynomial

$$x^4 - x^3 - 4\, x^2 + 3\, x + 2 = (x-2)(x^3 + x^2 - 2\, x - 1),$$

which implies the following form of (3.16):

$$f_{n+3} + f_{n+2} - 2\, f_{n+1} - f_n = 2\,\big(f_{n+2} + f_{n+1} - 2\, f_n - f_{n-1}\big). \tag{3.17}$$

Since

$$f_3 + f_2 - 2\, f_1 - f_0 = 0$$

so, we obtain the required identity (3.13) from (3.17), (3.14) and (3.15). □

The first twelve elements of the sequences $\{f_n\}_{n=0}^{\infty}$, $\{g_n\}_{n=0}^{\infty}$ and $\{h_n\}_{n=0}^{\infty}$ are presented in Table 1.

Remark 3.5 We note, that

$$h_{n-1} = \mathcal{B}_n = \frac{1}{2}\Big(\big(\mathcal{A}_n\big)^2 - \mathcal{A}_{2n}\Big), \qquad n = 1, 2, \ldots \tag{3.18}$$

is an accelerator sequence for Catalan's constant (see [1] and A094648 [7]).

The next lemma contains a sequence of eight identities and simultanously six newly defined auxiliary sequences of real numbers $\{\widetilde{A}_n\}$, $\{\widetilde{B}_n\}$, $\{\widetilde{C}_n\}$, $\{F_n\}$, $\{G_n\}$ and $\{H_n\}$.

Lemma 3.6 *The following identities hold*

$$\begin{aligned} &4\,\cos(\tfrac{2\pi}{7})\,\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n + 4\,\cos(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^n + \\ &\qquad + 4\,\cos(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n = \\ &\qquad = f_n^2 + g_n^2 - h_n^2 + h_{2n+1} - 2\, h_{2n} - 4\, h_{2n-1} - f_{2n} := 2\, F_n, \end{aligned} \tag{3.19}$$

$$\begin{aligned} &2\,\cos(\tfrac{2\pi}{7})\,\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^n + 2\,\cos(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n + \\ &\qquad + 2\,\cos(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n = \\ &\qquad = F_n + h_n^2 - f_n^2 - h_{2n+1} + 2\, h_{2n} + 2\, h_{2n-1} := G_n, \end{aligned} \tag{3.20}$$

$$\begin{aligned} &2\,\cos(\tfrac{2\pi}{7})\,\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n + 2\,\cos(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n + \\ &\qquad + 2\,\cos(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^n = \\ &\qquad = f_n^2 - 2\,(h_{2n} + h_{2n-1}) - F_n = h_n^2 - h_{2n+1} - G_n := H_n, \end{aligned} \tag{3.21}$$

$$g_n^2 - 2\, h_{2n-1} - f_{2n} = (-1)^n\big(\mathcal{A}_n + \mathcal{A}_{n+1} - 7\, A_n\big) = G_n + F_n, \tag{3.22}$$

$$f_n^2 - 2\, h_{2n} - 2\, h_{2n-1} = (-1)^n\big(\mathcal{A}_n - \mathcal{A}_{n-2}\big) = H_n + F_n, \tag{3.23}$$

$$2\,\sin(\tfrac{2\pi}{7})\,\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^{2n}+2\,\sin(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{2n}+$$
$$+\,2\,\sin(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{2n}=$$
$$=a_{2n}-a_n\,h_{2n-1}+\tfrac{\sqrt{7}}{2}\,(h_{2n-1}^2-h_{4n-1}):=\widetilde{A}_n, \quad (3.24)$$

$$2\,\sin(\tfrac{2\pi}{7})\,\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{2n}+2\,\sin(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^{2n}+$$
$$+\,2\,\sin(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{2n}=$$
$$=b_{2n}-b_n\,h_{2n-1}+\tfrac{\sqrt{7}}{2}\,(h_{2n-1}^2-h_{4n-1}):=\widetilde{B}_n, \quad (3.25)$$

$$2\,\sin(\tfrac{2\pi}{7})\,\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{2n}+2\,\sin(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{2n}+$$
$$+\,2\,\sin(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^{2n}=$$
$$=c_{2n}-c_n\,h_{2n-1}+\tfrac{\sqrt{7}}{2}\,(h_{2n-1}^2-h_{4n-1}):=\widetilde{C}_n, \quad (3.26)$$

Now we are ready to present the final result of this section. All recurrent sequences defined in this section are applied below to the description of the coefficients of certain polynomials.

Theorem 3.7 *The following decompositions of polynomials hold:*

$$\big(\mathbb{X}-2\,\sin(\tfrac{2\pi}{7})\big(2\,\cos(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{4\pi}{7})\big(2\,\cos(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{8\pi}{7})\big(2\,\cos(\tfrac{4\pi}{7})\big)^n\big)=$$
$$=\begin{cases}\mathbb{X}^3-a_k\,\mathbb{X}^2+7\,B_{2k}\,\mathbb{X}+\sqrt{7}, & \textit{for } n=2\,k,\\ \mathbb{X}^3-\alpha_{k-1}\,\mathbb{X}^2-7\,B_{2k-1}\,\mathbb{X}+\sqrt{7}, & \textit{for } n=2\,k-1,\end{cases} \quad (3.27)$$

$$\big(\mathbb{X}-2\,\sin(\tfrac{4\pi}{7})\big(2\,\cos(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{8\pi}{7})\big(2\,\cos(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{2\pi}{7})\big(2\,\cos(\tfrac{4\pi}{7})\big)^n\big)=$$
$$=\begin{cases}\mathbb{X}^3-b_k\,\mathbb{X}^2+7\,(C_{2k}-B_{2k})\,\mathbb{X}+\sqrt{7}, & \textit{for } n=2\,k,\\ \mathbb{X}^3-\beta_{k-1}\,\mathbb{X}^2+7\,(B_{2k-1}-C_{2k-1})\,\mathbb{X}+\sqrt{7}, & \textit{for } n=2\,k-1,\end{cases} \quad (3.28)$$

$$\big(\mathbb{X}-2\,\sin(\tfrac{2\pi}{7})\big(2\,\cos(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{4\pi}{7})\big(2\,\cos(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{8\pi}{7})\big(2\,\cos(\tfrac{8\pi}{7})\big)^n\big)=$$
$$=\begin{cases}\mathbb{X}^3-c_k\,\mathbb{X}^2-7\,C_{2k}\,\mathbb{X}+\sqrt{7}, & \textit{for } n=2\,k,\\ \mathbb{X}^3-\gamma_{k-1}\,\mathbb{X}^2+7\,C_{2k-1}\,\mathbb{X}+\sqrt{7}, & \textit{for } n=2\,k-1,\end{cases} \quad (3.29)$$

$$\big(\mathbb{X}-2\,\cos(\tfrac{2\pi}{7})\big(2\,\cos(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{4\pi}{7})\big(2\,\cos(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{8\pi}{7})\big(2\,\cos(\tfrac{2\pi}{7})\big)^n\big)=$$
$$=\mathbb{X}^3-f_n\,\mathbb{X}^2+(-1)^n\,(7\,A_n-3\,\mathcal{A}_n)\,\mathbb{X}-1, \quad (3.30)$$

$$\big(\mathbb{X}-2\,\cos(\tfrac{8\pi}{7})\big(2\,\cos(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{2\pi}{7})\big(2\,\cos(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{4\pi}{7})\big(2\,\cos(\tfrac{2\pi}{7})\big)^n\big)=$$
$$=\mathbb{X}^3-g_n\,\mathbb{X}^2+(-1)^n\,(\mathcal{A}_n+\mathcal{A}_{n+1}-7\,A_n)\,\mathbb{X}-1, \quad (3.31)$$

$$\big(\mathbb{X}-2\,\cos(\tfrac{2\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^n\big)\times$$
$$\times\big(\mathbb{X}-2\,\cos(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n\big)=$$
$$=\mathbb{X}^3-F_n\,\mathbb{X}^2+(f_n+h_n)\,\mathbb{X}-1, \quad (3.32)$$

$$\big(\mathbb{X}-2\,\cos(\tfrac{2\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n\big)\times$$
$$\times\big(\mathbb{X}-2\,\cos(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n\big)=$$
$$=\mathbb{X}^3-G_n\,\mathbb{X}^2+(f_n+g_n)\,\mathbb{X}-1, \quad (3.33)$$

$$\big(\mathbb{X}-2\,\cos(\tfrac{2\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n\big)\times$$
$$\times\big(\mathbb{X}-2\,\cos(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^n\big)=$$
$$=\mathbb{X}^3-H_n\,\mathbb{X}^2+(g_n+h_n)\,\mathbb{X}-1. \quad (3.34)$$

Proof: Ad (3.30) We have

$$4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})=2\,\cos(\tfrac{2\pi}{7})+2\,\cos(\tfrac{4\pi}{7})$$

hence, by (1.4) we obtain

$$\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{n+1}=\big(2\,\cos(\tfrac{2\pi}{7})+2\,\cos(\tfrac{4\pi}{7})\big)\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n=$$
$$=\Big[\big(2\,\cos(\tfrac{2\pi}{7})+2\,\cos(\tfrac{4\pi}{7})+2\,\cos(\tfrac{8\pi}{7})\big)-\big(2\,\cos(\tfrac{2\pi}{7})+2\,\cos(\tfrac{8\pi}{7})\big)+2\,\cos(\tfrac{2\pi}{7})\Big]\times$$
$$\times\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n=-\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n-4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n+$$
$$+\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{n-1}.$$

Treating, in a similar way, the following products:

$$\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^{n+1}\qquad\text{and}\qquad\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{n+1},$$

and adding all three received decompositions, by (2.1), we obtain

$$(-1)^n\,\big(\mathcal{A}_{n+1}-\mathcal{A}_n-\mathcal{A}_{n-1}\big)=4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n+$$
$$+4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^n+4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n.$$

But, by [10], we have

$$\mathcal{A}_{n+3}-2\,\mathcal{A}_{n+2}-\mathcal{A}_{n+1}+\mathcal{A}_n\equiv 0,$$

hence

$$\mathcal{A}_{n+1}-\mathcal{A}_n-\mathcal{A}_{n-1}=\mathcal{A}_n-\mathcal{A}_{n-2},$$

which implies decomposition (3.30). □

Remark 3.8 It should also be noted that the following interesting identity holds:

$$4\,\mathcal{A}_n-\mathcal{A}_{n-2}=7\,A_n. \quad (3.35)$$

4 The second group of special cases of (2.11)

Let us set

$$
\begin{aligned}
u_n &= 2\,\cos(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n + 2\,\cos(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n + 2\,\cos(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n, && (4.1)\\
v_n &= 2\,\cos(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n + 2\,\cos(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n + 2\,\cos(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n, && (4.2)\\
w_n &= 2\,\cos(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n + 2\,\cos(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n + 2\,\cos(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n, && (4.3)\\
x_n &= 2\,\sin(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n + 2\,\sin(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n + 2\,\sin(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n, && (4.4)\\
y_n &= 2\,\sin(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n + 2\,\sin(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n + 2\,\sin(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n, && (4.5)\\
z_n &= \big(2\,\sin(\tfrac{2\pi}{7})\big)^{n+1} + \big(2\,\sin(\tfrac{4\pi}{7})\big)^{n+1} + \big(2\,\sin(\tfrac{8\pi}{7})\big)^{n+1}, && (4.6)
\end{aligned}
$$

for $n \in \mathbb{N}$ and $u_0 = v_0 = w_0 = -1$ and $z_0 = y_0 = z_0 = \sqrt{7}$, $z_1 = 7$ (see Table 3, and for sequence $\{-w_{2n+1}/\sqrt{7}\}$ see A115146 [7], and see A079309 [7] for sequence $\{z_{2n}/\sqrt{7}\}$).

Then, as may by verified without difficulty, the following recurrence relations hold

$$
\begin{cases}
u_{n+1} = x_n,\\
v_{n+1} = -y_n - z_n = x_n - \sqrt{7}\,z_{n-1},\\
w_{n+1} = y_n - x_n,\\
x_{n+1} = u_n - w_n,\\
y_{n+1} = w_n - v_n,\\
z_{n+1} = 2\,z_{n-1} - v_n,
\end{cases}
\tag{4.7}
$$

for every $n \in \mathbb{N}$. Hence, we easily obtain

$$
\begin{cases}
x_{n+2} = x_n - w_{n+1} = 2\,x_n - y_n,\\
y_{n+2} = y_n - v_{n+1} - x_n = 2\,y_n - x_n + z_n,\\
z_{n+2} = y_n + 3\,z_n,
\end{cases}
\tag{4.8}
$$

for every $n \in \mathbb{N}$ and finally the recurrence relation (see also equation (1.2)):

$$
z_{n+6} - 7\,z_{n+4} + 14\,z_{n+2} - 7\,z_n = 0, \tag{4.9}
$$

which also satisfies the remaining sequences discussed in this section: $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$.

Remark 4.1 The characteristic polynomial of equation (4.9) (after rescaling) has the form of (1.2) and was recognized by Johannes Kepler (1571–1630). The roots of this polynomial are equal to $|A_1A_2|^2$, $|A_1A_3|^2$ and $|A_1A_4|^2$, where $A_1A_2\ldots A_7$ is a regular convex heptagon inscribed in the unit circle [2].

Theorem 4.2 *The following decompositions of polynomials hold:*

$$
\begin{aligned}
&\big(\mathbb{X} - 2\,\cos(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X} - 2\,\cos(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X} - 2\,\cos(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n\big) =\\
&\qquad = \mathbb{X}^3 - u_n\,\mathbb{X}^2 + \tfrac{1}{2}\big(u_n^2 - v_{2n} - 2\,z_{2n-1}\big)\,\mathbb{X} - (-\sqrt{7}\,)^n, \qquad (4.10)
\end{aligned}
$$

$$\big(\mathbb{X}-2\,\cos(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n\big)= \\ =\mathbb{X}^3-v_n\,\mathbb{X}^2+\tfrac{1}{2}\big(v_n^2-w_{2n}-2\,z_{2n-1}\big)\,\mathbb{X}-(-\sqrt{7}\,)^n, \quad (4.11)$$

$$\big(\mathbb{X}-2\,\cos(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n\big)= \\ =\mathbb{X}^3-w_n\,\mathbb{X}^2+\tfrac{1}{2}\big(w_n^2-u_{2n}-2\,z_{2n-1}\big)\,\mathbb{X}-(-\sqrt{7}\,)^n, \quad (4.12)$$

$$\big(\mathbb{X}-2\,\sin(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n\big)= \\ =\mathbb{X}^3-x_n\,\mathbb{X}^2+\tfrac{1}{2}\big(x_n^2+w_{2n}-2\,z_{2n-1}\big)\,\mathbb{X}-(-\sqrt{7}\,)^{n+1}, \quad (4.13)$$

$$\big(\mathbb{X}-2\,\sin(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n\big)= \\ =\mathbb{X}^3-y_n\,\mathbb{X}^2+\tfrac{1}{2}\big(y_n^2+u_{2n}-2\,z_{2n-1}\big)\,\mathbb{X}-(-\sqrt{7}\,)^{n+1}, \quad (4.14)$$

$$\big(\mathbb{X}-\big(2\,\sin(\tfrac{2\pi}{7})\big)^{n+1}\big)\big(\mathbb{X}-\big(2\,\sin(\tfrac{4\pi}{7})\big)^{n+1}\big)\big(\mathbb{X}-\big(2\,\sin(\tfrac{8\pi}{7})\big)^{n+1}\big)= \\ =\mathbb{X}^3-z_n\,\mathbb{X}^2+\tfrac{1}{2}\big(z_n^2-z_{2n+1}\big)\,\mathbb{X}-(-\sqrt{7}\,)^{n+1}. \quad (4.15)$$

Additionally, primarily to describe the generating functions of the sequences $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ (see Section 7) six new recurrence sequences were introduced, which, from a certain point of view, are conjugated with sequences $\{u_n\}$, …, $\{z_n\}$. Let us set

$$u_n^*=2\,\cos(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n+2\,\cos(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n+ \\ +2\,\cos(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n, \quad (4.16)$$

$$v_n^*=2\,\cos(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n+2\,\cos(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n+ \\ +2\,\cos(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n, \quad (4.17)$$

$$w_n^*=2\,\cos(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n+2\,\cos(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n+ \\ +2\,\cos(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n, \quad (4.18)$$

$$x_n^*=2\,\sin(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n+2\,\sin(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n+ \\ +2\,\sin(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n, \quad (4.19)$$

$$y_n^*=2\,\sin(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n+2\,\sin(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n+ \\ +2\,\sin(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n, \quad (4.20)$$

$$z_n^*=2\,\sin(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n+2\,\sin(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n+ \\ +2\,\sin(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n, \quad (4.21)$$

for $n\in\mathbb{N}$ and $u_0^*=v_0^*=w_0^*=-1$ and $x_0^*=y_0^*=z_0^*=\sqrt{7}$ (see Table 4).

An easy computation shows that the following recurrence relations hold

$$\begin{cases} u_{n+1}^*=u_n^*+w_n^*-v_n^*-z_{n-1}^2+z_{2n-1}, \\ v_{n+1}^*=z_{n-1}^2-z_{2n-1}-u_n^*, \\ w_{n+1}^*=u_n^*-w_n^*, \end{cases} \quad (4.22)$$

for every $n \geq 1$. Hence, we obtain

$$u^*_{n+1} + v^*_{n+1} = w^*_n - v^*_n, \tag{4.23}$$
$$u^*_n = w^*_{n+1} + w^*_n, \tag{4.24}$$
$$w^*_{n+2} + w^*_{n+1} - w^*_n = -v^*_{n+1} - v^*_n, \tag{4.25}$$
$$w^*_{n+3} - 3\,w^*_{n+1} - w^*_n = z_{2n+1} + z_{2n-1} - z_n^2 - z_{n-1}^2. \tag{4.26}$$

for every $n \geq 1$.

Also the following recurrence relations hold

$$\begin{cases} x^*_{n+1} = 2\,y^*_n - z^*_n, \\ y^*_{n+1} = 2\,x^*_n + y^*_n - z^*_n, \\ z^*_{n+1} = -x^*_n - y^*_n - z^*_n, \end{cases} \tag{4.27}$$

for every $n \in \mathbb{N}$, which implies the identities

$$y^*_{n+1} = -x^*_{n+2} + 2\,x^*_{n+1} + 7\,x^*_n, \tag{4.28}$$
$$x^*_{n+2} + 7\,x^*_n + 7\,x^*_{n-1} = 0 \tag{4.29}$$

for every $n \geq 1$.

Theorem 4.3 *The following decompositions of polynomials hold:*

$$\begin{aligned}&\big(\mathbb{X} - 2\,\cos(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X} - 2\,\cos(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big) \times \\ &\quad\times \big(\mathbb{X} - 2\,\cos(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n\big) = \\ &\quad\quad = \mathbb{X}^3 - u^*_n\,\mathbb{X}^2 + (-\sqrt{7}\,)^n\,\big(u_n + v_n\big)\,\mathbb{X} - 7^n,\end{aligned} \tag{4.30}$$

$$\begin{aligned}&\big(\mathbb{X} - 2\,\cos(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X} - 2\,\cos(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big) \times \\ &\quad\times \big(\mathbb{X} - 2\,\cos(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big) = \\ &\quad\quad = \mathbb{X}^3 - v^*_n\,\mathbb{X}^2 + (-\sqrt{7}\,)^n\,\big(v_n + w_n\big)\,\mathbb{X} - 7^n,\end{aligned} \tag{4.31}$$

$$\begin{aligned}&\big(\mathbb{X} - 2\,\cos(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X} - 2\,\cos(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n\big) \times \\ &\quad\times \big(\mathbb{X} - 2\,\cos(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big) = \\ &\quad\quad = \mathbb{X}^3 - w^*_n\,\mathbb{X}^2 + (-\sqrt{7}\,)^n\,\big(u_n + w_n\big)\,\mathbb{X} - 7^n,\end{aligned} \tag{4.32}$$

$$\begin{aligned}&\big(\mathbb{X} - 2\,\sin(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X} - 2\,\sin(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big) \times \\ &\quad\times \big(\mathbb{X} - 2\,\sin(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big) = \\ &\quad\quad = \mathbb{X}^3 - x^*_n\,\mathbb{X}^2 + (-\sqrt{7}\,)^n\,\big(w_n - v_n\big)\,\mathbb{X} - 7^n,\end{aligned} \tag{4.33}$$

$$\big(\mathbb{X} - 2\,\sin(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X} - 2\,\sin(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n\big)\times$$
$$\times\big(\mathbb{X} - 2\,\sin(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big) =$$
$$= \mathbb{X}^3 - y_n^*\,\mathbb{X}^2 + (-\sqrt{7}\,)^n\,\big(u_n - w_n\big)\,\mathbb{X} - 7^n, \quad (4.34)$$

$$\big(\mathbb{X} - 2\,\sin(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X} - 2\,\sin(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big)\times$$
$$\times\big(\mathbb{X} - 2\,\sin(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n\big) =$$
$$= \mathbb{X}^3 - z_n^*\,\mathbb{X}^2 + (-\sqrt{7}\,)^n\,\big(v_n - w_n\big)\,\mathbb{X} - 7^n. \quad (4.35)$$

Lemma 4.4 *The following two groups of identities hold:*

$$\begin{aligned}
u_n^2 &= 2\,z_{2n-1} + v_{2n} + v_n^* + u_n^*,\\
v_n^2 &= 2\,z_{2n-1} + w_{2n} + v_n^* + w_n^*,\\
w_n^2 &= 2\,z_{2n-1} + u_{2n} + u_n^* + w_n^*,\\
x_n^2 &= 2\,z_{2n-1} - w_{2n} + w_n^* - v_n^*,\\
y_n^2 &= 2\,z_{2n-1} - u_{2n} + u_n^* - w_n^*,\\
z_n^2 &= 2\,z_{2n-1} + 2\,v_n^{'} - 2\,u_n^*,
\end{aligned} \quad (4.36)$$

and

$$\begin{aligned}
\big(u_n^*\big)^2 &= z_{2n-1}^2 - z_{4n-1} + v_n^* + 2\,\big(-\sqrt{7}\,\big)^n\big(u_n + v_n\big),\\
\big(v_n^*\big)^2 &= z_{2n-1}^2 - z_{4n-1} + w_n^* + 2\,\big(-\sqrt{7}\,\big)^n\big(v_n + w_n\big),\\
\big(w_n^*\big)^2 &= z_{2n-1}^2 - z_{4n-1} + u_n^* + 2\,\big(-\sqrt{7}\,\big)^n\big(u_n + w_n\big),\\
\big(x_n^*\big)^2 &= z_{2n-1}^2 - z_{4n-1} - w_n^* + 2\,\big(-\sqrt{7}\,\big)^n\big(w_n - v_n\big),\\
\big(y_n^*\big)^2 &= z_{2n-1}^2 - z_{4n-1} - u_n^* + 2\,\big(-\sqrt{7}\,\big)^n\big(u_n - w_n\big),\\
\big(z_n^*\big)^2 &= z_{2n-1}^2 - z_{4n-1} - v_n^* + 2\,\big(-\sqrt{7}\,\big)^n\big(v_n - u_n\big).
\end{aligned} \quad (4.37)$$

5 Some Ramanujan-type formulas

Let $p, q, r \in \mathbb{R}$. Shevelev [4] (see also [2]) proved that if z_1, z_2, z_3 are roots of the polynomial

$$w(z) = z^3 + p\,z^2 + q\,z + r$$

and z_1, z_2, z_3 are all reals and at least the following condition holds

$$p\,\sqrt[3]{r} + 3\,\big(\sqrt[3]{r}\big)^2 + q = 0, \quad (5.1)$$

then the following formula holds

$$\sqrt[3]{z_1} + \sqrt[3]{z_2} + \sqrt[3]{z_3} = \sqrt[3]{-p - 6\,\sqrt[3]{r} + \sqrt[3]{(p + 6\,\sqrt[3]{r})^3 - (p - 3\,\sqrt[3]{r})^3}} =$$
$$= \sqrt[3]{-p - 6\,\sqrt[3]{r} + 3\,\sqrt[3]{9\,r + 3\,p\,(\sqrt[3]{r})^2 + p^2\,\sqrt[3]{r}}} \quad (5.2)$$

(all radicals are determined to be real).

It was verified that only following polynomials (among all discussed in this paper) satisfy the condition (5.1); simultanously below the respective form of identity (5.2) is presented:

polynomial (1.4) and polynomial (2.1) for $n = 1$ (it is the classical Ramanujan formula):

$$\sqrt[3]{2\,\cos(\tfrac{2\pi}{7})} + \sqrt[3]{2\,\cos(\tfrac{4\pi}{7})} + \sqrt[3]{2\,\cos(\tfrac{8\pi}{7})} = \sqrt[3]{5 - 3\,\sqrt[3]{7}}; \tag{5.3}$$

polynomial (3.31) for $n = 4$:

$$\cos(\tfrac{2\pi}{7})\,\sqrt[3]{\sec(\tfrac{8\pi}{7})} + \cos(\tfrac{4\pi}{7})\,\sqrt[3]{\sec(\tfrac{2\pi}{7})} + \cos(\tfrac{8\pi}{7})\,\sqrt[3]{\sec(\tfrac{4\pi}{7})} = \frac{1}{2}\,\sqrt[3]{18\,(2 - \sqrt[3]{7})}; \tag{5.4}$$

polynomial (3.31) for $n = 3$:

$$2\,\cos(\tfrac{2\pi}{7})\,\sqrt[3]{2\cos(\tfrac{4\pi}{7})} + 2\,\cos(\tfrac{4\pi}{7})\,\sqrt[3]{2\cos(\tfrac{8\pi}{7})} + 2\,\cos(\tfrac{8\pi}{7})\,\sqrt[3]{2\cos(\tfrac{2\pi}{7})} = \\ = \sqrt[3]{-2 - 3\,\sqrt[3]{49}}; \tag{5.5}$$

polynomial (3.32) for $n = 2, 3$, respectively:

$$\sqrt[3]{\frac{\cos(\frac{2\pi}{7})}{1+\cos(\frac{8\pi}{7})}} + \sqrt[3]{\frac{\cos(\frac{4\pi}{7})}{1+\cos(\frac{2\pi}{7})}} + \sqrt[3]{\frac{\cos(\frac{8\pi}{7})}{1+\cos(\frac{4\pi}{7})}} = \sqrt[3]{11 - 3\,\sqrt[3]{49}}, \tag{5.6}$$

$$\sec(\tfrac{4\pi}{7})\,\sqrt[3]{2\cos(\tfrac{2\pi}{7})} + \sec(\tfrac{8\pi}{7})\,\sqrt[3]{2\cos(\tfrac{4\pi}{7})} + \sec(\tfrac{2\pi}{7})\,\sqrt[3]{2\cos(\tfrac{8\pi}{7})} = -2\,\sqrt[3]{9\,(1 + \sqrt[3]{7})}; \tag{5.7}$$

polynomial (3.30) for $n = 1$, polynomial (3.31) for $n = 1$ and polynomial (3.34) for $n = 2$:

$$\sqrt[3]{\sec(\tfrac{2\pi}{7})} + \sqrt[3]{\sec(\tfrac{4\pi}{7})} + \sqrt[3]{\sec(\tfrac{8\pi}{7})} = \sqrt[3]{8 - 6\,\sqrt[3]{7}}; \tag{5.8}$$

polynomial (4.11) for $n = 6$:

$$\sin^2(\tfrac{2\pi}{7})\,\sqrt[3]{2\cos(\tfrac{4\pi}{7})} + \sin^2(\tfrac{4\pi}{7})\,\sqrt[3]{2\cos(\tfrac{8\pi}{7})} + \sin^2(\tfrac{8\pi}{7})\,\sqrt[3]{2\cos(\tfrac{2\pi}{7})} = \\ = -\frac{1}{4}\,\sqrt[3]{63\,(1 + \sqrt[3]{7})}; \tag{5.9}$$

polynomial (4.13) for $n = 2$:

$$\sqrt[3]{\frac{\sin(\frac{2\pi}{7})}{\sin(\frac{8\pi}{7})}} + \sqrt[3]{\frac{\sin(\frac{4\pi}{7})}{\sin(\frac{2\pi}{7})}} + \sqrt[3]{\frac{\sin(\frac{8\pi}{7})}{\sin(\frac{4\pi}{7})}} = \sqrt[3]{5 - 3\,\sqrt[3]{7}}; \tag{5.10}$$

polynomial (4.14) for $n = 2$:

$$\sqrt[3]{\frac{\sin(\frac{2\pi}{7})}{\sin(\frac{4\pi}{7})}} + \sqrt[3]{\frac{\sin(\frac{4\pi}{7})}{\sin(\frac{8\pi}{7})}} + \sqrt[3]{\frac{\sin(\frac{8\pi}{7})}{\sin(\frac{2\pi}{7})}} = \sqrt[3]{4 - 3\,\sqrt[3]{7}}; \tag{5.11}$$

polynomial (4.14) for $n = 5$:

$$\sin^2(\tfrac{2\pi}{7}) \sqrt[3]{\frac{\sin(\frac{8\pi}{7})}{\sin(\frac{2\pi}{7})}} + \sin^2(\tfrac{4\pi}{7}) \sqrt[3]{\frac{\sin(\frac{2\pi}{7})}{\sin(\frac{4\pi}{7})}} + \sin^2(\tfrac{8\pi}{7}) \sqrt[3]{\frac{\sin(\frac{4\pi}{7})}{\sin(\frac{8\pi}{7})}} = \frac{1}{4} \sqrt[3]{77 - 21 \sqrt[3]{49}}; \quad (5.12)$$

polynomial (4.30) for $n = 3$:

$$\sqrt[6]{7} \left(\csc(\tfrac{2\pi}{7}) \sqrt[3]{2\cos(\tfrac{2\pi}{7})} + \csc(\tfrac{4\pi}{7}) \sqrt[3]{2\cos(\tfrac{4\pi}{7})} + \csc(\tfrac{8\pi}{7}) \sqrt[3]{2\cos(\tfrac{8\pi}{7})} \right) = \\ = 2 \sqrt[3]{2 + 3 \sqrt[3]{49}}; \quad (5.13)$$

polynomial (4.32) for $n = 3$:

$$\sqrt[6]{7} \left(\csc(\tfrac{4\pi}{7}) \sqrt[3]{2\cos(\tfrac{2\pi}{7})} + \csc(\tfrac{8\pi}{7}) \sqrt[3]{2\cos(\tfrac{4\pi}{7})} + \csc(\tfrac{2\pi}{7}) \sqrt[3]{2\cos(\tfrac{8\pi}{7})} \right) = \\ = 2 \sqrt[3]{3 \sqrt[3]{7} - 5}; \quad (5.14)$$

polynomial (4.32) for $n = 6$:

$$\csc^2(\tfrac{4\pi}{7}) \sqrt[3]{2\cos(\tfrac{2\pi}{7})} + \csc^2(\tfrac{8\pi}{7}) \sqrt[3]{2\cos(\tfrac{4\pi}{7})} + \csc^2(\tfrac{2\pi}{7}) \sqrt[3]{2\cos(\tfrac{8\pi}{7})} = \\ = -4 \sqrt[3]{\frac{2 + 3 \sqrt[3]{49}}{7}}. \quad (5.15)$$

Remark 5.1 Shevelev [4] presented the identities (5.10) and (5.11) in the following alternative form:

$$\sqrt[3]{2 + \sec(\tfrac{2\pi}{7})} + \sqrt[3]{2 + \sec(\tfrac{4\pi}{7})} + \sqrt[3]{2 + \sec(\tfrac{8\pi}{7})} = \sqrt[3]{6 \sqrt[3]{7} - 10} \quad (5.16)$$

and

$$\sqrt[3]{1 + 2\,\cos(\tfrac{2\pi}{7})} + \sqrt[3]{1 + 2\,\cos(\tfrac{4\pi}{7})} + \sqrt[3]{1 + 2\,\cos(\tfrac{8\pi}{7})} = \\ = \sqrt[3]{\frac{2\cos(\frac{2\pi}{7})}{2\cos(\frac{2\pi}{7}) + 1}} + \sqrt[3]{\frac{2\cos(\frac{4\pi}{7})}{2\cos(\frac{4\pi}{7}) + 1}} + \sqrt[3]{\frac{2\cos(\frac{8\pi}{7})}{2\cos(\frac{8\pi}{7}) + 1}} = \sqrt[3]{3 \sqrt[3]{7} - 4}, \quad (5.17)$$

respectively.

Remark 5.2 By (5.9), (5.3) and by applied the formula $2\,\sin^2\alpha = 1 - \cos(2\alpha)$ we obtain

$$2\,\cos\left(\tfrac{2\pi}{7}\right) \sqrt[3]{2\,\cos\left(\tfrac{2\pi}{7}\right)} + 2\,\cos\left(\tfrac{4\pi}{7}\right) \sqrt[3]{2\,\cos\left(\tfrac{4\pi}{7}\right)} + 2\,\cos\left(\tfrac{8\pi}{7}\right) \sqrt[3]{2\,\cos\left(\tfrac{8\pi}{7}\right)} = \\ = \sqrt[3]{63\left(1 + \sqrt[3]{7}\right)} - 2 \sqrt[3]{5 - 3 \sqrt[3]{7}}. \quad (5.18)$$

Moreover, from (5.3), (5.8) and the equality $8\,\cos(\frac{2\pi}{7})\,\cos(\frac{4\pi}{7})\,\cos(\frac{8\pi}{7}) = 1$ we get

$$\sqrt[3]{\left(2\,\cos\left(\tfrac{2\pi}{7}\right)\right)^2} + \sqrt[3]{\left(2\,\cos\left(\tfrac{4\pi}{7}\right)\right)^2} + \sqrt[3]{\left(2\,\cos\left(\tfrac{8\pi}{7}\right)\right)^2} = \\ = \sqrt[3]{\left(5 - 3 \sqrt[3]{7}\right)^2} - 2 \sqrt[3]{4 - 3 \sqrt[3]{7}}. \quad (5.19)$$

6 The sine-Fibonacci numbers of order 7

The quasi-Fibonacci numbers of order 7 introduced by Wituła et al. in [10] and further developed in Section 1 constitute the simplest, only one-parameter type of the so-called cosine-Fibonacci numbers of order 7. The name "cosine-Fibonacci numbers" is derived from the form of the decomposition of formulas: $(1+\delta\,(\xi+\xi^6))^n = (1+2\,\delta\,\cos(\frac{2\pi}{7}))^n, \ldots$ In this Section we shall introduce and analyze a sine variety of these numbers, created in the course of decomposing formulas: $(1+\delta\,(\xi-\xi^6))^n = (1+2\,i\,\delta\,\sin(\frac{2\pi}{7}))^n, \ldots$

Theorem 6.1 *Let $\xi,\delta\in\mathbb{C}$, $\xi^7=1$ and $\xi\neq 1$. Then, for every $n\in\mathbb{N}$, there exist polynomials $p_n, r_n, s_n, k_n, l_n \in \mathbb{Z}[\delta]$, called here the sine-Fibonacci numbers of order 7, so that*

$$\begin{aligned}
(1+\delta\,(\xi\ \ -\xi^6))^n &= p_n(\delta)+r_n(\delta)\,(\xi\ \ -\xi^6)+s_n(\delta)\,(\xi^2-\xi^5)+\\
&\qquad + k_n(\delta)\,(\xi\ \ +\xi^6)+l_n(\delta)\,(\xi^2+\xi^5), \qquad (6.1)\\
(1+\delta\,(\xi^2-\xi^5))^n &= p_n(\delta)+r_n(\delta)\,(\xi^2-\xi^5)+s_n(\delta)\,(\xi^4-\xi^3)+\\
&\qquad + k_n(\delta)\,(\xi^2+\xi^5)+l_n(\delta)\,(\xi^4+\xi^3), \qquad (6.2)\\
(1+\delta\,(\xi^4-\xi^3))^n &= p_n(\delta)+r_n(\delta)\,(\xi^4-\xi^3)+s_n(\delta)\,(\xi\ \ -\xi^6)+\\
&\qquad + k_n(\delta)\,(\xi^4+\xi^3)+l_n(\delta)\,(\xi\ \ +\xi^6). \qquad (6.3)
\end{aligned}$$

We have $p_1(\delta)=1$, $r_1(\delta)=\delta$ and $s_1(\delta)=k_1(\delta)=l_1(\delta)=0$ (see Table 7, where the initial values of these polynomial are presented). These polynomials are connected by recurrence relations

$$\begin{cases}
p_{n+1}(\delta)=p_n(\delta)-\delta\,s_n(\delta)-2\,\delta\,r_n(\delta)-i\sqrt{7}\,\delta l_n(\delta),\\
r_{n+1}(\delta)=r_n(\delta)+\delta\,p_n(\delta),\\
s_{n+1}(\delta)=s_n(\delta)+\delta\,k_n(\delta)+\delta\,l_n(\delta),\\
k_{n+1}(\delta)=k_n(\delta)-2\,\delta\,s_n(\delta),\\
l_{n+1}(\delta)=l_n(\delta)+\delta\,r_n(\delta)-\delta\,s_n(\delta),
\end{cases} \qquad (6.4)$$

for $n\in\mathbb{N}$. Hence, the following relationships can be deduced:

$$\delta\,p_n(\delta)=r_{n+1}(\delta)-r_n(\delta), \qquad (6.5)$$

$$2\,\delta\,s_n(\delta)=k_{n+1}(\delta)-k_n(\delta), \qquad (6.6)$$

$$2\,\delta^2\,l_n(\delta)=-k_{n+2}(\delta)+2\,k_{n+1}(\delta)-(1+2\,\delta^2)\,k_n(\delta), \qquad (6.7)$$

$$2\,\delta^3\,r_n(\delta)=-k_{n+3}(\delta)+3\,k_{n+2}(\delta)-3\,(1+\delta^2)\,k_{n+1}(\delta)+(1+3\,\delta^2)\,k_n(\delta) \qquad (6.8)$$

and, at last, the main identity

$$\begin{aligned}
k_{n+5}(\delta)-5\,k_{n+4}+(10+5\,\delta^2)\,k_{n+3}(\delta)+(i\sqrt{7}\,\delta^3-15\,\delta^2-10)\,k_{n+2}(\delta)+\\
+(7\,\delta^4-i\,2\sqrt{7}\,\delta^3+15\,\delta^2+5)\,k_{n+1}(\delta)+\\
+(i\,2\sqrt{7}\,\delta^5-7\,\delta^4+i\sqrt{7}\,\delta^3-5\,\delta^2-1)\,k_n(\delta)=0. \qquad (6.9)
\end{aligned}$$

This identity satisfies, also by (6.5)–(6.8), the remaining polynomials: $p_n(\delta)$, $r_n(\delta)$, $s_n(\delta)$, and $l_n(\delta)$, $n\in\mathbb{N}$. The characteristic polynomial corresponding to identity (6.9) has the *following decomposition:*

$$\mathbb{X}^5-5\,\mathbb{X}^4+(10+5\,\delta^2)\,\mathbb{X}^3+(i\,\sqrt{7}\,\delta^3-15\,\delta^2-10)\,\mathbb{X}^2+(7\,\delta^4-i\,2\,\sqrt{7}\,\delta^3+15\,\delta^2+5)\,\mathbb{X}+ \\ +i\,2\,\sqrt{7}\,\delta^5-7\,\delta^4+i\,\sqrt{7}\,\delta^3-5\,\delta^2-1=(\mathbb{X}-1-\delta\,(\xi-\xi^6))(\mathbb{X}-1-\delta\,(\xi^2-\xi^5))\times \\ \times(\mathbb{X}-1-\delta\,(\xi^4-\xi^3))(\mathbb{X}-1+\delta\,(\xi+\xi^2+\xi^4))(\mathbb{X}-1+\delta\,(\xi^3+\xi^5+\xi^6))= \\ =\big(\mathbb{X}^3+(-3-i\,\sqrt{7}\,\delta)\,\mathbb{X}^2+(3+i\,2\,\sqrt{7}\,\delta)\,\mathbb{X}-i\,\sqrt{7}\,\delta^3-i\,\sqrt{7}\,\delta-1\big)\times \\ \times\big(\mathbb{X}^2+(-2+i\,\sqrt{7}\,\delta)\,\mathbb{X}+1-2\,\delta^2-i\,\sqrt{7}\,\delta\big). \quad (6.10)$$

Hence, we obtain, for example, the following explicit form of $k_n(\delta)$:

$$k_n(\delta)=a\,(1+\delta\,(\xi-\xi^6))^n+b\,(1+\delta\,(\xi^2-\xi^5))^n+ \\ +c\,(1+\delta\,(\xi^4-\xi^3))^n+d\,(1-\delta\,(\xi+\xi^2+\xi^4))^n+e\,(1+\delta\,(\xi^3+\xi^5+\xi^6))^n, \quad (6.11)$$

where

$$\begin{aligned} a&=-2\big(5+(\xi^2+\xi^5)+7\,(\xi^3+\xi^4)\big)^{-1}=\tfrac{2}{301}\big(46\,(\xi+\xi^6)+5\,(\xi^3+\xi^4)-11\big),\\ b&=2\big(2+7\,(\xi^2+\xi^5)+6\,(\xi^3+\xi^4)\big)^{-1}=\tfrac{2}{187}\big(19\,(\xi+\xi^6)+46\,(\xi^2+\xi^5)-5\big),\\ c&=-2\big(4+6\,(\xi^2+\xi^5)-(\xi^3+\xi^4)\big)^{-1}=\tfrac{2}{301}\big(5\,(\xi^2+\xi^5)+46\,(\xi^3+\xi^4)-11\big),\\ d&=-2\big(-5+2\,(\xi+\xi^2+\xi^4)\big)^{-1}=\tfrac{2}{43}\big(5-2\,(\xi^3+\xi^5+\xi^6)\big),\\ e&=2\big(7+2\,(\xi+\xi^2+\xi^4)\big)^{-1}=\tfrac{2}{43}\big(7+2\,(\xi^3+\xi^5+\xi^6)\big). \end{aligned}$$

Corollary 6.2 *The following identity holds:*

$$\Omega_n(\delta):=(1+\delta\,(\xi-\xi^6))^n+(1+\delta\,(\xi^2-\xi^5))^n+(1+\delta\,(\xi^4-\xi^3))^n= \\ =3\,p_n(\delta)-k_n(\delta)-l_n(\delta)+i\,\sqrt{7}\big(r_n(\delta)+s_n(\delta)\big). \quad (6.12)$$

Definition 6.3 *To simplify the notation, we shall write Ω_n instead of $\Omega_n(1)$.*

Theorem 6.4 *The following decompositions of the polynomials hold:*

$$q_n(\mathbb{X};\delta):= \\ =\big(\mathbb{X}-\big(1+\delta\,(2\,i\,\sin(\tfrac{2\pi}{7}))^n\big)\big)\big(\mathbb{X}-\big(1+\delta\,(2\,i\,\sin(\tfrac{4\pi}{7}))^n\big)\big)\big(\mathbb{X}-\big(1+\delta\,(2\,i\,\sin(\tfrac{8\pi}{7}))^n\big)\big)= \\ =\mathbb{X}^3-\Omega_n(\delta)\,\mathbb{X}^2+\tfrac{1}{2}\big((\Omega_n(\delta))^2-\Omega_{2n}(\delta)\big)\,\mathbb{X}-\big(1+i\,\sqrt{7}\,\delta+i\,\sqrt{7}\,\delta^3\big)^n, \quad (6.13)$$

$$\big(x-\big(\cot(\tfrac{2\pi}{7})\big)^n\big)\big(x-\big(\cot(\tfrac{4\pi}{7})\big)^n\big)\big(x-\big(\cot(\tfrac{8\pi}{7})\big)^n\big)= \\ =x^3-\big(\tfrac{3}{\sqrt{7}}\big)^n\,\mathcal{A}_n(\tfrac{2}{3})\,x^2+\Omega_n(\tfrac{2\,i}{\sqrt{7}})\,x+\frac{(-1)^{n-1}}{\big(\sqrt{7}\big)^n}. \quad (6.14)$$

Moreover, the following identity holds:

$$\Omega_n^2(\delta)=\Omega_{2n}(\delta)+2\,\big(3\,p_n^2(\delta)-2\,l_n^2(\delta)-2\,k_n^2(\delta)-2\,p_n(\delta)\,k_n(\delta)-2\,p_n(\delta)\,l_n(\delta)-7\,r_n(\delta)\,s_n(\delta)+ \\ +3\,k_n(\delta)\,l_n(\delta)+i\,2\,\sqrt{7}\,(p_n(\delta)\,r_n(\delta)+p_n(\delta)\,s_n(\delta)-r_n(\delta)\,k_n(\delta)-s_n(\delta)\,l_n(\delta))+ \\ +i\,\sqrt{7}(r_n(\delta)\,l_n(\delta)-k_n(\delta)\,s_n(\delta))\big)= \\ =\Omega_{2n}(\delta)+4\,p_n(\delta)\,\Omega_n(\delta)+2\,\big(-3\,p_n^2(\delta)-2\,l_n^2(\delta)-2\,k_n^2(\delta)-7\,r_n(\delta)\,s_n(\delta)+3\,k_n(\delta)\,l_n(\delta)+ \\ +i\,\sqrt{7}(r_n(\delta)\,l_n(\delta)-2\,r_n(\delta)\,k_n(\delta)-k_n(\delta)\,s_n(\delta)-2\,s_n(\delta)\,l_n(\delta))\big). \quad (6.15)$$

7 Generating functions

The generating functions of almost all sequences discussed and defined in this paper are presented in this section.

By (1.9) and (2.2) we obtain (for $k \in \mathbb{N}$):

$$\sum_{n=0}^{\infty} \mathcal{A}_{kn}(\delta)\, \mathbb{X}^n = \big(1 - \big(1 + \delta\,(\xi + \xi^6)\big)^k\, \mathbb{X}\big)^{-1} + \big(1 - \big(1 + \delta\,(\xi^2 + \xi^5)\big)^k\, \mathbb{X}\big)^{-1} +$$
$$+ \big(1 - \big(1 + \delta\,(\xi^3 + \xi^4)\big)^k\, \mathbb{X}\big)^{-1} = \frac{3 - 2\,\mathcal{A}_k(\delta)\,\mathbb{X} + \mathcal{B}_k(\delta)\,\mathbb{X}^2}{1 - \mathcal{A}_k(\delta)\,\mathbb{X} + \mathcal{B}_k(\delta)\,\mathbb{X}^2 - (1 - \delta - 2\,\delta^2 + \delta^3)^k\,\mathbb{X}^3}, \quad (7.1)$$

and, in the sequel, for $k = 1$ we get

$$\sum_{n=0}^{\infty} \mathcal{A}_{n}(\delta)\, \mathbb{X}^n = \big(1 - \big(1 + \delta\,(\xi + \xi^6)\big)\, \mathbb{X}\big)^{-1} + \big(1 - \big(1 + \delta\,(\xi^2 + \xi^5)\big)\, \mathbb{X}\big)^{-1} +$$
$$+ \big(1 - \big(1 + \delta\,(\xi^3 + \xi^4)\big)\, \mathbb{X}\big)^{-1} = \frac{\mathbb{X}^2\, p_7'\big(1/\mathbb{X};\delta\big)}{\mathbb{X}^3\, p_7\big(1/\mathbb{X};\delta\big)} =$$
$$= \frac{3 + 2\,(\delta - 3)\,\mathbb{X} + (3 - 2\,\delta - 2\,\delta^2)\,\mathbb{X}^2}{1 + (\delta - 3)\,\mathbb{X} + (3 - 2\,\delta - 2\,\delta^2)\,\mathbb{X}^2 + (-1 + \delta + 2\,\delta^2 - \delta^3)\,\mathbb{X}^3}, \quad (7.2)$$

where $p_7(\mathbb{X}, \delta) = \mathbb{X}^3 + (\delta - 3)\mathbb{X}^2 + (3 - 2\delta - 2\delta^2)\mathbb{X} + (-1 + \delta + 2\delta^2 - \delta^3)$ (see [10]).

By (1.10) and (2.7) we get (for $k \in \mathbb{N}$):

$$\sum_{n=0}^{\infty} \mathcal{B}_{kn}(\delta)\, \mathbb{X}^n = \frac{\mathbb{X}^2\, r_k'\big(1/\mathbb{X};\delta\big)}{\mathbb{X}^3\, r_k\big(1/\mathbb{X};\delta\big)} =$$
$$= \frac{3 - 2\,\mathcal{B}_k(\delta)\,\mathbb{X} + (1 - \delta - 2\,\delta^2 + \delta^3)\,\mathcal{A}_k(\delta)\,\mathbb{X}^2}{1 - \mathcal{B}_k(\delta)\,\mathbb{X} + (1 - \delta - 2\,\delta^2 + \delta^3)\,\mathcal{A}_k(\delta)\,\mathbb{X}^2 - (1 - \delta - 2\,\delta^2 + \delta^3)^{2k}\,\mathbb{X}^3}, \quad (7.3)$$

where $r_n(\mathbb{X}, \delta)$ is defined in (2.7). By (3.19) from [10] we obtain

$$\sum_{n=0}^{\infty} A_{n}(\delta)\, \mathbb{X}^n = \frac{2 - \xi^3 - \xi^4}{1 - \big(1 + \delta\,(\xi + \xi^6)\big)\,\mathbb{X}} + \frac{2 - \xi - \xi^6}{1 - \big(1 + \delta\,(\xi^2 + \xi^5)\big)\,\mathbb{X}} +$$
$$+ \frac{2 - \xi^2 - \xi^5}{1 - \big(1 + \delta\,(\xi^3 + \xi^4)\big)\,\mathbb{X}} = \frac{1 + (-2 + \delta)\,\mathbb{X} + (1 - \delta)\,\mathbb{X}^2}{\mathbb{X}^3\, p_7\big(1/\mathbb{X};\delta\big)}; \quad (7.4)$$

by (3.18) and (3.17) from [10] we obtain respectively:

$$\sum_{n=0}^{\infty} B_{n}(\delta)\, \mathbb{X}^n = \frac{\xi + \xi^6 - \xi^3 - \xi^4}{1 - \big(1 + \delta\,(\xi + \xi^6)\big)\,\mathbb{X}} + \frac{\xi^2 + \xi^5 - \xi - \xi^6}{1 - \big(1 + \delta\,(\xi^2 + \xi^5)\big)\,\mathbb{X}} +$$
$$+ \frac{\xi^3 + \xi^4 - \xi^2 - \xi^5}{1 - \big(1 + \delta\,(\xi^3 + \xi^4)\big)\,\mathbb{X}} = \frac{\delta\,\mathbb{X} + (\delta^2 - \delta)\,\mathbb{X}^2}{\mathbb{X}^3\, p_7\big(1/\mathbb{X};\delta\big)} \quad (7.5)$$

and

$$\sum_{n=0}^{\infty} C_n(\delta)\,\mathbb{X}^n = \sum_{n=0}^{\infty}\big(3A_n(\delta) - B_n(\delta) - \mathcal{A}_n(\delta)\big)\,\mathbb{X}^n = \frac{\delta^2\,\mathbb{X}^2}{\mathbb{X}^3\,p_7\big(1/\mathbb{X};\delta\big)}. \tag{7.6}$$

Equally, on the grounds of (3.1)–(3.3) and (3.24)–(3.26), we obtain the following formulas:

$$\sum_{n=0}^{\infty} a_{kn}\,\mathbb{X}^n = \frac{\sqrt{7} - (b_k + c_k)\,\mathbb{X} + \widetilde{A}_k\,\mathbb{X}^2}{1 - \mathcal{B}_{2k}\,\mathbb{X} + \mathcal{A}_{2k}\,\mathbb{X}^2 - \mathbb{X}^3}, \tag{7.7}$$

$$\sum_{n=0}^{\infty} b_{kn}\,\mathbb{X}^n = \frac{\sqrt{7} - (a_k + c_k)\,\mathbb{X} + \widetilde{B}_k\,\mathbb{X}^2}{1 - \mathcal{B}_{2k}\,\mathbb{X} + \mathcal{A}_{2k}\,\mathbb{X}^2 - \mathbb{X}^3}, \tag{7.8}$$

$$\sum_{n=0}^{\infty} c_{kn}\,\mathbb{X}^n = \frac{\sqrt{7} - (a_k + b_k)\,\mathbb{X} + \widetilde{C}_k\,\mathbb{X}^2}{1 - \mathcal{B}_{2k}\,\mathbb{X} + \mathcal{A}_{2k}\,\mathbb{X}^2 - \mathbb{X}^3}. \tag{7.9}$$

The special cases (for $k = 1$) can also be generated in the following selective ways: by (3.1) and (1.5) we obtain

$$\sum_{n=0}^{\infty} a_n\,\mathbb{X}^n = \frac{2\,\sin\left(\frac{2\pi}{7}\right)}{1 - 4\,\cos^2\left(\frac{8\pi}{7}\right)\mathbb{X}} + \frac{2\,\sin\left(\frac{4\pi}{7}\right)}{1 - 4\,\cos^2\left(\frac{2\pi}{7}\right)\mathbb{X}} + \frac{2\,\sin\left(\frac{8\pi}{7}\right)}{1 - 4\,\cos^2\left(\frac{4\pi}{7}\right)\mathbb{X}} = $$

$$= \frac{\sqrt{7} - 2\sqrt{7}\,\mathbb{X} - \sqrt{7}\,\mathbb{X}^2}{1 - 5\,\mathbb{X} + 6\,\mathbb{X}^2 - \mathbb{X}^3}; \tag{7.10}$$

however, by (3.2) and (3.4) we obtain

$$\sum_{n=0}^{\infty} b_n\,\mathbb{X}^n = \sum_{n=0}^{\infty}\big(a_{n+1} + 2\,a_n\big)\,\mathbb{X}^n = \frac{1 + 2\,\mathbb{X}}{\mathbb{X}}\Big(\sum_{n=0}^{\infty} a_n\,\mathbb{X}^n\Big) - \frac{\sqrt{7}}{\mathbb{X}} =$$

$$= \frac{2\,\sin\left(\frac{4\pi}{7}\right)}{1 - 4\,\cos^2\left(\frac{8\pi}{7}\right)\mathbb{X}} + \frac{2\,\sin\left(\frac{8\pi}{7}\right)}{1 - 4\,\cos^2\left(\frac{2\pi}{7}\right)\mathbb{X}} + \frac{2\,\sin\left(\frac{2\pi}{7}\right)}{1 - 4\,\cos^2\left(\frac{4\pi}{7}\right)\mathbb{X}} =$$

$$= \frac{\sqrt{7} - 3\sqrt{7}\,\mathbb{X} + 3\sqrt{7}\,\mathbb{X}^2}{1 - 5\,\mathbb{X} + 6\,\mathbb{X}^2 - \mathbb{X}^3} \tag{7.11}$$

and

$$\sum_{n=0}^{\infty} c_n\,\mathbb{X}^n = \sum_{n=0}^{\infty}\big(a_n + 2\,b_n - b_{n+1}\big)\,\mathbb{X}^n = \frac{\sqrt{7} - 5\sqrt{7}\,\mathbb{X} + 4\sqrt{7}\,\mathbb{X}^2}{1 - 5\,\mathbb{X} + 6\,\mathbb{X}^2 - \mathbb{X}^3}. \tag{7.12}$$

By (6.12) and (6.13) we get

$$\sum_{n=0}^{\infty} \Omega_{kn}(\delta)\,\mathbb{X}^n = \Big(1 - \big(1 + \delta\,(\xi - \xi^6)\big)^k\,\mathbb{X}\Big)^{-1} + \Big(1 - \big(1 + \delta\,(\xi^2 - \xi^5)\big)^k\,\mathbb{X}\Big)^{-1} +$$

$$+ \Big(1 - \big(1 + \delta\,(\xi^3 - \xi^4)\big)^k\,\mathbb{X}\Big)^{-1} = \frac{\mathbb{X}^2\,q_k'\big(1/\mathbb{X};\delta\big)}{\mathbb{X}^3\,q_k\big(1/\mathbb{X};\delta\big)} =$$

$$= \frac{3 - 2\,\Omega_k(\delta)\,\mathbb{X} + \frac{1}{2}\,\big((\Omega_k(\delta))^2 - \Omega_{2k}(\delta)\big)\,\mathbb{X}^2}{1 - \Omega_k(\delta)\,\mathbb{X} + \frac{1}{2}\,\big((\Omega_k(\delta))^2 - \Omega_{2k}(\delta)\big)\,\mathbb{X}^2 - (1 + i\sqrt{7}\,\delta + i\sqrt{7}\,\delta^3)^k\,\mathbb{X}^3}, \tag{7.13}$$

and, the special case, for $k = 1$ (by (6.12) and (6.10)):

$$\sum_{n=0}^{\infty} \Omega_n(\delta)\, \mathbb{X}^n = \big(1 - \big(1 + \delta\,(\xi - \xi^6)\big)\,\mathbb{X}\big)^{-1} + \big(1 - \big(1 + \delta\,(\xi^2 - \xi^5)\big)\,\mathbb{X}\big)^{-1} + \\ + \big(1 - \big(1 + \delta\,(\xi^4 - \xi^3)\big)\,\mathbb{X}\big)^{-1} = \frac{\mathbb{X}^2\, q_1'\big(1/\mathbb{X};\delta\big)}{\mathbb{X}^3\, q_1\big(1/\mathbb{X};\delta\big)} = \\ = \frac{3 - 2\,(3 + i\sqrt{7}\,\delta)\,\mathbb{X} + (3 + i\,2\sqrt{7}\,\delta)\,\mathbb{X}^2}{1 - (3 + i\sqrt{7}\,\delta)\,\mathbb{X} + (3 + i\,2\sqrt{7}\,\delta)\,\mathbb{X}^2 - (1 + i\sqrt{7}\,\delta + i\sqrt{7}\,\delta^3)\,\mathbb{X}^3}, \quad (7.14)$$

where $q_k(\mathbb{X};\delta)$ is defined in (6.13).

By (1.4), (3.9)–(3.11) and (3.19)–(3.21) the following formulas may be obtained:

$$\sum_{n=0}^{\infty} f_{kn}\, \mathbb{X}^n = \frac{2\,\cos\left(\frac{2\pi}{7}\right)}{1 - \big(2\,\cos(\frac{4\pi}{7})\big)^k\,\mathbb{X}} + \frac{2\,\cos\left(\frac{4\pi}{7}\right)}{1 - \big(2\,\cos(\frac{8\pi}{7})\big)^k\,\mathbb{X}} + \frac{2\,\cos\left(\frac{8\pi}{7}\right)}{1 - \big(2\,\cos(\frac{2\pi}{7})\big)^k\,\mathbb{X}} = \\ = \frac{-1 - (g_k + h_k)\,\mathbb{X} + F_k\,\mathbb{X}^2}{1 - h_{k-1}\,\mathbb{X} + \frac{1}{2}\big(h_{k-1}^2 - h_{2k-1}\big)\,\mathbb{X}^2 - \mathbb{X}^3}, \quad (7.15)$$

$$\sum_{n=0}^{\infty} g_{kn}\, \mathbb{X}^n = \frac{2\,\cos\left(\frac{2\pi}{7}\right)}{1 - \big(2\,\cos(\frac{8\pi}{7})\big)^k\,\mathbb{X}} + \frac{2\,\cos\left(\frac{4\pi}{7}\right)}{1 - \big(2\,\cos(\frac{2\pi}{7})\big)^k\,\mathbb{X}} + \frac{2\,\cos\left(\frac{8\pi}{7}\right)}{1 - \big(2\,\cos(\frac{4\pi}{7})\big)^k\,\mathbb{X}} = \\ = \frac{-1 - (f_k + h_k)\,\mathbb{X} + G_k\,\mathbb{X}^2}{1 - h_{k-1}\,\mathbb{X} + \frac{1}{2}\big(h_{k-1}^2 - h_{2k-1}\big)\,\mathbb{X}^2 - \mathbb{X}^3}, \quad (7.16)$$

$$\sum_{n=0}^{\infty} h_{kn-1}\, \mathbb{X}^n = \frac{1}{1 - \big(2\,\cos(\frac{2\pi}{7})\big)^k\,\mathbb{X}} + \frac{1}{1 - \big(2\,\cos(\frac{4\pi}{7})\big)^k\,\mathbb{X}} + \frac{1}{1 - \big(2\,\cos(\frac{8\pi}{7})\big)^k\,\mathbb{X}} = \\ = \frac{3 - 2\,h_{k-1}\,\mathbb{X} + \frac{1}{2}\big(h_{k-1}^2 - h_{2k-1}\big)\,\mathbb{X}^2}{1 - h_{k-1}\,\mathbb{X} + \frac{1}{2}\big(h_{k-1}^2 - h_{2k-1}\big)\,\mathbb{X}^2 - \mathbb{X}^3} \overset{by\ (2.1)}{=} \frac{3 - 2\,\mathcal{B}_k\,\mathbb{X} + (-1)^k\,\mathcal{A}_k\,\mathbb{X}^2}{1 - \mathcal{B}_k\,\mathbb{X} + (-1)^k\,\mathcal{A}_k\,\mathbb{X}^2 - \mathbb{X}^3}. \quad (7.17)$$

$(h_{-1} := 1)$.

We note that the special case of the formulas (7.15) and (7.16) for $k = 1$ can be treated in another way (polynomial (1.4) will be denoted here by $p_7(\mathbb{X})$):

$$\sum_{n=0}^{\infty} h_n\, \mathbb{X}^n = \frac{\xi + \xi^6}{1 - (\xi + \xi^6)\,\mathbb{X}} + \frac{\xi^2 + \xi^5}{1 - (\xi^2 + \xi^5)\,\mathbb{X}} + \frac{\xi^3 + \xi^4}{1 - (\xi^3 + \xi^4)\,\mathbb{X}} = \\ = -\frac{\big(\mathbb{X}^3\, p_7\big(1/\mathbb{X}\big)\big)'}{\mathbb{X}^3\, p_7\big(1/\mathbb{X}\big)} = \frac{-1 + 4\,\mathbb{X} + 3\,\mathbb{X}^2}{1 + \mathbb{X} - 2\,\mathbb{X}^2 - \mathbb{X}^3}, \quad (7.18)$$

hence by (3.12):

$$\sum_{n=0}^{\infty} g_n\,\mathbb{X}^n = -1 + \sum_{n=1}^{\infty} \big(h_{n+1} - 2\,h_{n-1}\big)\,\mathbb{X}^n =$$
$$= -1 + \mathbb{X}^{-1} \sum_{n=0}^{\infty} h_n\,\mathbb{X}^n - \mathbb{X}^{-1}\big(h_0 + h_1\,\mathbb{X}\big) - 2\,\mathbb{X} \sum_{n=0}^{\infty} h_n\,\mathbb{X}^n =$$
$$= -6 + \mathbb{X}^{-1} + \frac{1 - 2\,\mathbb{X}^2}{\mathbb{X}} \sum_{n=0}^{\infty} h_n\,\mathbb{X}^n = \frac{-1 - 3\,\mathbb{X} + 3\,\mathbb{X}^2}{1 + \mathbb{X} - 2\,\mathbb{X}^2 - \mathbb{X}^3} \tag{7.19}$$

and, finally, using (3.12) again we get

$$\sum_{n=0}^{\infty} f_n\,\mathbb{X}^n = \sum_{n=0}^{\infty} \big(g_{n+1} - h_n\big)\,\mathbb{X}^n =$$
$$= \mathbb{X}^{-1} \sum_{n=0}^{\infty} g_n\,\mathbb{X}^n + \mathbb{X}^{-1} - \sum_{n=0}^{\infty} h_n\,\mathbb{X}^n = -\frac{1 + 3\,\mathbb{X} + 4\,\mathbb{X}^2}{1 + \mathbb{X} - 2\,\mathbb{X}^2 - \mathbb{X}^3}. \tag{7.20}$$

By (1.1), (4.4)–(4.6), (4.15) and (4.19)–(4.21) we have

$$\sum_{n=0}^{\infty} x_{kn}\,\mathbb{X}^n = \frac{2\,\sin\left(\frac{2\pi}{7}\right)}{1 - \big(2\,\sin(\frac{8\pi}{7})\big)^k\,\mathbb{X}} + \frac{2\,\sin\left(\frac{4\pi}{7}\right)}{1 - \big(2\,\sin(\frac{2\pi}{7})\big)^k\,\mathbb{X}} + \frac{2\,\sin\left(\frac{8\pi}{7}\right)}{1 - \big(2\,\sin(\frac{4\pi}{7})\big)^k\,\mathbb{X}} =$$
$$= \frac{\sqrt{7} - (y_k + z_k)\,\mathbb{X} + x_k^*\,\mathbb{X}^2}{1 - z_{k-1}\,\mathbb{X} + \frac{1}{2}\big(z_{k-1}^2 - z_{2k-1}\big)\,\mathbb{X}^2 - \big(-\sqrt{7}\big)^k\,\mathbb{X}^3}, \tag{7.21}$$

$$\sum_{n=0}^{\infty} y_{kn}\,\mathbb{X}^n = \frac{2\,\sin\left(\frac{2\pi}{7}\right)}{1 - \big(2\,\sin(\frac{4\pi}{7})\big)^k\,\mathbb{X}} + \frac{2\,\sin\left(\frac{4\pi}{7}\right)}{1 - \big(2\,\sin(\frac{8\pi}{7})\big)^k\,\mathbb{X}} + \frac{2\,\sin\left(\frac{8\pi}{7}\right)}{1 - \big(2\,\sin(\frac{2\pi}{7})\big)^k\,\mathbb{X}} =$$
$$= \frac{\sqrt{7} - (x_k + z_k)\,\mathbb{X} + y_k^*\,\mathbb{X}^2}{1 - z_{k-1}\,\mathbb{X} + \frac{1}{2}\big(z_{k-1}^2 - z_{2k-1}\big)\,\mathbb{X}^2 - \big(-\sqrt{7}\big)^k\,\mathbb{X}^3}, \tag{7.22}$$

$$\sum_{n=0}^{\infty} z_{kn}\,\mathbb{X}^n = \frac{2\,\sin\left(\frac{2\pi}{7}\right)}{1 - \big(2\,\sin(\frac{2\pi}{7})\big)^k\,\mathbb{X}} + \frac{2\,\sin\left(\frac{4\pi}{7}\right)}{1 - \big(2\,\sin(\frac{4\pi}{7})\big)^k\,\mathbb{X}} + \frac{2\,\sin\left(\frac{8\pi}{7}\right)}{1 - \big(2\,\sin(\frac{8\pi}{7})\big)^k\,\mathbb{X}} =$$
$$= \frac{\sqrt{7} - (x_k + y_k)\,\mathbb{X} + z_k^*\,\mathbb{X}^2}{1 - z_{k-1}\,\mathbb{X} + \frac{1}{2}\big(z_{k-1}^2 - z_{2k-1}\big)\,\mathbb{X}^2 - \big(-\sqrt{7}\big)^k\,\mathbb{X}^3}, \tag{7.23}$$

and ($z_{-1} := 1$):

$$\sum_{n=0}^{\infty} z_{kn-1}\,\mathbb{X}^n = \Big(1 - \big(2\,\sin(\tfrac{2\pi}{7})\big)^k\,\mathbb{X}\Big)^{-1} + \Big(1 - \big(2\,\sin(\tfrac{4\pi}{7})\big)^k\,\mathbb{X}\Big)^{-1} +$$
$$+ \Big(1 - \big(2\,\sin(\tfrac{8\pi}{7})\big)^k\,\mathbb{X}\Big)^{-1} = \frac{3 - 2\,z_{k-1}\,\mathbb{X} + \frac{1}{2}\big(z_{k-1}^2 - z_{2k-1}\big)\,\mathbb{X}^2}{1 - z_{k-1}\,\mathbb{X} + \frac{1}{2}\big(z_{k-1}^2 - z_{2k-1}\big)\,\mathbb{X}^2 - \big(-\sqrt{7}\big)^k\,\mathbb{X}^3}. \tag{7.24}$$

By (1.1), (1.4), (4.1)–(4.3), (4.15), (4.16)–(4.18), (4.30)–(4.32) and (4.6), respectively, we get

$$\sum_{n=0}^{\infty} u_{kn}\,\mathbb{X}^n = \frac{2\,\cos\left(\frac{2\pi}{7}\right)}{1-\left(2\,\sin(\frac{2\pi}{7})\right)^k\mathbb{X}} + \frac{2\,\cos\left(\frac{4\pi}{7}\right)}{1-\left(2\,\sin(\frac{4\pi}{7})\right)^k\mathbb{X}} + \frac{2\,\cos\left(\frac{8\pi}{7}\right)}{1-\left(2\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} = \\ = \frac{-1-(v_k+w_k)\,\mathbb{X}+u_k^*\,\mathbb{X}^2}{1-z_{k-1}\,\mathbb{X}+\frac{1}{2}\left(z_{k-1}^2-z_{2k-1}\right)\mathbb{X}^2-\left(-\sqrt{7}\right)^k\mathbb{X}^3}, \quad (7.25)$$

$$\sum_{n=0}^{\infty} v_{kn}\,\mathbb{X}^n = \frac{2\,\cos\left(\frac{4\pi}{7}\right)}{1-\left(2\,\sin(\frac{2\pi}{7})\right)^k\mathbb{X}} + \frac{2\,\cos\left(\frac{8\pi}{7}\right)}{1-\left(2\,\sin(\frac{4\pi}{7})\right)^k\mathbb{X}} + \frac{2\,\cos\left(\frac{2\pi}{7}\right)}{1-\left(2\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} = \\ = \frac{-1-(u_k+w_k)\,\mathbb{X}+v_k^*\,\mathbb{X}^2}{1-z_{k-1}\,\mathbb{X}+\frac{1}{2}\left(z_{k-1}^2-z_{2k-1}\right)\mathbb{X}^2-\left(-\sqrt{7}\right)^k\mathbb{X}^3}, \quad (7.26)$$

$$\sum_{n=0}^{\infty} w_{kn}\,\mathbb{X}^n = \frac{2\,\cos\left(\frac{8\pi}{7}\right)}{1-\left(2\,\sin(\frac{2\pi}{7})\right)^k\mathbb{X}} + \frac{2\,\cos\left(\frac{2\pi}{7}\right)}{1-\left(2\,\sin(\frac{4\pi}{7})\right)^k\mathbb{X}} + \frac{2\,\cos\left(\frac{4\pi}{7}\right)}{1-\left(2\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} = \\ = \frac{-1-(u_k+v_k)\,\mathbb{X}+w_k^*\,\mathbb{X}^2}{1-z_{k-1}\,\mathbb{X}+\frac{1}{2}\left(z_{k-1}^2-z_{2k-1}\right)\mathbb{X}^2-\left(-\sqrt{7}\right)^k\mathbb{X}^3}, \quad (7.27)$$

$$\sum_{n=0}^{\infty} u_{kn}^*\,\mathbb{X}^n = \frac{2\,\cos\left(\frac{2\pi}{7}\right)}{1-\left(4\,\sin(\frac{4\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} + \frac{2\,\cos\left(\frac{4\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} + \\ + \frac{2\,\cos\left(\frac{8\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\pi}{7})\,\sin(\frac{4\pi}{7})\right)^k\mathbb{X}} = \frac{-1-(v_k^*+w_k^*)\,\mathbb{X}+\left(-\sqrt{7}\right)^k u_k\,\mathbb{X}^2}{1-\frac{1}{2}\left(z_{k-1}^2-z_{2k-1}\right)\mathbb{X}+\left(-\sqrt{7}\right)^k z_{k-1}\,\mathbb{X}^2-7^k\,\mathbb{X}^3}, \quad (7.28)$$

$$\sum_{n=0}^{\infty} v_{kn}^*\,\mathbb{X}^n = \frac{2\,\cos\left(\frac{2\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\pi}{7})\,\sin(\frac{4\pi}{7})\right)^k\mathbb{X}} + \frac{2\,\cos\left(\frac{4\pi}{7}\right)}{1-\left(4\,\sin(\frac{4\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} + \\ + \frac{2\,\cos\left(\frac{8\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} = \frac{-1-(u_k^*+w_k^*)\,\mathbb{X}+\left(-\sqrt{7}\right)^k v_k\,\mathbb{X}^2}{1-\frac{1}{2}\left(z_{k-1}^2-z_{2k-1}\right)\mathbb{X}+\left(-\sqrt{7}\right)^k z_{k-1}\,\mathbb{X}^2-7^k\,\mathbb{X}^3}, \quad (7.29)$$

$$\sum_{n=0}^{\infty} w_{kn}^*\,\mathbb{X}^n = \frac{2\,\cos\left(\frac{2\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} + \frac{2\,\cos\left(\frac{4\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\pi}{7})\,\sin(\frac{4\pi}{7})\right)^k\mathbb{X}} + \\ + \frac{2\,\cos\left(\frac{8\pi}{7}\right)}{1-\left(4\,\sin(\frac{4\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} = \frac{-1-(u_k^*+v_k^*)\,\mathbb{X}+\left(-\sqrt{7}\right)^k w_k\,\mathbb{X}^2}{1-\frac{1}{2}\left(z_{k-1}^2-z_{2k-1}\right)\mathbb{X}+\left(-\sqrt{7}\right)^k z_{k-1}\,\mathbb{X}^2-7^k\,\mathbb{X}^3}. \quad (7.30)$$

By (1.1), (4.19)–(4.21), (4.33)–(4.35) and (4.6), we obtain

$$\sum_{n=0}^{\infty} x^*_{kn}\,\mathbb{X}^n = \frac{2\,\sin\left(\frac{2\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\pi}{7})\,\sin(\frac{4\pi}{7})\right)^k\mathbb{X}} + \frac{2\,\sin\left(\frac{4\pi}{7}\right)}{1-\left(4\,\sin(\frac{4\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} + \\ + \frac{2\,\sin\left(\frac{8\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} = \frac{\sqrt{7}-(y^*_k+z^*_k)\,\mathbb{X}+\left(-\sqrt{7}\right)^k x_k\,\mathbb{X}^2}{1-\frac{1}{2}\left(z^2_{k-1}-z_{2k-1}\right)\mathbb{X}+\left(-\sqrt{7}\right)^k z_{k-1}\,\mathbb{X}^2-7^k\,\mathbb{X}^3}, \quad (7.31)$$

$$\sum_{n=0}^{\infty} y^*_{kn}\,\mathbb{X}^n = \frac{2\,\sin\left(\frac{2\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} + \frac{2\,\sin\left(\frac{4\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\pi}{7})\,\sin(\frac{4\pi}{7})\right)^k\mathbb{X}} + \\ + \frac{2\,\sin\left(\frac{8\pi}{7}\right)}{1-\left(4\,\sin(\frac{4\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} = \frac{\sqrt{7}-(x^*_k+z^*_k)\,\mathbb{X}+\left(-\sqrt{7}\right)^k y_k\,\mathbb{X}^2}{1-\frac{1}{2}\left(z^2_{k-1}-z_{2k-1}\right)\mathbb{X}+\left(-\sqrt{7}\right)^k z_{k-1}\,\mathbb{X}^2-7^k\,\mathbb{X}^3}, \quad (7.32)$$

$$\sum_{n=0}^{\infty} z^*_{kn}\,\mathbb{X}^n = \frac{2\,\sin\left(\frac{2\pi}{7}\right)}{1-\left(4\,\sin(\frac{4\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} + \frac{2\,\sin\left(\frac{4\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} + \\ + \frac{2\,\sin\left(\frac{8\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\pi}{7})\,\sin(\frac{4\pi}{7})\right)^k\mathbb{X}} = \frac{\sqrt{7}-(x^*_k+y^*_k)\,\mathbb{X}+\left(-\sqrt{7}\right)^k z_k\,\mathbb{X}^2}{1-\frac{1}{2}\left(z^2_{k-1}-z_{2k-1}\right)\mathbb{X}+\left(-\sqrt{7}\right)^k z_{k-1}\,\mathbb{X}^2-7^k\,\mathbb{X}^3}. \quad (7.33)$$

8 Jordan decomposition

For sequences $A_n(\delta)$, $B_n(\delta)$ and $C_n(\delta)$ we have the equality

$$\begin{bmatrix} A_{n+1}(\delta) \\ B_{n+1}(\delta) \\ C_{n+1}(\delta) \end{bmatrix} = \mathcal{W}(\delta) \begin{bmatrix} A_n(\delta) \\ B_n(\delta) \\ C_n(\delta) \end{bmatrix}, \qquad n \in \mathbb{N}, \quad (8.1)$$

where

$$\mathcal{W}(\delta) = \begin{bmatrix} 1 & 2\,\delta & -\delta \\ \delta & 1 & 0 \\ 0 & \delta & 1-\delta \end{bmatrix}. \quad (8.2)$$

Matrix $\mathcal{W}(\delta)$ is a diagonalized matrix, and the following decomposition can be obtained

$$\mathcal{W}(\delta) = A \cdot \begin{bmatrix} 1+\delta\,(\xi+\xi^6) & 0 & 0 \\ 0 & 1+\delta\,(\xi^2+\xi^5) & 0 \\ 0 & 0 & 1+\delta\,(\xi^3+\xi^4) \end{bmatrix} \cdot A^{-1}, \quad (8.3)$$

where

$$A = \begin{bmatrix} 1+(\xi+\xi^6)^{-1} & 1+(\xi^2+\xi^5)^{-1} & 1+(\xi^3+\xi^4)^{-1} \\ (\xi+\xi^6)^{-2}+(\xi+\xi^6)^{-1} & (\xi^2+\xi^5)^{-2}+(\xi^2+\xi^5)^{-1} & (\xi^3+\xi^4)^{-2}+(\xi^3+\xi^4)^{-1} \\ (\xi+\xi^6)^{-2} & (\xi^2+\xi^5)^{-2} & (\xi^3+\xi^4)^{-2} \end{bmatrix}$$

and

$$A^{-1} = \frac{1}{7}\begin{bmatrix} (\xi\ +\xi^6)^2 & (\xi\ +\xi^6)^3 & (\xi\ +\xi^6)^4 \\ (\xi^2+\xi^5)^2 & (\xi^2+\xi^5)^3 & (\xi^2+\xi^5)^4 \\ (\xi^3+\xi^4)^2 & (\xi^3+\xi^4)^3 & (\xi^3+\xi^4)^4 \end{bmatrix} \cdot \begin{bmatrix} -3 & 2 & 5 \\ 1 & 1 & -2 \\ 2 & -1 & -2 \end{bmatrix}.$$

It should be noticed, that characteristic polynomial $w(\lambda)$ of matrix $\mathcal{W}(\delta)$ is equal to

$$w(\lambda) = -\delta^3\, p_7\Big(\frac{\lambda-1}{\delta}\Big).$$

9 Tables

Table 1:

n	$a_n/\sqrt{7}$	$b_n/\sqrt{7}$	$c_n/\sqrt{7}$	$\alpha_n/\sqrt{7}$	$\beta_n/\sqrt{7}$	$\gamma_n/\sqrt{7}$	f_n	g_n	h_n
0	1	1	1	0	−2	1	−1	−1	−1
1	3	2	0	−2	−5	3	−2	−2	5
2	8	7	−2	−9	−15	8	−4	3	−4
3	23	24	−9	−33	−47	23	−1	−8	13
4	70	80	−33	−113	−150	70	−9	12	−16
5	220	263	−113	−376	−483	220	3	−25	38
6	703	859	−376	−1235	−1562	703	−22	41	−57
7	2265	2797	−1235	−4032	−5062	2265	19	−79	117
8	7327	9094	−4032	−13126	−16421	7327	−60	136	−193
9	23748	29547	−13126	−42673	−53295	23748	76	−253	370
10	77043	95968	−42673	−138641	−173011	77043	−177	446	−639
11	250054	311652	−138641	−450293	−561706	250054	269	−816	1186

Table 2:

n	0	1	2	3	4	5	6	7	8	9	10	11
ε_n	1	0	−2	2	2	−6	2	10	−14	−6	34	−22
ω_n	0	1	−1	−1	3	−1	−5	7	3	−17	11	23
Ψ_n	2	−1	−3	5	1	−11	9	13	−31	5	57	−67

Table 3:

n	u_n	v_n	w_n	x_n	y_n	z_n
0	-1	-1	-1	$\sqrt{7}$	$\sqrt{7}$	$\sqrt{7}$
1	$\sqrt{7}$	$-2\sqrt{7}$	0	0	0	7
2	0	-7	0	$\sqrt{7}$	$2\sqrt{7}$	$4\sqrt{7}$
3	$\sqrt{7}$	$-6\sqrt{7}$	$\sqrt{7}$	0	7	21
4	0	-28	7	0	$7\sqrt{7}$	$14\sqrt{7}$
5	0	$-21\sqrt{7}$	$7\sqrt{7}$	-7	35	70
6	-7	-105	42	$-7\sqrt{7}$	$28\sqrt{7}$	$49\sqrt{7}$
7	$-7\sqrt{7}$	$-77\sqrt{7}$	$35\sqrt{7}$	-49	147	245
8	-49	-392	196	$-42\sqrt{7}$	$112\sqrt{7}$	$175\sqrt{7}$
9	$-42\sqrt{7}$	$-287\sqrt{7}$	$154\sqrt{7}$	-245	588	882
10	-245	-1470	833	$-196\sqrt{7}$	$441\sqrt{7}$	$637\sqrt{7}$
11	$-196\sqrt{7}$	$-1078\sqrt{7}$	$637\sqrt{7}$	-1078	2303	3234

Table 4:

n	u_n^*	v_n^*	w_n^*	$x_n^*/\sqrt{7}$	$y_n^*/\sqrt{7}$	$z_n^*/\sqrt{7}$
0	-1	-1	-1	1	1	1
1	-7	7	0	1	2	-3
2	-14	7	-7	7	7	0
3	-56	42	-7	14	21	-14
4	-147	98	-49	56	63	-21
5	-490	343	-98	147	196	-98
6	-1421	980	-392	490	588	-245
7	-4459	3087	-1029	1421	1813	-833
8	-13377	9261	-3430	4459	5488	-2401
9	-41160	28469	-9947	13377	16807	-7546
10	-124852	86436	-31213	41160	51107	-22638
11	-381759	264110	-93639	124852	156065	-69629

Table 5:

n	F_n	G_n	H_n	$\widetilde{A}_n/\sqrt{7}$	$\widetilde{B}_n/\sqrt{7}$	$\widetilde{C}_n/\sqrt{7}$
0	-1	-1	-1	1	1	1
1	-4	3	3	-1	3	4
2	5	-9	-2	-8	15	19
3	-15	20	6	-42	76	95
4	31	-46	-11	-213	384	479
5	-72	103	26	-1076	1939	2418
6	160	-232	-57	-5433	9790	12208
7	-361	521	129	-27431	49429	61637
8	810	-1171	-289	-138497	249563	311200
9	-1821	2631	650	-699260	1260023	1571223
10	4091	-5912	-1460	-3530506	6361752	7932975
11	-9193	13284	3281	-17825233	32119960	40052935

Table 6:

n	$\Omega_n(\delta)$
0	3
1	$i\sqrt{7}\delta+3$
2	$-7\delta^2+2i\sqrt{7}\delta+3$
3	$-4i\sqrt{7}\delta^3-21\delta^2+3i\sqrt{7}\delta+3$
4	$21\delta^4-16i\sqrt{7}\delta^3-42\delta^2+4i\sqrt{7}\delta+3$
5	$14i\sqrt{7}\delta^5+105\delta^4-40i\sqrt{7}\delta^3-70\delta^2+5i\sqrt{7}\delta+3$
6	$-70\delta^6+84i\sqrt{7}\delta^5+315\delta^4-80i\sqrt{7}\delta^3-105\delta^2+6i\sqrt{7}\delta+3$
7	$-49i\sqrt{7}\delta^7-490\delta^6+294i\sqrt{7}\delta^5+735\delta^4-140i\sqrt{7}\delta^3-147\delta^2+7i\sqrt{7}\delta+3$

Table 7:

n	$p_n(\delta)$
0	1
1	1
2	$-2\delta^2+1$
3	$-i\sqrt{7}\delta^3-6\delta^2+1$
4	$3\delta^4-4i\sqrt{7}\delta^3-12\delta^2+1$
5	$5i\sqrt{7}\delta^5+15\delta^4-10i\sqrt{7}\delta^3-20\delta^2+1$
6	$-8\delta^6+30i\sqrt{7}\delta^5+45\delta^4-20i\sqrt{7}\delta^3-30\delta^2+1$
7	$-17i\sqrt{7}\delta^7-56\delta^6+105i\sqrt{7}\delta^5+105\delta^4-35i\sqrt{7}\delta^3-42\delta^2+1$

n	$r_n(\delta)$
0	0
1	δ
2	2δ
3	$-2\delta^3+3\delta$
4	$-i\sqrt{7}\delta^4-8\delta^3+4\delta$
5	$3\delta^5-5i\sqrt{7}\delta^4-20\delta^3+5\delta$
6	$5i\sqrt{7}\delta^6+18\delta^5-15i\sqrt{7}\delta^4-40\delta^3+6\delta$
7	$-8\delta^7+35i\sqrt{7}\delta^6+63\delta^5-35i\sqrt{7}\delta^4-70\delta^3+7\delta$

n	$s_n(\delta)$	$k_n(\delta)$
0	0	0
1	0	0
2	0	0
3	δ^3	0
4	$4\delta^3$	$-2\delta^4$
5	$-5\delta^5+10\delta^3$	$-10\delta^4$
6	$-i\sqrt{7}\delta^6-30\delta^5+20\delta^3$	$10\delta^6-30\delta^4$
7	$18\delta^7-7i\sqrt{7}\delta^6-105\delta^5+35\delta^3$	$2i\sqrt{7}\delta^7+70\delta^6-70\delta^4$

n	$l_n(\delta)$
0	0
1	0
2	δ^2
3	$3\delta^2$
4	$-3\delta^4+6\delta^2$
5	$-i\sqrt{7}\delta^5-15\delta^4+10\delta^2$
6	$8\delta^6-6i\sqrt{7}\delta^5-45\delta^4+15\delta^2$
7	$6i\sqrt{7}\delta^7+56\delta^6-21i\sqrt{7}\delta^5-105\delta^4+21\delta^2$

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References

[1] D. M. Bradley, A class of series acceleration formulae for Catalan's constant, *Ramanujan J.* **3** (1999), 149–173.

[2] R. Grzymkowski and R. Wituła, *Calculus Methods in Algebra, Part One*, WPKJS, 2000 (in Polish).

[3] K. Ireland, M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer Verlag, 1982.

[4] V. S. Shevelev, Three Ramanujan's formulas, *Kvant* **6** (1988), 52–55 (in Russian).

[5] D. O. Shklyarsky, N. N. Chentzov, I. M. Yaglom, *The USSR Olympiad Problem Book*, W. H. Freeman & Co., 1962.

[6] D. O. Shklyarsky, N. N. Chentsov, I. M. Yaglom, *Selected Problems and Theorems in Elementary Mathematics, Arithmetic and Algebra*, Mir, 1979.

[7] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, 2007.

[8] D. Surowski and P. McCombs, Homogeneous polynomials and the minimal polynomial of $\cos(2\pi/n)$, *Missouri J. Math. Sci.* **15** (2003), 4–14.

[9] W. Watkins and J. Zeitlin, The minimal polynomials of $\cos(2\pi/n)$, *Amer. Math. Monthly* **100** (1993), 471–474.

[10] R. Wituła, D. Słota and A. Warzyński, Quasi-Fibonacci numbers of the seventh order, *J. Integer Seq.* **9** (2006), Article 06.4.3.

[11] A. M. Yaglom and I. M. Yaglom, An elementary proof of the Wallis, Leibniz and Euler formulas for π, *Uspekhi Matem. Nauk* **VIII** (1953), 181–187 (in Russian).

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