Journal of Integer Sequences, Vol. 13 (2010),
Article 10.1.5

Divisibility by 3 of Even Multiperfect Numbers of Abundancy 3 and 4

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Abstract

We say a number is flat if it can be written as a non-trivial power of 2 times an odd squarefree number. The power is the "exponent" and the number of odd primes the "length". Let N be flat and 4-perfect with exponent a and length m. If $a \not\equiv 1 \bmod 12$, then a is even. If a is even and $3 \nmid N$ then m is also even. If $a \equiv 1 \bmod 12$ then $3 \mid N$ and m is even. If N is flat and 3-perfect and $3 \nmid N$, then if $a \not\equiv 1 \bmod 12$, a is even. If $a \equiv 1 \bmod 12$ then m is odd. If N is flat and 3 or 4-perfect then it is divisible by at least one Mersenne prime, but not all odd prime divisors are Mersenne. We also give some conditions for the divisibility by 3 of an arbitrary even 4-perfect number.

1 Introduction

We say a natural number N is multiperfect of abundancy k (or k-perfect) if $\sigma(N) = kN, (k \geq 2)$, where $\sigma(N)$ denotes the sum of all of the divisors of N. A perfect number is a multiperfect number of abundancy 2.

A number of writers have taken an interest in the presence or absence of divisibility by 3 for classes of multiperfect numbers. Carmichael [3] showed that an even multiperfect number with exactly 4 primes must be divisible by 3, and that there are only two of these numbers, one with abundancy 3 and the other with abundancy 4. If the abundancy is 3 then the number is $c_3 := 2^9 \cdot 3 \cdot 11 \cdot 31$, and if the abundancy is 4 then the number is $d_1 := 2^5 \cdot 3^3 \cdot 5 \cdot 7$. Kishore [5, 6] and also Hagis [4] showed that any odd perfect number not divisible by 3 must have at least 11 different prime factors. We also have the immediate relationship that if $3 \nmid N$, N is 3-perfect if and only if $3N$ is 4-perfect.

Now consider the table of all known multiperfect numbers of abundancy 3:

$$\begin{aligned}
c_1 &= 2^3 \cdot 3 \cdot 5, \\
c_2 &= 2^5 \cdot 3 \cdot 7, \\
c_3 &= 2^9 \cdot 3 \cdot 11 \cdot 31, \\
c_4 &= 2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73, \\
c_5 &= 2^{13} \cdot 3 \cdot 11 \cdot 43 \cdot 127, \\
c_6 &= 2^{14} \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151.
\end{aligned}$$

Definition 1. We say a number N is **flat** if its odd part is square free, i.e. if N can be written in the form $N = 2^a \cdot p_1 \cdots p_m$ where $a \geq 0$, $m \geq 0$ and $p_1 < p_2 < \cdots < p_m$, where the p_i are odd primes. If N is flat then the value of a is called its **exponent** and the value of m its **length**.

All the known 3-perfect numbers are flat and divisibility of the numbers in this table by 3 is precisely correlated with the parity of the exponent.

Similarly one could consider 4-perfect numbers with the flat shape, for example d_7 and d_{10} in the partial listing of 4-perfect numbers below:

$$\begin{aligned}
d_1 &= 2^5 \cdot 3^3 \cdot 5 \cdot 7, \\
d_2 &= 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13, \\
d_3 &= 2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19, \\
d_4 &= 2^9 \cdot 3^3 \cdot 5 \cdot 11 \cdot 31, \\
d_5 &= 2^7 \cdot 3^3 \cdot 5^2 \cdot 17 \cdot 31, \\
d_6 &= 2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31, \\
d_7 &= 2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73, \\
d_8 &= 2^{10} \cdot 3^3 \cdot 5^2 \cdot 23 \cdot 31 \cdot 89, \\
d_9 &= 2^{13} \cdot 3^3 \cdot 5 \cdot 11 \cdot 43 \cdot 127, \\
d_{10} &= 2^{14} \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151.
\end{aligned}$$

There are 36 known examples of 4-perfect numbers, of any shape, and they are all even and all divisible by 3. (None of the other over 2000 known multiperfect numbers other than the six 3-perfect and two 4-perfect numbers given above, as of the year this is being written, are flat.) We are able to demonstrate in part the apparent relationships between flatness, the divisibility of the number by 3, the parity of the exponent, and its length.

Theorem 2. *Let N be flat and 4-perfect with exponent a and length m. If $a \not\equiv 1 \bmod 12$, then a is even. If a is even and $3 \nmid N$ then m is also even. If $a \equiv 1 \bmod 12$ then $3 \mid N$ and m is even.*

Theorem 3. *Let N be flat and 3-perfect with exponent a and length m and with $3 \nmid N$. If $a \not\equiv 1 \bmod 12$ then a is even. If $a \equiv 1 \bmod 12$ then m is odd and every odd prime divisor of N is congruent to 1 modulo 3.*

Although the table of examples suggests that all even multiperfect numbers of abundancy 4 are divisible by 3, we are not able to show this completely, but have the following conditions:

Theorem 4. *Let N be 4-perfect and even and let $N = 2^a p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ be its standard prime factorization. Then in the following three cases N is divisible by 3:*
(A) If a is odd,
(B) If there exists an i with α_i odd and $p_i \equiv 2 \bmod 3$,
(C) If there exists an i with $\alpha_i \equiv 2 \bmod 3$ and $p_i \equiv 1 \bmod 3$.
If $3 \mid N$ with a even then necessarily at least one of (A) or (B) or (C) hold.

It is also of some interest to observe the existence of Mersenne primes in the factorizations of the multiperfect numbers. Of course every 2-perfect number must be divisible by a Mersenne prime. We are able to show this persists for flat multiperfect numbers of multiplicities 3 and 4, but that non-Mersenne primes must always be present:

Theorem 5. *Let N be even, flat and multiperfect. (A) If the multiplicity is not greater than 4 then N is divisible by at least one Mersenne prime. (B) If all odd prime divisors of N are Mersenne primes then N is perfect.*

2 Proofs of Theorems 2 and 3

In order to prove Theorem 2, we need a number of lemmas. First an elementary divisibility result:

Lemma 6. *Let d and n be positive integers and p be a prime. Then $d+1 \mid n+1$ if and only if $\sigma(p^d) \mid \sigma(p^n)$.*

Lemma 7. *If N is a flat 4-perfect number with exponent a, then $a \not\equiv 3 \bmod 4$ and $a \not\equiv 5 \bmod 6$.*

Proof. (1) Let $a \equiv 3 \bmod 4$ and $N = 2^a \cdot p_1 \cdots p_m$. Since $4 \mid a+1$ we have, by Lemma 6, $15 = 2^4 - 1 \mid 2^{a+1} - 1$ so we can write

$$\begin{aligned}
(2^{a+1}-1)(p_1+1)\cdots(p_m+1) &= 2^{a+2} p_1 \cdots p_m, \\
15 \cdot \frac{2^{a+1}-1}{15}(p_1+1)\cdots(p_m+1) &= 2^{a+2} 3 \cdot 5 \cdot p_3 \cdots p_m, \\
15 \cdot \frac{2^{a+1}-1}{15} \cdot 2^2 \cdot 2 \cdot 3 \cdot (p_3+1)\cdots(p_m+1) &= 2^{a+2} 3 \cdot 5 \cdot p_3 \cdots p_m,
\end{aligned}$$

and therefore 3^2 divides the right hand side, a contradiction, showing that $a \not\equiv 3 \bmod 4$.

(2) Let $a \equiv 5 \bmod 6$ and $N = 2^a \cdot p_1 \cdots p_m$. Then

$$(2^{a+1} - 1)(p_1 + 1) \cdots (p_m + 1) = 2^{a+2} p_1 \cdots p_m.$$

Since $a + 1 \equiv 0 \bmod 6$, so $5 + 1 \mid a + 1$, so $63 = 3^2 \cdot 7 = \sigma(2^5) \mid \sigma(2^a) = 2^{a+1} - 1$. So, $3^2 \mid p_1 \cdots p_m$, which is a contradiction. Therefore, $a \not\equiv 5 \bmod 6$. □

Lemma 8. *If N is a flat 4-perfect number with exponent a then $a \not\equiv 9 \bmod 12$.*

Proof. Let $\sigma(N) = 4N$ and $a = 12b + 9$ with $b \geq 0$, then

$$\begin{aligned}
\sigma(N) &= \sigma(2^a)\sigma(p_1) \cdots \sigma(p_m) \\
&= (2^{a+1} - 1)(p_1 + 1) \cdots (p_m + 1) \\
&= (2^{6b+5} + 1)(2^{6b+5} - 1)(p_1 + 1) \cdots (p_m + 1) \\
&= 2^{12b+11} p_1 p_2 \cdots p_m.
\end{aligned}$$

Note that $3 \mid 2^{6b+5} + 1$. If for any i, $(2 \leq i \leq m)$, $p_i \equiv 2 \bmod 3$, then $3 \mid p_i + 1$, which implies N has too many 3's. So we can say $p_i \equiv 1 \bmod 3$, for all i, $(2 \leq i \leq m)$.

Since

$$2^{6b+5} + 1 = 3(21x + 11) = 3(3y + 2),$$

where $x = 2^{6b-1} + 2^{6b-7} + \cdots + 2^5$ and $y = 7x + 3$, then $p_1 = 3$, and $3y + 2$ is the product of some odd prime factors of N. So $3y + 2 = \prod_{i \in I} p_i$, where $I \subseteq \{2, 3, \cdots, m\}$. Since $3y + 2 \equiv 2 \bmod 3$, but $\prod_{i \in I} p_i \equiv 1 \bmod 3$, we reach a contradiction.

Therefore, N is not a 4-perfect number, if $a \equiv 9 \bmod 12$. □

Lemma 9. *Let N be flat, 4-perfect with exponent a, $N = 2^a p_1 \cdots p_m$. If $a \equiv 1 \bmod 12$ then $3 \mid N$, for $2 \leq i \leq m$, $p_i \equiv 1 \bmod 3$, and m is even.*

Proof. Suppose $a = 12b + 1$, $b \geq 0$. We can assume $b \geq 1$. Because $\sigma(N) = 4N$ we have

$$\begin{aligned}
\sigma(N) &= (2^{12b+2} - 1)(p_1 + 1)(p_2 + 1) \cdots (p_m + 1) \\
&= 2^{12b+3} p_1 p_2 \cdots p_m, \text{ and} \\
\sigma(2^a) &= 2^{12b+2} - 1 \\
&= 3(21x + 1) \\
&= 3(3y + 1),
\end{aligned}$$

where $x = 2^{12b-4} + 2^{12b-10} + \cdots + 2^2$ and $y = 7x$.

So, $p_1 = 3$, and

$$\frac{(2^{12b+2} - 1)}{3}(p_2 + 1) \cdots (p_m + 1) = 2^{12b+1} p_2 \cdots p_m \tag{1}$$

If any $p_i \equiv 2 \bmod 3$, with $2 \leq i \leq m$, then $p_i + 1 \equiv 0 \bmod 3$, implying there would be too many 3's, so for all i with $2 \leq i \leq m$, we must have $p_i \equiv 1 \bmod 3$. Now taking the equation (1) modulo 3, we get

$$\prod_{i=2}^{m}(p_i + 1) \equiv 2^{m-1} \equiv 2^a \equiv 2 \bmod 3$$

and therefore $2^m \equiv 1 \bmod 3$ so m must be even. □

Lemma 10. *Let N be flat and 4-perfect with even exponent and suppose also $3 \nmid N$. Then the length of N is even.*

Proof. Let $N = 2^a p_1 \cdots p_m$ and $a = 2b$. Since $3 \nmid N$, for $1 \le i \le m$ each $p_i \equiv 1 \bmod 3$, and if $2^{\beta_i} \| p_i + 1$, since $(p_i + 1)2^{-\beta_i}$ is a product of primes congruent to 1 modulo 3, it must also be congruent to 1 modulo 3. Thus each β_i is odd. Since $\beta_1 + \cdots + \beta_m = 2b + 2$, m must be even. □

Now, we can provide the proof of Theorem 2 as follows:

Proof. Suppose $N = 2^a p_1 p_2 \cdots p_m$ is a 4-perfect number. By Lemma 7, we know $a \not\equiv 5 \bmod 6$, which implies $a \not\equiv 5 \bmod 12$ and $a \not\equiv 11 \bmod 12$. By Lemma 8 $a \not\equiv 9 \bmod 12$. By Lemma 7 again, since $a \not\equiv 3 \bmod 4$, we have $a \not\equiv 3 \bmod 12$ and $a \not\equiv 7 \bmod 12$. Therefore, since by hypothesis $a \not\equiv 1 \bmod 12$, a must be even. By Lemma 10 if a is even and $3 \nmid N$, then m is even. By Lemma 9 if $a \equiv 1 \bmod 12$ then $3 \mid N$ and m is even. □

Lemma 11. *If N is a flat 3-perfect number with odd exponent and $3 \nmid N$ then every odd prime divisor of N is congruent to 1 modulo 3.*

Proof. Let N be flat and 3-perfect with $N = 2^a \cdot p_1 \cdots p_m$, where the exponent a is odd, and suppose that $3 \nmid N$. Then

$$3 \cdot 2^a \cdot p_1 \cdots p_m = \sigma(2^a) \cdot (p_1 + 1) \cdots (p_m + 1). \tag{2}$$

Since a is odd and $2^2 \equiv 1 \bmod 3$, then $\sigma(2^a) = 2^{a+1} - 1 \equiv 0 \bmod 3$. Therefore

$$2^a \cdot p_1 \cdots p_m = \frac{\sigma(2^a)}{3}(p_1 + 1) \cdots (p_m + 1). \tag{3}$$

Since $3 \nmid N$, then for all $i = 1, 2, \cdots, m$, $p_i \neq 3$.

If there exists a prime factor p_i of N with $p_i \equiv 2 \bmod 3$, for some $i \in \{1, 2, \cdots, m\}$, then in the right hand side of equation (3), $(p_i + 1) \equiv 0 \bmod 3$, giving $3 \mid N$, a contradiction. □

Theorem 3 is a corollary of Theorem 2:

Proof. Let $M = 3N$. Then M is a flat 4-perfect number with the same exponent a as N. By Theorem 2, when $a \not\equiv 1 \bmod 12$, a is even. When $a \equiv 1 \bmod 12$, again by Theorem 2 the length of M is even so the length of N is odd and, by Lemma 11, every odd prime divisor of N is congruent to 1 modulo 3. □

3 Proof of Theorem 4

Definition 12. Let p and q be distinct primes. The **exponent of q modulo p,** $\exp_p q$, is the minimum natural number l such that $p \mid q^l - 1$, [1, Chapter 10].

For example $\exp_2 q = 1$ for all odd primes q. If $q \neq 3$, $\exp_3 q = q \bmod 3 = \frac{3-(q|3)}{2}$, where we have used the least positive residue and $(a|b)$ is the Legendre symbol. If $p > q$ then $\exp_p q > 1$. If $\exp_p q > 1$ and $\alpha > 1$ then $\exp_p q \mid \alpha$ if and only if $p \mid q^\alpha - 1$.

Definition 13. If p is a prime and N a natural number let $v_p(N)$ be the exponent of the highest power of p dividing N, or 0 if p does not divide N.

We are able to get the following result from Theorems 94 and 95 in Nagell [7, pp. 164–166] or Pomerance [8, p. 269]:

Lemma 14. *If p is an odd prime and $x > 1$ an integer with $p \mid x - 1$ then for every $e \geq 1$*

$$v_p\left(\frac{x^e - 1}{x - 1}\right) = v_p(e).$$

Lemma 15. *(Prime factorization of $\sigma(q^e)$)*

Let $i \geq 1$ and p be any odd prime, q a prime with $q \geq 2$ such that $p \neq q$. Then
(1) if $\exp_p q = 1$ then $p \mid \sigma(q^e)$ if and only if $p \mid e + 1$, and
(2) if $\exp_p q > 1$ then $p \mid \sigma(q^e)$ if and only if $\exp_p q \mid e + 1$.

Proof. (1) Now $\exp_p q = 1$ if and only if $p \mid q - 1$. By Lemma 14

$$v_p(\sigma(q^e)) = v_p(\frac{q^{e+1} - 1}{q - 1}) = v_p(e + 1)$$

and both implications of this part follow directly.

(2) If $\exp_p q > 1$ we have $p \nmid q - 1$ so $q - 1 \not\equiv 0 \bmod p$. Hence $p \mid \sigma(q^e) \Leftrightarrow p \mid q^{e+1} - 1 \Leftrightarrow \exp_p q \mid e + 1$. □

Now we are able to prove Theorem 4.

Proof. (A) If $N = 2^a \cdot p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ and a is odd, then since

$$(2^{a+1} - 1)\sigma(p_1^{\alpha_1}) \cdots \sigma(p_m^{\alpha_m}) = 2^{a+2} p_1^{\alpha_1} \cdots p_m^{\alpha_m}$$

and $3 \mid 2^{a+1} - 1$, one of the p_i must be 3, so $3 \mid N$.

(B) If one of the α_i is odd and the corresponding $p_i \equiv 2 \bmod 3$, then, since $2 \mid \alpha_i + 1$, by Lemma 6, $1 + p_i \mid \sigma(p_i^{\alpha_i})$, so again $3 \mid N$.

(C) Let us suppose that 3 does not divide N. Let $b = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ so $3 \nmid b$. And, because of point (A) we may assume that a is even. Then the hypothesis $\sigma(N) = 4N$ gives

$$(2^{a+1} - 1)\sigma(b) = 2^{a+2} b$$

which implies

$$\sigma(b) = 2b + \frac{2b}{2^{a+1} - 1}.$$

Suppose $b \equiv 2 \bmod 3$. Then since each divisor d of b satisfies $3 \nmid d$, each sum $b/d + d \equiv 0 \bmod 3$. But from the equation above, $\sigma(b) \equiv 0 \bmod 2$, so, since each divisor of b is odd, b has an even number of divisors. Arrange them in pairs $\{b/d, d\}$ and add to show that $3 \mid \sigma(b)$ leading to $3 \mid b$, a contradiction.

Since $3 \nmid b$, by what we have just shown this means $b \equiv 1 \bmod 3$. Then by the given hypothesis and definition of b, there is a $p_i \equiv 1 \bmod 3$ and, by (B) if any of the $p_i \equiv 2 \bmod 3$, then its corresponding α_i is even (otherwise $3 \mid b$).

Now consider the equation $\sigma(N) = 4N$:

$$(2^{a+1}-1)\sigma(p_1^{\alpha_1})\cdots\sigma(p_m^{\alpha_m}) = 2^{a+2}\cdot b$$

with a even, and take this equation modulo 3. This leads to

$$(1+\alpha_1)\cdots(1+\alpha_l) \equiv 1 \bmod 3,$$

where, if needed, we have reordered the α_i to place the non-empty set of those with $p_i \equiv 1 \bmod 3$ first, and l is the number of primes congruent to 1 modulo 3. But given an $\alpha_i \equiv 2 \bmod 3$ we obtain $0 \equiv 1 \bmod 3$, a contradiction which implies therefore $3 \mid b$, so finally $3 \mid N$.

For the necessary condition assume $N = 2^{2a}\cdot p_1^{\alpha_1}\cdots p_m^{\alpha_m}$ and $3 \mid N$. Because $2^{2a+1}-1 \equiv 1 \bmod 3$ we must have an i with $3 \mid \sigma(p_i^{\alpha_i})$. If $\exp_3 p_i = 1 \Leftrightarrow p_i \equiv 1 \bmod 3$, we must have, by Lemma 15, $3 \mid \alpha_i + 1$ so $\alpha_i \equiv 2 \mod 3$ which is (C). If however $\exp_3 p_i = 2$ then $3 \nmid p_i - 1$ and $3 \mid p_i^2 - 1$, so we must have $2 \mid \alpha_i + 1$ so α_i is odd and $p_i \equiv 2 \bmod 3$, which is (B). □

Lemma 16. *Let N be a flat 3-perfect integer, not divisible by 3 and whose exponent a is even. Let $N = 2^a p_1 p_2 \cdots p_m$ be its standard prime factorization.*

(A) There exists an unique $j \in \{1, 2, \cdots, m\}$ such that 3 divides $p_j + 1$.

(B) For each i, $1 \le i < j$, $p_i + 1$ divides N.

Proof. (A) We have $(2^{a+1}-1)(p_1+1)(p_2+1)\cdots(p_m+1) = 3N$. Since a is even, $3 \nmid 2^{a+1}-1$. Since $3 \nmid N$, there is an unique $j \in \{1, 2, \cdots, m\}$ such that $3 \mid p_j + 1$.

(B) By point (A), $p_i + 1$ is coprime with 3 and $p_i + 1$ divides $\sigma(N) = 3N$. Thus $p_i + 1 \mid N$. □

4 Mersenne Prime Divisors

Recall the statement of Theorem 5: Let N be even, flat and multiperfect. (A) If the multiplicity is not greater than 4 then N is divisible by at least one Mersenne prime. (B) If all odd prime divisors of N are Mersenne primes then N is perfect.

Proof. Let $N = 2^a p_1 \cdots p_m$ with $m \ge 1$.

(A) We can assume that $3 \nmid N$. If the multiplicity $k = 2$ then $N = 2^{p-1}M_p$ where p is prime and M_p is a Mersenne prime.

Let $k = 4$. Write

$$(2^{a+1}-1)(p_1+1)\cdots(p_m+1) = 2^{a+2}p_1\cdots p_m.$$

If p_1 is not Mersenne, the least odd divisor of $p_1 + 1$ is an odd prime $q < p_1$ which divides $p_1 \cdots p_m$ and, therefore, divides N. This contradicts the fact that p_1 is the least odd divisor of N. Thus p_1 is Mersenne.

Now let $k = 3$. If a is odd, write

$$\left(\frac{2^{a+1}-1}{3}\right)(p_1+1)\cdots(p_m+1) = 2^a p_1\cdots p_m.$$

Like in the case $k = 4$, we deduce from this equation that p_1 is Mersenne.

If a is even, by Lemma 16, there exists an unique j such that 3 divides $p_j + 1$. Then either 3 is the unique odd prime divisor of $p_j + 1$, either there is an odd prime $q_1 > 3$ which divides $p_j + 1$.

In the latter case let us suppose that no prime factor of N less than p_j is Mersenne. Then there exists an odd prime factor q_2 of $q_1 + 1$. By Lemma 16, $q_2 \mid q_1 + 1 \mid N$. Thus, $q_2 < q_1$ is an another odd prime factor of N less than p_j. Repeating this construction we get a decreasing sequence (q_n) of odd prime divisors of N. This is absurd.

In the former case, since $p_1 < p_2 \cdots < p_m$ and $p_m \nmid p_i + 1$ for $1 \le i \le m$, we must have $p_m \mid 2^{a+1} - 1$, so, assuming $j_o < m$, we can write

$$\left(\frac{2^{a+1}-1}{p_m}\right)\left(\frac{p_1+1}{2^{\alpha_1}}\right)\cdots\left(\frac{p_{j_o}+1}{3\cdot 2^{\alpha_o}}\right)\cdots\left(\frac{p_m+1}{2^{\alpha_m}}\right) = p_1 \cdots p_{m-1},$$

where each $\alpha_i = v_2(p_i+1)$, so that each term on the left is odd. Each of the terms $(p_i+1)/2^{\alpha_i}$ for $i \neq j_o$ on the left hand side can be written as a non-empty product of distinct primes from $\{p_1, \cdots, p_{m-1}\}$ and there are exactly $m - 1$ such terms. Therefore we can cancel each from both sides to derive $2^{a+1} - 1 = p_m$, so p_m is Mersenne. Also, we note that $(p_1 + 1)/2^{\alpha_1}$ must be 1, meaning that p_1 is a Mersenne prime.

If however $j_o = m$, then since there are at least two odd prime divisors [2] then the smallest odd prime divisor of N is Mersenne: canceling the 3's in the expression

$$\sigma(2^a)(p_1 + 1) \cdots 3 \cdot 2^{\alpha_o} = 3 \cdot 2^a p_1 \cdots p_m$$

shows that all potential odd prime divisors of $p_1 + 1$ are too large to actually occur so p_1 must be Mersenne.

(B) Let $\sigma(N) = kN$ for some $k \ge 2$ and suppose that all of the p_i are Mersenne. Then

$$\sigma(2^a)(p_1 + 1) \cdots (p_m + 1) = k \cdot 2^a \cdot p_1 \cdots p_m.$$

There exist primes q_i such that $p_i = 2^{q_i} - 1$. Therefore

$$(2^{a+1} - 1) \cdot 2^{q_1} \cdots 2^{q_m} = k \cdot 2^a \cdot (2^{q_1} - 1) \cdots (2^{q_m} - 1)$$

so $a \le q_1 + \cdots + q_m$ and for each i, $2^{q_i} - 1 \mid 2^{a+1} - 1$. But then Lemma 6 implies $q_i \mid a + 1$, and, since necessarily these q_i's are distinct primes, $q_1 \cdots q_m \mid a + 1$ giving

$$q_1 \cdots q_m \le a + 1 \le q_1 + q_2 + \cdots q_m + 1.$$

It follows (say by induction on m) that $m = 1$, therefore $N = 2^a \cdot p_1$. Then

$$(2^{a+1} - 1)(p_1 + 1) = k2^a p_1$$

implies $p_1 \mid 2^{a+1} - 1$ and $2^a \mid p_1 + 1$, so $2^a \le p_1 + 1 \le 2^{a+1}$ or

$$1 \le \frac{p_1 + 1}{2^a} \le 2.$$

If $(p_1 + 1)/2^a = 1$ then $p_1 + 1 = 2^a$, so $p_1 = 2^a - 1$ and $2^a - 1 \mid 2^{a+1} - 1$, which implies $a = 1$. It leads to the perfect number 6. If $(p_1 + 1)/2^a = 2$, then $p_1 = 2^{a+1} - 1$ giving $k = 2$, so N is perfect. □

5 Comments

The six flat 3-perfect numbers given in the introduction have been known for over 100 years. There are no flat multiperfect numbers known of abundancy 5 or more, so in addition to the conjecture that all even 4-perfect numbers, flat or otherwise, are divisible by 3, an additional problem in this area is to find an upper bound for the possible multiplicities of flat multiperfect numbers. We have not been able to do this.

6 Acknowledgements

We gratefully acknowledge the helpful contributions made by two referees to improve an earlier version of this paper. The second author was supported by a grant from the New Zealand Institute for Mathematics and its Applications (NZIMA).

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2000 *Mathematics Subject Classification*: Primary 11A05; Secondary 11A51.
Keywords: multiperfect number, flat number, abundancy.

(Concerned with sequences A005820, A027687.)

Received June 30 2009; revised versions received October 12 2009; January 7 2010. Published in *Journal of Integer Sequences*, January 8 2010.

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