Journal of Integer Sequences, Vol. 12 (2009),
Article 09.1.4

Fourier Expansions and Integral Representations for Genocchi Polynomials

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Abstract

In this paper, by using the Lipschitz summation formula, we obtain Fourier expansions and integral representations for the Genocchi polynomials. Some other new and interesting results are also shown.

1 Introduction

It is well-known that Genocchi polynomials $G_n(x)$ are defined [8] by

$$\frac{2ze^{xz}}{e^z+1} = \sum_{n=0}^{\infty} G_n(x)\frac{z^n}{n!}, \quad |z| < \pi. \tag{1}$$

In particular, the quantities $G_n \triangleq G_n(0)$ for $n \geq 0$ are called Genocchi numbers, with $G_{2n+1} = 0$ for $n \geq 1$ and, for example,

$$G_0 = 0, \quad G_1 = 1, \quad G_2 = -1, \quad G_4 = 1, \quad G_6 = -3, \quad G_8 = 17, \quad G_{10} = -155, \quad G_{12} = 2073.$$

This is Sloane's sequence A001469.

The n-th Genocchi function $\widehat{G}_n(x)$ may be introduced in the following way: for $0 \le x < 1$ and $n \ge 0$,

$$\widehat{G}_n(x) \triangleq G_n(x) \quad \text{and} \quad \widehat{G}_n(x+1) = -\widehat{G}_n(x); \tag{2}$$

for $x \in \mathbb{R}$ and $r \in \mathbb{Z}$,

$$\widehat{G}_n(x) = (-1)^{\lfloor x \rfloor} G_n(\{x\}) \quad \text{and} \quad \widehat{G}_n(x+r) = (-1)^r \widehat{G}_n(x), \tag{3}$$

where the symbols $\{x\}$ and $\lfloor x \rfloor$ denote the fractional part of x and the greatest integer not exceeding x respectively. Sometimes we also call $\widehat{G}_n(x)$ the periodic Genocchi polynomials. For convenience, in what follows, we would still employ $G_n(x)$ to stand for the periodic Genocchi polynomials, when no confusion appears in the context.

It is also well-known that Euler polynomials $E_n(x)$ for $n \ge 0$ may be defined [1, 2] by

$$\frac{2e^{xz}}{e^z+1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!}, \quad |z| < \pi. \tag{4}$$

By (1) and (4), we can obtain the following relation between Euler polynomials $E_n(x)$ and Genocchi polynomials $G_n(x)$:

$$G_n(x) = nE_{n-1}(x), \tag{5}$$

or

$$E_n(x) = \frac{1}{n+1} G_{n+1}(x). \tag{6}$$

In this paper, by using the Lipschitz summation formula, we establish Fourier expansions and integral representations for Genocchi polynomials and present an explicit formula for Genocchi polynomials at rational arguments.

2 Fourier expansions for Genocchi polynomials

Recall [5] that the Lipschitz summation formula states that

$$\sum_{n+\mu>0} \frac{e^{2\pi i(n+\mu)\tau}}{(n+\mu)^{1-\alpha}} = \frac{\Gamma(\alpha)}{(-2\pi i)^\alpha} \sum_{k \in \mathbb{Z}} \frac{e^{-2\pi i k\mu}}{(\tau+k)^\alpha}, \tag{7}$$

where $\alpha \in \mathbb{C}$, either $\Re(\alpha) > 1$ for $\mu \in \mathbb{Z}$ or $\Re(\alpha) > 0$ for $\mu \in \mathbb{R} \setminus \mathbb{Z}$, τ belongs to the upper half of the complex plane, and Γ is Euler gamma function.

In virtue of the Lipschitz summation formula (7), we obtain Fourier expansions for Genocchi polynomials as follows.

Theorem 1. *For either $n = 0$ and $0 < x < 1$ or $n > 0$ and $0 \le x \le 1$,*

$$G_n(x) = \frac{2 \cdot n!}{(\pi i)^n} \sum_{k \in \mathbb{Z}} \frac{e^{(2k-1)\pi i x}}{(2k-1)^n} \tag{8}$$

$$= \frac{4 \cdot n!}{\pi^n} \sum_{k=0}^{\infty} \frac{\cos[(2k+1)\pi x - n\pi/2]}{(2k+1)^n}. \tag{9}$$

Proof. For $0 \le x \le 1$, utilization of (1) and the generalized binomial theorem yields

$$\sum_{k=0}^{\infty} G_k(x) \frac{(2\pi i\tau)^{k-1}}{k!} = \frac{2e^{2\pi i\tau x}}{e^{2\pi i\tau}+1} = 2\sum_{k=0}^{\infty}(-1)^k e^{2\pi i(k+x)\tau}, \quad |\tau| < \frac{1}{2}. \tag{10}$$

Differentiating $n-1$ times with respect to the variable τ on both sides of (10) gives

$$\sum_{k=n}^{\infty} G_k(x) \frac{(2\pi i)^{k-1}\tau^{k-n}}{k(k-n)!} = 2(2\pi i)^{n-1}\sum_{k=0}^{\infty}(-1)^k (k+x)^{n-1} e^{2\pi i(k+x)\tau}. \tag{11}$$

On the other hand, replacing τ by $\tau + \frac{1}{2}$ and letting $\alpha = n$ and $\mu = x$ in Lipschitz summation formula (7) lead to

$$\frac{(n-1)!}{(-\pi i)^n}\sum_{k\in\mathbb{Z}} \frac{e^{-(2k+1)\pi i x}}{(2k+2\tau+1)^n} = \sum_{k=0}^{\infty}(-1)^k (k+x)^{n-1} e^{2\pi i(k+x)\tau}. \tag{12}$$

Combining (11) and (12) reveals

$$\sum_{k=n}^{\infty} G_k(x) \frac{(2\pi i)^{k-1}\tau^{k-n}}{k(k-n)!} = \frac{(-1)^n 2^n (n-1)!}{\pi i}\sum_{k\in\mathbb{Z}} \frac{e^{-(2k+1)\pi i x}}{(2k+2\tau+1)^n}. \tag{13}$$

Taking $\tau \to 0$ in (13) gives the equation (8).

Since $i^{-n} = e^{-\frac{n\pi i}{2}}$, the equation (9) follows as a direct consequence of (8). □

The following corollary is a straightforward consequence of Theorem 1.

Corollary 2. *For either $n = 0$ and $0 < x < 1$ or $n > 0$ and $0 \le x \le 1$,*

$$G_{2n}(x) = (-1)^n \frac{4\cdot(2n)!}{\pi^{2n}}\sum_{k=0}^{\infty}\frac{\cos[(2k+1)\pi x]}{(2k+1)^{2n}} \tag{14}$$

and

$$G_{2n-1}(x) = (-1)^{n-1} \frac{4\cdot(2n-1)!}{\pi^{2n-1}}\sum_{k=0}^{\infty}\frac{\sin[(2k+1)\pi x]}{(2k+1)^{2n-1}}. \tag{15}$$

Remark 3. From Fourier expansions of Euler polynomials (see [1, 3, 6]) and the equation (5), We can directly derive the formula (9) and Corollary 2. Conversely, we can also recover some known Fourier expansions of Euler polynomials by applying the relation (6), Theorem 1 and Corollary 2.

3 Integral representations for Genocchi polynomials

Now we are in a position to state and prove the uniform integral representations for Genocchi polynomials as follows.

Theorem 4. *For $n \in \mathbb{N}$ and $0 \le \Re(x) \le 1$,*

$$G_n(x) = 2n \int_0^\infty \frac{e^{\pi t}\cos(\pi x - n\pi/2) - e^{-\pi t}\cos(\pi x + n\pi/2)}{\cosh(2\pi t) - \cos(2\pi x)} t^{n-1}\,\mathrm{d}t. \tag{16}$$

Proof. Utilizing

$$G_n(x) = \frac{4 \cdot n!}{\pi^n} \sum_{k=0}^\infty \frac{\cos[(2k+1)\pi x - n\pi/2]}{(2k+1)^n}$$

and

$$\int_0^\infty t^n e^{-at}\,\mathrm{d}t = \frac{n!}{a^{n+1}} \tag{17}$$

for $n \ge 0$ and $\Re(a) > 0$ reveals that

$$\begin{aligned}
G_n(x) &= \frac{4n}{\pi^n} \sum_{k=0}^\infty \cos\left[(2k+1)\pi x - \frac{n\pi}{2}\right] \int_0^\infty t^{n-1} e^{-(2k+1)t}\,\mathrm{d}t \\
&= \frac{4n}{\pi^n} \int_0^\infty t^{n-1} \sum_{k=0}^\infty e^{-(2k+1)t} \cos\left[(2k+1)\pi x - \frac{n\pi}{2}\right] \mathrm{d}t \\
&= \frac{4n}{\pi^n} \int_0^\infty \left\{ \cos\left(\frac{n\pi}{2}\right) \sum_{k=0}^\infty e^{-(2k+1)t} \cos[(2k+1)\pi x] \right. \\
&\quad \left. + \sin\left(\frac{n\pi}{2}\right) \sum_{k=0}^\infty e^{-(2k+1)t} \sin[(2k+1)\pi x] \right\} t^{n-1}\,\mathrm{d}t.
\end{aligned}$$

By making use of

$$\sum_{k=0}^\infty e^{-(2k+1)t} \sin[(2k+1)x] = \frac{\sin x \cosh t}{\cosh(2t) - \cos(2x)}$$

and

$$\sum_{k=0}^\infty e^{-(2k+1)t} \cos[(2k+1)x] = \frac{\cos x \sinh t}{\cosh(2t) - \cos(2x)} \tag{18}$$

for $t > 0$, which may be deduced from

$$\sum_{k=0}^\infty e^{(xi-t)(2k+1)} = \frac{\cos x \sinh t + i \sin x \cosh t}{\cosh(2t) - \cos(2x)}$$

for $t > 0$, and applying the transformation $t = \pi s$, the desired formula (16) follows. □

It is easy to see that Theorem 4 implies the following integral representations for Genocchi polynomials.

Corollary 5. *For $n \in \mathbb{N}$ and $0 \leq \Re(x) \leq 1$,*

$$G_{2n-1}(x) = 4(2n-1)(-1)^{n-1} \int_0^\infty \frac{\sin(\pi x)\cosh(\pi t)}{\cosh(2\pi t) - \cos(2\pi x)} t^{2n-2}\, \mathrm{d}t \tag{19}$$

and

$$G_{2n}(x) = 8n(-1)^n \int_0^\infty \frac{\cos(\pi x)\sinh(\pi t)}{\cosh(2\pi t) - \cos(2\pi x)} t^{2n-1}\, \mathrm{d}t. \tag{20}$$

Remark 6. The uniform integral representations for Genocchi polynomials are not found in the classical literatures such as [1, 4, 6]. So the formula (16) is presumably new.

Remark 7. Our method used in this section can also be applied to establish uniform Fourier expansions and uniform integral representations for both Bernoulli and Euler polynomials.

Remark 8. Please note that Theorem 1 can be derived from Theorem 4.

4 Corollaries of uniform integral representations

Finally, we present some corollaries of Theorem 4.

Corollary 9. *For $n \in \mathbb{N}$ and $0 \leq \Re(x) \leq 1$,*

$$G_n(x) = (-1)^{n-1} \frac{4n}{\pi^n} \int_0^1 \frac{\cos(\pi x - n\pi/2) - t^2 \cos(\pi x + n\pi/2)}{t^4 - 2t^2\cos(2\pi x) + 1} (\log t)^{n-1}\, \mathrm{d}t. \tag{21}$$

Proof. Substituting

$$\cosh(2\pi t) = \frac{e^{2\pi t} + e^{-2\pi t}}{2}$$

into (16) gives

$$G_n(x) = 4n \int_0^\infty \frac{e^{\pi t}\cos(\pi x - n\pi/2) - e^{-\pi t}\cos(\pi x + n\pi/2)}{e^{2\pi t} + e^{-2\pi t} - 2\cos(2\pi x)} t^{n-1}\, \mathrm{d}t. \tag{22}$$

Further carrying out the transformation $u = e^{-\pi t}$ in (22) yields the desired formula (21). ☐

It is easy to see that the following formulas can be deduced from Corollary 9.

Corollary 10. *For $n \in \mathbb{N}$ and $0 \leq \Re(x) \leq 1$,*

$$G_{2n-1}(x) = (-1)^{n-1} \frac{4(2n-1)}{\pi^{2n-1}} \int_0^1 \frac{(1+t^2)\sin(\pi x)}{t^4 - 2t^2\cos(2\pi x) + 1} (\log t)^{2n-2}\, \mathrm{d}t \tag{23}$$

and

$$G_{2n}(x) = (-1)^{n-1} \frac{8n}{\pi^{2n}} \int_0^1 \frac{(1-t^2)\cos(\pi x)}{t^4 - 2t^2\cos(2\pi x) + 1} (\log t)^{2n-1}\, \mathrm{d}t. \tag{24}$$

In [4, p. 35, (21)], it was listed that

$$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!} B_{2n}, \tag{25}$$

where $\zeta(s)$ is the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s > 0 \tag{26}$$

and B_n for $n \ge 0$ are Bernoulli numbers defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi. \tag{27}$$

By (25) and

$$G_{2n} = 2(1 - 2^{2n}) B_{2n}, \tag{28}$$

it follows that

$$G_{2n} = \frac{(-1)^{n-1} 2^2 (1 - 2^{2n})(2n)!}{(2\pi)^{2n}} \zeta(2n). \tag{29}$$

Combining the formula

$$\int_a^x G_n(t) \,\mathrm{d}t = \frac{G_{n+1}(x) - G_{n+1}(a)}{n+1}$$

in [8] and the integral formulas

$$\int \frac{2t(1-t^2)\cos x}{t^4 - 2t^2\cos(2x) + 1} \,\mathrm{d}x = \arctan\left(\frac{2t\sin x}{1-t^2}\right) + C,$$
$$\int \frac{4t(1+t^2)\sin x}{t^4 - 2t^2\cos(2x) + 1} \,\mathrm{d}x = \log\left(\frac{t^2 - 2t\cos x + 1}{t^2 + 2t\cos x + 1}\right) + C,$$
$$\int_0^1 \frac{\log(1+t)(\log t)^{n-1}}{t} \,\mathrm{d}t = (-1)^n (n-1)! \left(\frac{1}{2^n} - 1\right) \zeta(n+1),$$
$$\int_0^1 \frac{\log(1-t)(\log t)^{n-1}}{t} \,\mathrm{d}t = (-1)^n (n-1)! \zeta(n+1)$$

in [7] with (29) and Corollary 10 gives the following corollary.

Corollary 11. *For $n \in \mathbb{N}$ and $0 \le \Re(x) \le 1$,*

$$G_{2n+1}(x) = (-1)^{n-1} \frac{4n(2n+1)}{\pi^{2n+1}} \int_0^1 \arctan\left[\frac{2t\sin(\pi x)}{1-t^2}\right] \frac{(\log t)^{2n-1}}{t} \,\mathrm{d}t \tag{30}$$

and

$$G_{2n}(x) = (-1)^{n-1} \frac{2n(2n-1)}{\pi^{2n}} \int_0^1 \log\left[\frac{t^2 - 2t\cos(\pi x) + 1}{t^2 + 2t\cos(\pi x) + 1}\right] \frac{(\log t)^{2n-2}}{t} \,\mathrm{d}t. \tag{31}$$

By (19), (24) and (31), the following integral representations for Genocchi numbers can be obtained.

Corollary 12. *For $n \geq 0$,*

$$G_{2n} = (-1)^n 4n \int_0^\infty t^{2n-1} \operatorname{csch}(\pi t)\, \mathrm{d}t$$
$$= (-1)^{n+1} \frac{8n}{\pi^{2n}} \int_0^1 \frac{(\log t)^{2n-1}}{1-t^2}\, \mathrm{d}t$$
$$= (-1)^{n-1} \frac{4n(2n-1)}{\pi^{2n}} \int_0^1 \log\left(\frac{1-t}{1+t}\right) \frac{(\log t)^{2n-2}}{t}\, \mathrm{d}t.$$

Finally, we give an explicit formula for Genocchi polynomials at rational arguments.

Theorem 13. *For $n, q \in \mathbb{N}$ and $p \in \mathbb{Z}$,*

$$G_n\left(\frac{p}{q}\right) = \frac{4 \cdot n!}{(2q\pi)^n} \sum_{j=1}^{q} \zeta\left(n, \frac{2j-1}{2q}\right) \cos\left[\frac{(2j-1)p\pi}{q} - \frac{n\pi}{2}\right], \tag{32}$$

where

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \tag{33}$$

for $\Re(s) > 1$ and $a \notin \mathbb{Z}_0^-$ is Hurwitz zeta function (see [4, 6]).

Proof. The formula (9) can be rewritten as

$$G_n(x) = \frac{4 \cdot n!}{\pi^n} \sum_{k=1}^{\infty} \frac{\cos[n\pi/2 - (2k-1)\pi x]}{(2k-1)^n}.$$

By (33) and the elementary series identity

$$\sum_{k=1}^{\infty} f(k) = \sum_{j=1}^{q} \sum_{k=0}^{\infty} f(qk+j), \quad q \in \mathbb{N}, \tag{34}$$

we obtain the desired formula (32) by setting $x = \frac{p}{q}$. This completes the proof. □

From Theorem 13, we can easily deduce the following corollary.

Corollary 14. *For $n, q \in \mathbb{N}$ and $p \in \mathbb{Z}$,*

$$G_{2n}\left(\frac{p}{q}\right) = (-1)^n \frac{4 \cdot (2n)!}{(2q\pi)^{2n}} \sum_{j=1}^{q} \zeta\left(2n, \frac{2j-1}{2q}\right) \cos\left[\frac{(2j-1)p\pi}{q}\right] \tag{35}$$

and

$$G_{2n-1}\left(\frac{p}{q}\right) = (-1)^{n-1} \frac{4 \cdot (2n-1)!}{(2q\pi)^{2n-1}} \sum_{j=1}^{q} \zeta\left(2n-1, \frac{2j-1}{2q}\right) \sin\left[\frac{(2j-1)p\pi}{q}\right]. \tag{36}$$

Remark 15. We can directly obtain the formulas (35) and (36) by using the relation (5) and the formulas (12a) and (12b) in [3]. Similarly, we can also derive the corresponding formula for Euler polynomials at rational arguments by applying the relation (6), Theorem 13 and Corollary 14.

5 Acknowledgements

The author would like to thank Professor Jeffrey Shallit and Professor Sen-Lin Guo in Canada, Professor Feng Qi in China and the anonymous referees for their valuable comments and suggestions for improving the original version of this manuscript.

The present investigation was supported, in part, by the *PCSIRT Project of the Ministry of Education of China* under Grant IRT0621 and *Innovation Program of Shanghai Municipal Education Committee of China* under Grant 08ZZ24 and *Henan Innovation Project For University Prominent Research Talents of China* under Grant 2007KYCX0021.

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2000 *Mathematics Subject Classification*: Primary 11B83; Secondary 42A16.
Keywords: Genocchi numbers, Genocchi polynomials, Fourier expansion, integral representation, Lipschitz summation formula.

(Concerned with sequence A001469.)

Received June 24 2008; revised versions received August 29 2008; October 4 2008. Published in *Journal of Integer Sequences*, December 14 2008.

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