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Integrals and Polygamma Representations for Binomial Sums

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Abstract

We consider sums involving the product of reciprocal binomial coefficient and polynomial terms and develop some double integral identities. In particular cases it is possible to express the sums in closed form, give some general results, recover some known results in Coffey and produce new identities.

1 Introduction

In a recent paper Coffey [8] considers summations over digamma and polygamma functions and develops many results, namely two of his propositions, in terms of the Riemann zeta function $\zeta(\cdot)$, are respectively equations (59) and (66a)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{p+1}(n+1)} = (-1)^p (1 - 2\ln 2) + \sum_{m=1}^{p} (-1)^{p+m} \left(2^{-m} - 1\right) \zeta(m+1) \tag{1}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^{p+1}(n+1)} = (-1)^p + \sum_{m=1}^{p} (-1)^{p+m} \zeta(m+1). \tag{2}$$

Coffey [8] also constructs new integral representations for these sums. The major aim of this paper is to investigate general binomial sums with various parameters that then enables one to give more general representations of (1) and (2), thereby generalizing the propositions of

Coffey, both in closed form in terms of zeta functions and digamma functions at possible rational values of the argument, and in double integral form. The following definitions will be useful. The Psi, or digamma function $\psi(z)$, is defined by

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z} \right) - \gamma$$

where γ denotes the Euler-Mascheroni constant and $\Gamma(z)$ is the Gamma function. Similarly

$$\psi(z+1) = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z} \right) - \gamma, \tag{3}$$

and

$$2\psi(2z) = \psi(z) + \psi\left(z + \frac{1}{2}\right) + 2\ln 2. \tag{4}$$

Sums of reciprocals of binomial coefficients appear in the calculation of massive Feynman diagrams [13] within several different approaches: for instance, as solutions of differential equations for Feynman amplitudes, through a naive ε-expansion of hypergeometric functions within Mellin-Barnes technique or in the framework of recently proposed algebraic approach [12]. There has recently been a renewed interest in the study of series involving binomial coefficients and a number of authors have obtained either closed form representation or integral representation for some particular cases of these series. The interested reader is referred to [1, 2, 3, 4, 5, 6, 9, 11, 16, 17, 18, 19, 20, 21, 22, 23, 25, 26, 27].

2 The main results

The following Lemma and well-known definition will be useful in the proof of the main theorem.

Definition 1. Let $|z| \leq 1$, $m \geq 1$ and $q \neq -1, -2, -3, \ldots$. Then

$$\sum_{r=1}^{\infty} \frac{z^r}{(q+r)^m} = \frac{(-1)^{m-1}}{(m-1)!} \int_0^1 \frac{z\ y^q\ (\ln(y))^{m-1}}{1-zy} dy. \tag{5}$$

The next Lemma deals with two infinite sums.

Lemma 2. *Let a and r be positive real numbers. Then*

$$\sum_{n=1}^{\infty} \frac{1}{n(an+r)} = \frac{H^{(1)}_{\frac{r}{a}}}{r} \quad \text{and} \tag{6}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(an+r)} = \frac{1}{r} \left\{ H^{(1)}_{\frac{r}{2a}} - H^{(1)}_{\frac{r}{a}} \right\}. \tag{7}$$

Proof.

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n\left(an+r\right)} &= \frac{1}{r}\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\frac{r}{a}}\right) \text{ and from (3)} \\
&= \frac{1}{r}\left[\gamma+\psi\left(\frac{r}{a}+1\right)\right] \\
&= \frac{H_{\frac{r}{a}}^{(1)}}{r}\text{; hence (6) is attained.}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n\left(an+r\right)} &= \frac{1}{2r}\left[\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\frac{r}{2a}}\right)-\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\frac{r-a}{2a}}\right)-\sum_{n=1}^{\infty}\frac{1}{n\left(2n-1\right)}\right] \\
&= \frac{1}{2r}\left[-\gamma+\psi\left(\frac{r}{2a}+1\right)+\gamma-\psi\left(\frac{r}{2a}+\frac{1}{2}\right)-2\ln 2\right] \\
&= \frac{1}{2r}\left[\psi\left(\frac{r}{2a}+1\right)-\psi\left(\frac{r}{2a}+\frac{1}{2}\right)-2\ln 2\right],
\end{aligned}$$

from the definition (4)

$$\psi\left(\frac{r}{2a}+\frac{1}{2}\right)=2\psi\left(\frac{r}{a}+1\right)-\psi\left(\frac{r}{2a}+1\right)-2\ln 2;\ \text{hence}$$

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n\left(an+r\right)} &= \frac{1}{2r}\left[2\psi\left(\frac{r}{2a}+1\right)-2\psi\left(\frac{r}{a}+1\right)\right] \\
&= \frac{1}{r}\left[H_{\frac{r}{2a}}^{(1)}-\gamma-H_{\frac{r}{a}}^{(1)}+\gamma\right]\text{, therefore (7) follows.}
\end{aligned}$$

□

Remark 3. In the following Corollaries and remarks we encounter harmonic numbers at possible rational values of the argument, of the form $H_{\frac{r}{a}}^{(\alpha)}$ where $r=1,2,3,...,k,\ \alpha=1,2,3,...$ and $k\in\mathbb{N}$. The polygamma function $\psi^{(\alpha)}\left(z\right)$ is defined as:

$$\psi^{(\alpha)}\left(z\right)=\frac{d^{\alpha+1}}{dz^{\alpha+1}}\left[\log\Gamma\left(z\right)\right]=\frac{d^{\alpha}}{dz^{\alpha}}\left[\psi\left(z\right)\right],\ z\neq\left\{0,-1,-2,-3,...\right\}.$$

To evaluate $H_{\frac{r}{a}}^{(\alpha)}$ we have available a relation in terms of the polygamma function $\psi^{(\alpha)}\left(z\right)$, for rational arguments z,

$$H_{\frac{r}{a}}^{(\alpha+1)}=\zeta\left(\alpha+1\right)+\frac{(-1)^{\alpha}}{\alpha!}\psi^{(\alpha)}\left(\frac{r}{a}+1\right)$$

we also define

$$H_{\frac{r}{a}}^{(1)}=\gamma+\psi\left(\frac{r}{a}+1\right),\ \text{and } H_{0}^{(\alpha)}=0.$$

The evaluation of the polygamma function $\psi^{(\alpha)}\left(\frac{r}{a}\right)$ at rational values of the argument can be explicitly done via a formula as given by Kölbig [15], (see also [14]), or Choi and Cvijovic [7] in terms of the polylogarithmic or other special functions. Some specific values are given as, many others are listed in the book [24]:

$$\psi^{(n)}\left(\frac{1}{2}\right) = (-1)^n\, n!\left(2^{n+1}-1\right)\zeta(n+1)$$

$$H^{(1)}_{\frac{1}{4}} = 4 - \frac{\pi}{2} - 3\ln(2)\,, \quad H^{(1)}_{\frac{3}{4}} = \frac{4}{3} + \frac{\pi}{2} - 3\ln(2)\,,$$

$$H^{(1)}_{\frac{1}{3}} = \frac{3}{2} - \frac{\pi}{2\sqrt{3}} - \frac{3\ln 3}{2}, \text{ and } H^{(1)}_{\frac{5}{6}} = \frac{6}{5} + \frac{\sqrt{3}\pi}{2} - \frac{3\ln(3)}{2} - 2\ln(2)\,.$$

We now state the following theorem.

Theorem 4. *Let a be a positive real number, $|t| \le 1$, $j \ge 0$, and $k \in \mathbb{N} \cup \{0\}$. Then*

$$S_{k+1}(a,\ j,\ t) = \sum_{n=1}^{\infty} \frac{t^n}{n^{k+1}\begin{pmatrix} an+j+1 \\ j+1 \end{pmatrix}} \tag{8}$$

$$= \begin{cases} \frac{(j+1)t(-1)^k}{k!}\int\limits_0^1\int\limits_0^1 \frac{(1-x)^j\ x^a(\ln(y))^k}{1-tx^a y}dxdy, & \text{for } k \ge 1 \\ at\int\limits_0^1 \frac{(1-x)^{j+1}\ x^{a-1}}{1-tx^a}dx, & \text{for } k = 0 \end{cases}.$$

$$= T_0\ {}_{a+k+1}F_{a+k}\left[\left.\begin{matrix} \overbrace{1,1,.....,1}^{(k+2)-terms}, \overbrace{\frac{a+1}{a},.......,\frac{2a-1}{a}}^{(a-1)-terms} \\ \underbrace{2,2,.....,2}_{k-terms}, \underbrace{\frac{a+j+2}{a},.......,\frac{a+j+a+1}{a}}_{a-terms} \end{matrix}\right| t\right] \tag{9}$$

where ${}_pF_q[\cdot]$ is the generalized hypergeometric function,

$$T_0 = t(j+1)\ \ B(j+1, a+1)\,,$$

and $B(\cdot,\cdot)$ is the beta function.

Proof. Consider

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{t^n}{n^{k+1}\begin{pmatrix} an+j+1 \\ j+1 \end{pmatrix}} &= \sum_{n=1}^{\infty} \frac{(j+1)\,t^n\ \ \Gamma(j+1)\,\Gamma(an+1)}{n^{k+1}\ \Gamma(an+j+2)} \\ &= (j+1)\sum_{n=1}^{\infty} \frac{t^n}{n^{k+1}} B(an+1, j+1) \end{aligned}$$

now replacing the beta function with its integral representation, we have

$$(j+1)\sum_{n=1}^{\infty}\frac{t^n}{n^{k+1}}B\left(an+1,j+1\right)=(j+1)\sum_{n=1}^{\infty}\frac{t^n}{n^{k+1}}\int\limits_0^1 x^{an}\left(1-x\right)^j dx.$$

By a justified changing the order of integration and summation we have,

$$\sum_{n=1}^{\infty}\frac{t^n}{n^{k+1}\begin{pmatrix} an+j+1\\ j+1\end{pmatrix}}=(j+1)\int\limits_0^1 (1-x)^j\sum_{n=1}^{\infty}\frac{(tx^a)^n}{n^{k+1}}dx$$

$$=\frac{(j+1)\,t\,(-1)^k}{k!}\int\limits_0^1\int\limits_0^1\frac{(1-x)^j\ \ x^a\left(\ln\left(y\right)\right)^k}{1-tx^a y}dxdy,\ \text{ for } k\geq 1$$

upon utilizing Definition 1. The case of $k=0$ follows in a similar way so that

$$\begin{aligned} S_1\left(a,\ j,\ t\right) &= \sum_{n=1}^{\infty}\frac{t^n}{n\begin{pmatrix} an+j+1\\ j+1\end{pmatrix}}\\ &= at\int\limits_0^1\frac{(1-x)^{j+1}\ \ x^{a-1}}{1-tx^a}dx;\end{aligned}$$

hence the integrals in (8) are attained. By the consideration of the ratio of successive terms $\frac{U_{n+1}}{U_n}$ where

$$U_n=\frac{t^n}{n^{k+1}\begin{pmatrix} an+j+1\\ j+1\end{pmatrix}}$$

we obtain the result (9). □

The following interesting corollaries follow from Theorem 4.

Corollary 5. *Let $t=1$ and $a>0$. Also let $j\geq 0$ and $k\geq 1$ be integers. Then*

$$\begin{aligned} S_{k+1}\left(a,\ j,\ 1\right) &= \sum_{n=1}^{\infty}\frac{1}{n^{k+1}\begin{pmatrix} an+j+1\\ j+1\end{pmatrix}}\\ &= \frac{(j+1)\,(-1)^k}{k!}\int\limits_0^1\int\limits_0^1\frac{(1-x)^j\ \ x^a\left(\ln\left(y\right)\right)^k}{1-x^a y}dxdy,\ \textit{ for } k\geq 1 \qquad (10)\\ &= \sum_{s=0}^{k-1}A_s\,(j+1)!\,\zeta\left(k+1-s\right)+\sum_{r=1}^{j+1}(-1)^{r+k+1}\left(\frac{a}{r}\right)^k\begin{pmatrix} j+1\\ r\end{pmatrix}H^{(1)}_{\frac{r}{a}} \qquad (11)\end{aligned}$$

where

$$A_s = \lim_{n\to 0}\left[\frac{1}{s!}\frac{d^s}{dn^s}\left\{\frac{n^k}{n^k\ \prod_{r=1}^{j+1}(an+r)}\right\}\right],\ \ s=0,1,2,...k-1. \tag{12}$$

Proof. By expansion,

$$\sum_{n=1}^{\infty}\frac{1}{n^{k+1}\begin{pmatrix} an+j+1 \\ j+1 \end{pmatrix}} = \sum_{n=1}^{\infty}\frac{(j+1)!}{n^{k+1}(an+1)_{j+2}} = \sum_{n=1}^{\infty}\frac{(j+1)!}{n^{k+1}\prod_{r=1}^{j+1}(an+r)}$$
$$= \sum_{n=1}^{\infty}\frac{(j+1)!}{n}\left[\sum_{s=0}^{k-1}\frac{A_s}{n^{k-s}} + \sum_{r=1}^{j+1}\frac{B_r}{an+r}\right],$$

where

$$B_r = \lim_{n\to\left(-\frac{r}{a}\right)}\left\{\frac{an+r}{\prod_{r=1}^{j+1}(an+r)}\right\} = \frac{(-1)^{r+k+1}\,r}{(j+1)!}\left(\frac{a}{r}\right)^k\begin{pmatrix} j+1 \\ r \end{pmatrix},$$

A_s is defined by (12). Hence, after interchanging the sums, we have

$$\sum_{n=1}^{\infty}\frac{1}{n^{k+1}\begin{pmatrix} an+j+1 \\ j+1 \end{pmatrix}}$$
$$= (j+1)!\left[\sum_{s=0}^{k-1}A_s\sum_{n=1}^{\infty}\frac{1}{n^{k+1-s}} + \sum_{r=1}^{j+1}B_r\sum_{n=1}^{\infty}\frac{1}{n(an+r)}\right]$$
$$= (j+1)!\sum_{s=0}^{k-1}A_s\,\zeta(k+1-s) + \sum_{r=1}^{j+1}(-1)^{r+k+1}\left(\frac{a}{r}\right)^k\begin{pmatrix} j+1 \\ r \end{pmatrix}H^{(1)}_{\frac{r}{a}}$$

upon utilizing Lemma 2, which is the result (11). The degenerate case, for $j=-1$, gives the known result

$$\sum_{n=1}^{\infty}\frac{1}{n^{k+1}} = \zeta(k+1).$$

The integral (10) follows from the integral in (8). □

A similar result is evident for the case $t=-1$.

Corollary 6. *Let $t=-1$ and $a>0$. Also let $j\geq 0$ and $k\geq 1$ be integers. Then*

$$S_{k+1}\left(a,\ j,\ -1\right)=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n}}{n^{k+1}\left(\begin{array}{c}an+j+1\\ j+1\end{array}\right)}$$

$$=\frac{\left(j+1\right)\left(-1\right)^{k+1}}{k!}\int_{0}^{1}\int_{0}^{1}\frac{\left(1-x\right)^{j}\ \ x^{a}\left(\ln\left(y\right)\right)^{k}}{1+x^{a}y}dxdy,\ \ for\ k\geq 1$$

$$=\sum_{s=0}^{k-1}A_{s}\left(j+1\right)!\left(2^{s-k}-1\right)\zeta\left(k+1-s\right) \tag{13}$$

$$+\sum_{r=1}^{j+1}\left(-1\right)^{r+k+1}\left(\frac{a}{r}\right)^{k}\left(\begin{array}{c}j+1\\ r\end{array}\right)\left(H_{\frac{r}{a}}^{(1)}-H_{\frac{r}{2a}}^{(1)}\right)$$

Proof. The proof, uses (7) and follows the same details as that of Corollary 5, and will not be given here. □

The addition and subtraction of (11) and (13) gives us the following representations.

Remark 7. Let $a>0$ and let $j\geq 0$ and $k\geq 1$ be integers. Then

$$2^{1-k}\sum_{n=1}^{\infty}\frac{1}{n^{k+1}\left(\begin{array}{c}2an+j+1\\ j+1\end{array}\right)}$$

$$=\sum_{s=0}^{k-1}A_{s}\left(j+1\right)!\ 2^{s-k}\ \zeta\left(k+1-s\right)+\sum_{r=1}^{j+1}\left(-1\right)^{r+k+1}\left(\frac{a}{r}\right)^{k}\left(\begin{array}{c}j+1\\ r\end{array}\right)\left(2H_{\frac{r}{a}}^{(1)}-H_{\frac{r}{2a}}^{(1)}\right)$$

and

$$\sum_{n=1}^{\infty}\frac{1}{\left(2n-1\right)^{k+1}\left(\begin{array}{c}2an-a+j+1\\ j+1\end{array}\right)}$$

$$=\sum_{s=0}^{k-1}A_{s}\left(j+1\right)!\ \left(2-2^{s-k}\right)\ \zeta\left(k+1-s\right)+\sum_{r=1}^{j+1}\left(-1\right)^{r+k+1}\left(\frac{a}{r}\right)^{k}\left(\begin{array}{c}j+1\\ r\end{array}\right)H_{\frac{r}{2a}}^{(1)}.$$

We give the following example to illustrate some of the above identities.

Example 8. Let $k=4$, from (12)

$$A_{0}=\frac{1}{\left(j+1\right)!},\ \ A_{1}=-\frac{aH_{j+1}^{(1)}}{\left(j+1\right)!},\ \ A_{2}=\frac{a^{2}\left(\left(H_{j+1}^{(1)}\right)^{2}+H_{j+1}^{(2)}\right)}{2\left(j+1\right)!}$$

$$A_{3}=-\frac{a^{3}\left(\left(H_{j+1}^{(1)}\right)^{3}+3H_{j+1}^{(1)}H_{j+1}^{(2)}+2H_{j+1}^{(3)}\right)}{6\left(j+1\right)!}$$

therefore

$$\sum_{n=1}^{\infty}\frac{1}{n^5\begin{pmatrix} an+j+1 \\ j+1 \end{pmatrix}} = (j+1)!\left[A_0\zeta(5)+A_1\zeta(4)+A_2\zeta(3)+A_3\zeta(2)\right] + \sum_{r=1}^{j+1}(-1)^{r+1}\left(\frac{a}{r}\right)^4\begin{pmatrix} j+1 \\ r \end{pmatrix}H^{(1)}_{\frac{r}{a}}$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5\begin{pmatrix} an+j+1 \\ j+1 \end{pmatrix}} = (j+1)!\left[-\frac{15A_0}{16}\zeta(5)-\frac{7A_1}{8}\zeta(4)-\frac{3A_2}{4}\zeta(3)-\frac{A_3}{2}\zeta(2)\right] + \sum_{r=1}^{j+1}(-1)^{r}\left(\frac{a}{r}\right)^4\begin{pmatrix} j+1 \\ r \end{pmatrix}\left(H^{(1)}_{\frac{r}{a}}-H^{(1)}_{\frac{r}{2a}}\right).$$

Remark 9. The very special case of $a=1$ and $j=0$ allows one to evaluate

$$\begin{aligned} A_s &= \lim_{n\to 0}\left[\frac{1}{s!}\frac{d^s}{dn^s}\left\{\frac{n^k}{n^k\ (n+1)}\right\}\right],\ s=0,1,2,...k-1 \\ &= (-1)^s, \end{aligned}$$

and from (11) and (13) we can easily obtain (2) and (1).

A recurrence relation for a degenerate case, $j=0$, of Theorem 4 is embodied in the following corollary.

Corollary 10. *Let the conditions of Theorem 4 hold with $j=0$ and put*

$$\begin{aligned} S^0_{k+1} &: = S^0_{k+1}(a,t) = \sum_{n=1}^{\infty}\frac{t^n}{n^{k+1}(an+1)} \\ &= \frac{t}{a+1}\ {}_{k+3}F_{k+2}\left[\left.\begin{array}{c} \overbrace{1,1,.....,1}^{(k+2)-terms},\frac{a+1}{a} \\ \underbrace{2,2,.....,2}_{(k+1)-terms},\frac{2a+1}{a} \end{array}\right| t\right], \end{aligned}$$

then

$$S^0_{k+1}+aS^0_k=\mathrm{Li}_{k+1}(t),\ \textit{for } k\geq 1$$

with solution

$$\begin{aligned} S^0_{k+1} &= (-a)^k S^0_1+\sum_{r=1}^{k}(-a)^{r-1}\,t\ \Phi(t,k+2-r,1) \\ &= (-a)^k S^0_1+\sum_{r=1}^{k}(-a)^{r-1}\mathrm{Li}_{k+2-r}(t) \end{aligned}$$

where

$$S_1^0(a,t)=\sum_{n=1}^{\infty}\frac{t^n}{n(an+1)}=\frac{t}{a+1}\quad {}_3F_2\left[\begin{array}{c}1,1,\frac{a+1}{a}\\ 2,\frac{2a+1}{a}\end{array}\middle|\, t\right]$$

and Φ, Li *are the Lerch transcendent and the polylogarithm respectively.*

Proof. We notice that

$$S_{k+1}^0+aS_k^0=\sum_{n=1}^{\infty}\frac{t^n}{n^{k+1}}=\mathrm{Li}_{k+1}(t),$$

and hence the solution follows by iteration. □

Related results may be seen in Coffey [10, Lemmas 1 and 2].

Some examples are as follows:

• For $t=1$, we know that $\mathrm{Li}_{k+2-r}(1)=\zeta(k+2-r)$. Hence

$$S_{k+1}^0(a,1)=(-a)^k S_1^0(a,1)+\sum_{r=1}^{k}(-a)^{r-1}\zeta(k+2-r),\ \text{for } k\geq 1.$$

When $a=1$, we obtain Coffey's [8] result, by noting that, from (6), $S_1^0(1,1)=1$

$$S_{k+1}^0(1,1)=(-1)^k+\sum_{r=1}^{k}(-1)^{r-1}\ \zeta(k+2-r).$$

When $a=\frac{1}{2}$,

$$S_{k+1}^0\left(\frac{1}{2},1\right)=\frac{3(-1)^k}{2^{k+1}}+\sum_{r=1}^{k}\frac{(-1)^{r-1}}{2^{r-1}}\zeta(k+2-r).$$

Similarly

$$\begin{aligned}S_{k+1}^0(8,1) &= (-1)^k 2^{3(k+1)}-(-1)^k 2^{3k-1}\pi\sqrt{3+2\sqrt{2}}-(-1)^k 2^{3k+2}\ln(2)\\ &\quad -(-1)^k 2^{3k-3/2}\ln\left(3+2\sqrt{2}\right)+\sum_{r=1}^{k}(-2)^{3(r-1)}\zeta(k+2-r).\end{aligned}$$

• For $t=-1$, $\mathrm{Li}_{k+2-r}(-1)=\eta(k+2-r)=\left(2^{r-k-1}-1\right)\zeta(k+2-r)$, where $\eta(\cdot)$ is the Dirichlet Eta function. Hence

$$S_{k+1}^0(a,-1)=(-a)^k S_1^0(a,-1)+\sum_{r=1}^{k}(-a)^{r-1}\left(2^{r-k-1}-1\right)\zeta(k+2-r),\ \text{for } k\geq 1.$$

When $a=1$, we obtain Coffey's [8] result, by noting that

$$S_{k+1}^0(1,-1)=(-1)^k(1-2\ln 2)+\sum_{r=1}^{k}(-1)^{r-1}\ \left(2^{r-k-1}-1\right)\zeta(k+2-r).$$

When $a = 4$,

$$S^0_{k+1}(4,-1) = (-2)^k\left(4-\sqrt{2}\ln\left(1+\sqrt{2}\right)-\frac{\pi}{2}-\ln 2\right) + \sum_{r=1}^{k}(-1)^{r-1}\ 2^{2(r-1)}\left(2^{r-k-1}-1\right)\zeta(k+2-r),$$

and similarly

$$\begin{aligned} S^0_{k+1}\left(\frac{1}{8},-1\right) &= -(-1)^k\frac{533}{840}2^{-3k}+\sum_{r=1}^{k}\frac{(-1)^{r-1}}{8^{r-1}}\ \left(2^{r-k-1}-1\right)\zeta(k+2-r) \\ &= -\frac{8}{9}\ {}_{k+3}F_{k+2}\left[\begin{array}{c|} \overbrace{1,1,.....,1}^{(k+2)-terms},9 \\ \underbrace{2,2,.....,2}_{(k+1)-terms},10 \end{array} -1\right]. \end{aligned}$$

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