Journal of Integer Sequences, Vol. 13 (2010),
Article 10.8.1

A Survey of Gcd-Sum Functions

László Tóth
Department of Mathematics
University of Pécs
Ifjúság u. 6
7624 Pécs
Hungary
ltoth@gamma.ttk.pte.hu

Abstract

We survey properties of the gcd-sum function and of its analogs. As new results, we establish asymptotic formulae with remainder terms for the quadratic moment and the reciprocal of the gcd-sum function and for the function defined by the harmonic mean of the gcd's.

1 Introduction

The gcd-sum function, called also Pillai's arithmetical function is defined by

$$P(n) = \sum_{k=1}^{n} \gcd(k, n). \tag{1}$$

By grouping the terms according to the values of $\gcd(k, n)$ we have

$$P(n) = \sum_{d \mid n} d\, \phi(n/d), \tag{2}$$

where ϕ is Euler's function.

Properties of the function P, which arise from the representation (2), as well as various generalizations and analogs of it were investigated by several authors. It is maybe not surprising that some of these results were rediscovered for many times.

It follows from (2) that the arithmetic mean of $\gcd(1, n), \ldots, \gcd(n, n)$ is given by

$$A(n) = \frac{P(n)}{n} = \sum_{d \mid n} \frac{\phi(d)}{d}. \tag{3}$$

The harmonic mean of $\gcd(1,n),\ldots,\gcd(n,n)$ is

$$H(n) = n\left(\sum_{k=1}^{n}\frac{1}{\gcd(k,n)}\right)^{-1} = n^2\left(\sum_{d|n} d\,\phi(d)\right)^{-1}. \tag{4}$$

In the present paper we give a survey of the gcd-sum function and of its analogs. We also prove the following results concerning the functions A and H, which seem to have not appeared in the literature.

Our first result is an asymptotic formula with remainder term for the quadratic moment of the function A.

Let $\tau(n)$ denote, as usual, the number of divisors of n. Let α_4 be the exponent appearing in the divisor problem for the function $\tau_4(n) = \sum_{d_1d_2d_3d_4=n} 1$, that is

$$\sum_{n\le x}\tau_4(n) = x(K_1\log^3 x + K_2\log^2 x + K_3\log x + K_4) + \mathcal{O}(x^{\alpha_4+\varepsilon}), \tag{5}$$

for any $\varepsilon > 0$, where $K_1 = 1/6, K_2, K_3, K_4$ are constants. It is known that $\alpha_4 \le 1/2$ (result of Hardy and Littlewood) and it is conjectured that $\alpha_4 = 3/8$, cf. Titchmarsh [46, Ch. 12], Ivić et al. [22, Section 4]. If this conjecture were true, then it would follow that $\alpha_4 < 1/2$.

Theorem 1. *i) For any* $\varepsilon > 0$,

$$\sum_{n\le x} A^2(n) = x(C_1\log^3 x + C_2\log^2 x + C_3\log x + C_4) + \mathcal{O}(x^{1/2+\varepsilon}), \tag{6}$$

where C_1, C_2, C_3, C_4 *are constants,*

$$C_1 = \frac{1}{\pi^2}\prod_p\left(1+\frac{1}{p^3}-\frac{4}{p(p+1)}\right), \tag{7}$$

C_2, C_3, C_4 *are given by* (50) *in terms of the constants appearing in the asymptotic formula for* $\sum_{n\le x}\tau^2(n)$.

ii) Assume that $\alpha_4 < 1/2$. *Then the error term in* (6) *is* $\mathcal{O}(x^{1/2}\delta(x))$, *where*

$$\delta(x) = \exp(-c(\log x)^{3/5}(\log\log x)^{-1/5}), \tag{8}$$

with a positive constant c.

iii) If the Riemann hypothesis (RH) is true, then for any real x *sufficiently large the error term in* (6) *is* $\mathcal{O}(x^{(2-\alpha_4)/(5-4\alpha_4)}\eta(x))$, *where*

$$\eta(x) = \exp((\log x)^{1/2}(\log\log x)^{14}). \tag{9}$$

Remark 2. Let $M(x) = \sum_{n\le x}\mu(n)$ denote the Mertens function, where μ is the Möbius function. The error term of iii) comes from the estimate $M(x) \ll \sqrt{x}\,\eta(x)$, the best up to now, valid under RH and for x large, due to Soundararajan [42]. Note that in a preprint not yet published Balazard and Roton [3] have shown that the slightly better estimate $M(x) \ll \sqrt{x}\exp((\log x)^{1/2}(\log\log x)^{5/2+\varepsilon})$ holds assuming RH, for every $\varepsilon > 0$ sufficiently small.

Remark 3. If α_4 is near $3/8$ and RH is true, then the exponent $(2-\alpha_4)/(5-4\alpha_4)$ is near $13/28 \approx 0.4642$.

Our second result is regarding the function H.

Theorem 4. *For any $\varepsilon > 0$,*

$$\sum_{n\le x} \frac{H(n)}{n} = C_5 \log x + C_6 + \mathcal{O}\left(x^{-1+\varepsilon}\right), \tag{10}$$

where C_5 and C_6 are constants,

$$C_5 = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \prod_p \left(1 - \frac{p-1}{p^2-p+1} \sum_{a=1}^{\infty} \frac{p^{2a-1}+1}{p^{a-1}(p^{2a+1}+1)}\right). \tag{11}$$

Corollary 5. *For any $\varepsilon > 0$,*

$$\sum_{n\le x} H(n) = C_5 x + \mathcal{O}(x^{\varepsilon}), \tag{12}$$

hence the mean value of the function H is C_5.

Note that the arithmetic mean of the orders of elements in the cyclic group C_n of order n is

$$\alpha(n) = \frac{1}{n} \sum_{k=1}^{n} \frac{n}{\gcd(k,n)} = \frac{1}{n} \sum_{d\mid n} d\phi(d), \tag{13}$$

hence $H(n)/n = 1/\alpha(n)$. The function α and its average order were investigated by von zur Gathen et al. [17] and Bordellès [6].

The paper is organized as follows. Properties of the gcd-sum function P are presented in Section 2. Generalizations and connections to other functions are given in Section 3. Section 4 includes the proofs of Theorems 1 and 4. Several analogs of the gcd-sum function are surveyed in Section 5 and certain open problems are stated in Section 6. Finally, as added in proof, asymptotic formulae for $\sum_{n\le x} 1/P(n)$ and $\sum_{n\le x} 1/g(n)$, where g is any multiplicative analog of P discussed in the present paper are given in Section 7.

Throughout the paper we insist on the asymptotic properties of the functions. We remark that some other aspects, including arithmetical properties and generalizations of the gcd-sum function are surveyed by Bege [4, Ch. 3] and Haukkanen [20].

2 Properties of the gcd-sum function

According to (2), $P = E*\phi$ in terms of the Dirichlet convolution, with the notation $E(n) = n$. It follows that P is multiplicative and for any prime power p^a $(a \ge 1)$,

$$P(p^a) = (a+1)p^a - ap^{a-1}, \tag{14}$$

in particular $P(p) = 2p-1$, $P(p^2) = 3p^2 - 2p$, etc.

$(P(n))_{n\geq 1}$ is sequence A018804 in Sloane's Encyclopedia. It is noted there that $P(n)$ is the number of times the number 1 appears in the character table of the cyclic group C_n. Also, $P(n)$ is the number of incongruent solutions of the congruence $xy \equiv 0 \pmod{n}$.

The bounds $2n-1 \leq P(n) \leq n\tau(n)$ $(n \geq 1)$ follow at once by the definition (1) and (14), respectively. The Dirichlet series of P is given by

$$\sum_{n=1}^{\infty} \frac{P(n)}{n^s} = \frac{\zeta^2(s-1)}{\zeta(s)} \quad (\operatorname{Re} s > 2). \tag{15}$$

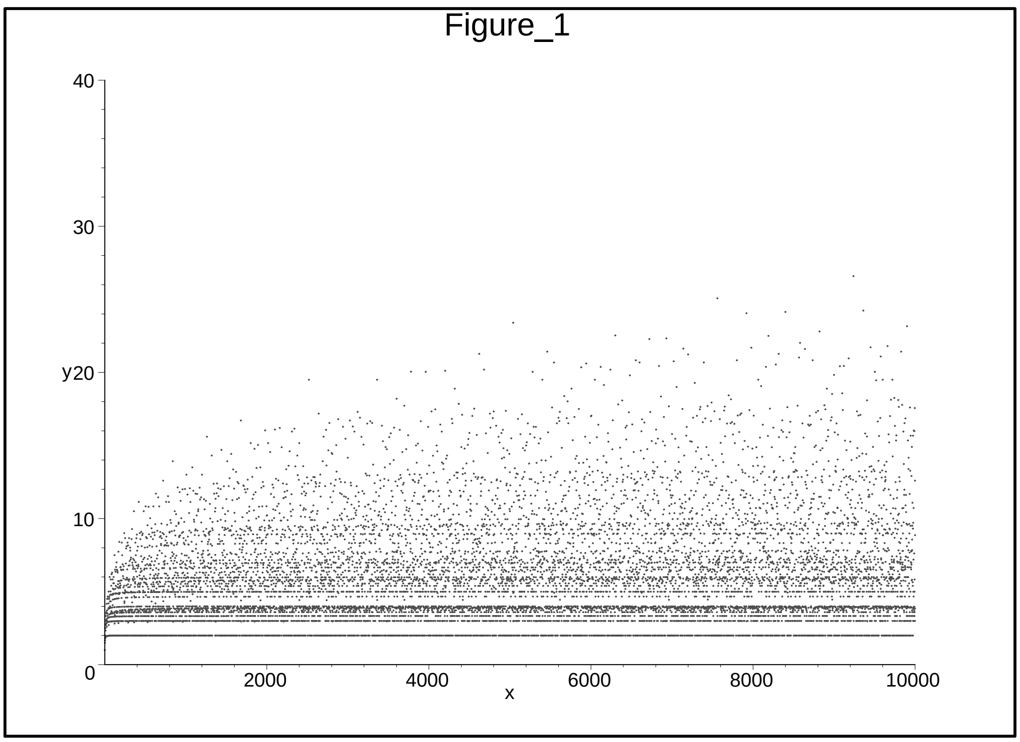
The convolution method applied for (2) leads to the asymptotic formulae

$$\sum_{n\leq x} P(n) = \frac{1}{2\zeta(2)} x^2 \log x + \mathcal{O}(x^2), \tag{16}$$

$$\sum_{n\leq x} \frac{P(n)}{n} = \frac{1}{\zeta(2)} x \log x + \mathcal{O}(x). \tag{17}$$

It follows that the average order of $A(n) = P(n)/n$ is $\log n/\zeta(2)$, that is, for $1 \leq k \leq n$ the average value of (k,n) is $\log n/\zeta(2)$, where $1/\zeta(2) = 6/\pi^2 \approx 0.607927$.

Figure 1 is a plot of the function $A(n)$ for $1 \leq n \leq 10\,000$, produced using Maple.



Observe that writing $\phi = E * \mu$, by (2) we have $P = E * E * \mu = E\tau * \mu$, that is

$$P(n) = \sum_{d|n} d\tau(d)\mu(n/d). \tag{18}$$

This follows also from (15). Note that P is a rational arithmetical function of order $(2,1)$, in the sense that P is the convolution of two completely multiplicative functions and of another one which is the inverse (under convolution) of a completely multiplicative function, cf. Haukkanen [19].

Using the representation (18) the following more precise asymptotic formula can be derived: for every $\varepsilon > 0$,

$$\sum_{n \le x} P(n) = \frac{x^2}{2\zeta(2)} \left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) + \mathcal{O}(x^{1+\theta+\varepsilon}), \tag{19}$$

where γ is Euler's constant and θ is the number appearing in Dirichlet's divisor problem, that is

$$\sum_{n \le x} \tau(n) = x \log x + (2\gamma - 1)x + \mathcal{O}(x^{\theta+\varepsilon}). \tag{20}$$

It is known that $1/4 \le \theta \le 131/416 \approx 0.3149$, where the upper bound, the best up to date, is the result of Huxley [21]. To be more precise, the result of Huxley [21] says that the error term in (20) is

$$\mathcal{O}(x^{131/416} (\log x)^{26947/8320}) \tag{21}$$

with $26947/8320 \approx 3.2388$.

The study of the gcd-sum function P and of its generalization P_f given by (29) goes back to the work of Cesàro in the years 1880'. The formula

$$\sum_{k=1}^{n} f(\gcd(k,n)) = \sum_{d \mid n} f(d) \phi(n/d), \tag{22}$$

valid for an arbitrary arithmetical function f, is sometimes referred to as Cesàro's formula, cf. Dickson [15, p. 127, 293], Sándor and Crstici [36, p. 182], Haukkanen [20].

The function P was rediscovered by Pillai [35] in 1933, showing formula (2) and that

$$\sum_{d \mid n} P(d) = n\tau(n) = \sum_{d \mid n} \sigma(d) \phi(n/d), \tag{23}$$

$\sigma(n)$ denoting, as usual, the sum of divisors of n.

Properties of P, including (2), (15) (16), (17) were discussed by Broughan [8] without referring to the work of Cesàro and Pillai.

Formulae (18) and (19) were obtained, even for a more general function, by Chidambaraswamy and Sitaramachandrarao [10]. They also proved the following result concerning the maximal order of $P(n)$:

$$\limsup_{n \to \infty} \frac{\log(P(n)/n) \log \log n}{\log n} = \log 2, \tag{24}$$

which is well known for the function $\tau(n)$ instead of $P(n)/n$. (18) and (19) were obtained later also by Bordellès [5].

In a recent paper Bordellès [7, Th. 8, eq. (xi)] pointed out that, according to (21), the error term of (19) is $\mathcal{O}(x^{547/416}(\log x)^{26947/8320})$.

The asymptotic formula (19) was obtained earlier by Kopetzky [25] with a weaker error term. The same formula (19) was derived also by Broughan [8, Th. 4.7] with the weaker error term $\mathcal{O}(x^{3/2}\log x)$, but the coefficient of x^2 is not correct ($\zeta^2(2)/2\zeta(3)$ is given).

One has

$$\sum_{m,n\le x} \gcd(m,n) = \frac{1}{\zeta(2)}x^2\log x + cx^2 + \mathcal{O}(x^{1+\theta+\varepsilon}), \tag{25}$$

with a suitable constant c, which follows from (19) using the connection formula between the two types of summation, namely $\sum_{n\le x} P(n) = \sum_{m\le n\le x}\gcd(m,n)$ and $\sum_{m,n\le x}\gcd(m,n)$, cf. Section 3. Formula (25) was given by Diaconis and Erdős [14] with the weaker error term $\mathcal{O}(x^{3/2}\log x)$.

The study of asymptotic formulae with error terms of $\sum_{n\le x} P(n)/n^s$ for real values of s was initiated by Broughan [8, 9] and continued by Tanigawa and Zhai [45].

Alladi [1] gave asymptotic formulae for $\sum_{k=1}^n(\gcd(k,n))^s$ and $\sum_{k=1}^n(\mathrm{lcm}[k,n])^s$ ($s\ge 1$). Sum functions of the gcd's and lcm's were also considered by Gould and Shonhiwa [18, 41] and Bordellès [6].

The function P appears in the number theory books of Andrews [2, p. 91, Problem 10], Niven and Zuckerman [32, Section 4.4, Problem 6] (the author of this survey met the function P for the first time in the Hungarian translation of this book) and McCarthy [29, p. 29, Problem 1.3].

See also the proposed problems of Shallit [39] (P is multiplicative and (14)), Teuffel [44] (formulae (2), (16) and asymptotic formulae for $\sum_{k=1}^n(\gcd(k,n))^s$ with $s\ge 2$) and Lau [26] (asymptotic formulae for $\sum_{1\le i,j\le n}\gcd(i,j)$ and $\sum_{1\le i,j\le n}\mathrm{lcm}[i,j]$).

In a recent paper de Koninck and Kátai [24] investigated two general classes of functions, one of them including $A(n)=P(n)/n$, and showed that

$$\sum_{p\le x} A(p-1) = Lx + \mathcal{O}(x(\log\log x)^{-1}), \tag{26}$$

where the sum is over the primes $p\le x$ and L is a constant given by

$$L = \frac{1}{2}\prod_p\left(1+\frac{1}{p(p-1)}\right)\sum_{n=1}^{\infty}\frac{F(n)\tau(n)}{n}\prod_{p\mid n}\left(1+\frac{p}{(p-1)^2}\right)^{-1}, \tag{27}$$

where F is the multiplicative function defined by $F(p^a) = -\frac{a/(a+1)}{p}$ for any prime power p^a ($a\ge 1$).

3 Generalizations, connections to other functions

The gcd-sum function P can be generalized in various directions. For example:

i) One can investigate the function

$$P_s(n) = \sum_{k=1}^n (\gcd(k,n))^s, \tag{28}$$

where s is a real number. More generally, for an arbitrary arithmetical function f let

$$P_f(n) = \sum_{k=1}^{n} f(\gcd(k,n)), \tag{29}$$

mentioned already in Section 2.

ii) A multidimensional version is the function

$$P_{(k)}(n) = \sum_{1\le i_1,\ldots,i_k\le n} \gcd(i_1,\ldots,i_k,n). \tag{30}$$

iii) If g is a nonconstant polynomial with integer coefficients let

$$P^{(g)}(n) = \sum_{k=1}^{n} \gcd(g(k),n). \tag{31}$$

iv) If A is a regular convolution and $(k,n)_A$ is the greatest divisor d of k such that $d \in A(n)$ (see for ex. McCarthy [29, Ch. 4]) let

$$P_A(n) = \sum_{k=1}^{n} (k,n)_A. \tag{32}$$

These generalizations can also be combined. The general function investigated by Tóth [50] includes all of i)-iv) given above (it is even more general). Tóth [48] considered a generalization defined for arithmetical progressions. We do not deal here with these generalizations, see [4, 10, 20, 25, 40, 56], but point out the following properties concerning functions of type P_f given by (29).

For an arbitrary arithmetical function f,

$$S_f(x) = \sum_{m,n\le x} f(\gcd(m,n)) = 2\sum_{n\le x}\sum_{k=1}^{n} f(\gcd(k,n)) - \sum_{n\le x} f(n) \tag{33}$$

$$= 2\sum_{n\le x} P_f(n) - \sum_{n\le x} f(n),$$

cf. Cohen [11, Lemma 3.1]. In that paper asymptotic formulae for $S_f(x)$ are deduced if $f(n) = \sum_{de=n} g(d)e^t$, where $t \ge 1$ and g is a bounded arithmetical function. For example, Cohen [11, Cor. 3.2] derived that

$$\sum_{m,n\le x} \phi(\gcd(m,n)) = \frac{x^2}{\zeta^2(2)}\left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta(2)}{2} - \frac{2\zeta'(2)}{\zeta(2)}\right) + R(x), \tag{34}$$

where $R(x) = \mathcal{O}(x^{3/2}\log x)$ and a similar result with the same error term for the function $\sigma(n)$. Cohen [12] improved these error terms into $R(x) = \mathcal{O}(x^{3/2})$ by an elementary method.

For $f = g * E$ one has $P_f = (g*\mu) * E\tau$, and simple convolution arguments show that for g bounded the error term for $\sum_{n\le x} P_f(n)$ is the same as in (19) and in (34), namely $\mathcal{O}(x^{1+\theta+\varepsilon})$. This was obtained also by Cohen [13], in a slightly different form.

Similar asymptotic formulae can be given for other choices of f. For example, let $f = \mu^2$. Then $P_{\mu^2}(p) = p$, $P_{\mu^2}(p^a) = p^a - p^{a-2}$ for any prime p and any $a \geq 2$. Furthermore, $P_{\mu^2}(n) = \sum_{d^2 e = n} \mu(d) e$ and obtain

$$\sum_{n \leq x} P_{\mu^2}(n) = \frac{x^2}{2\zeta(4)} + \mathcal{O}(x). \tag{35}$$

Bordellès [7, Th. 4] provides some general asymptotic results for $\sum_{n \leq x} P_f(n)$ with f belonging to certain classes of arithmetic functions. As special cases and among others, the following estimates are proven ([7, Th. 8, eq. (i),(ii),(iii),(v)]):

$$\sum_{n \leq x} P_{\mu_k}(n) = \frac{x^2}{2\zeta(2k)} + \mathcal{O}(x), \tag{36}$$

$$\sum_{n \leq x} P_{\tau_{(k)}}(n) = \frac{\zeta(2)}{2\zeta(2k)} x^2 + \mathcal{O}(x(\log x)^{2/3}), \tag{37}$$

$$\sum_{n \leq x} P_{\beta}(n) = \frac{\zeta(4)\zeta(6)}{2\zeta(12)} x^2 + \mathcal{O}(x), \tag{38}$$

$$\sum_{n \leq x} P_{a}(n) = \frac{x^2}{2} \prod_{j=2}^{\infty} \zeta(2j) + \mathcal{O}(x), \tag{39}$$

where μ_k is the characteristic function of the k-free integers, $\tau_{(k)}(n)$ is the number of k-free divisors of n ($k \geq 2$), $\beta(n)$ is the number of squarefull divisors of n and $a(n)$ represents the number of non-isomorphic abelian groups of order n. For $k = 2$, (36) gives (35).

Note that certain error terms given by Bordellès [7, Th. 8] can be improved. For example, the error term of (37) is given in [7] with an extra factor $(\log\log x)^{4/3}$. Here (37) yields by observing that $P_{\tau_{(k)}}(n) = \sum_{d^k e = n} \mu(d)\sigma(e)$ and using the following estimate of Walfisz: $\sum_{n \leq x} \sigma(n) = \frac{\zeta(2)}{2} x^2 + \mathcal{O}(x(\log x)^{2/3})$.

We have $\sum_{k=1}^{n} a^{\gcd(k,n)} = \sum_{d \mid n} a^d \phi(n/d) \equiv 0 \pmod{n}$ for any integers $a, n \geq 1$. This known congruence property has number theoretical and combinatorial proofs and interpretations, cf. Dickson [15, p. 78, 86].

The related formula

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} \gcd(k-1, n) = \tau(n)\phi(n) \quad (n \geq 1) \tag{40}$$

is due to Kesava Menon [23]. See also McCarthy [29, Ch. 1].

The products

$$h(n) = \prod_{k=1}^{n} \gcd(k,n), \quad h_f(n) = \prod_{k=1}^{n} f(\gcd(k,n)) \tag{41}$$

were considered by Loveless [27] (A067911). Note that the geometric mean of $\gcd(1,n)$, ..., $\gcd(n,n)$ is $G(n) = (h(n))^{1/n} = n/\prod_{d|n} d^{\phi(d)}$, which is a multiplicative function of n, cf. [27, Th. 6].

Using that $\log h(n) = \sum_{d|n} \phi(n/d) \log d$ we deduce

$$\sum_{n\le x} \log h(n) = -\frac{\zeta'(2)}{2\zeta(2)} x^2 + \mathcal{O}(x(\log x)^{8/3}(\log\log x)^{4/3}) \tag{42}$$

by applying the estimate of Walfisz for ϕ, namely

$$\sum_{n\le x} \phi(n) = \frac{1}{2\zeta(2)} x^2 + \mathcal{O}(x(\log x)^{2/3}(\log\log x)^{4/3}), \tag{43}$$

providing the best error up to date. (42) is given by Loveless [27, Th. 11, Corrig.] with a weaker error term, namely with $\mathcal{O}(x\log^3 x)$.

Some authors, including Diaconis and Erdős [14], Bege [4], Broughan [8] use or refer to a result of Saltykov – the error term in (43) is $\mathcal{O}(x(\log x)^{2/3}(\log\log x)^{1+\varepsilon})$ – which is not correct(!) as it was shown by Pétermann [33].

Note that Bordellès [6] obtained asymptotic formulae for another type of generalization, namely given by $g_k = \mu * E\tau_k$, where τ_k is the generalized (Dirichlet-Piltz) divisor function. It is more natural to define such functions $P^{(k)}$ in this way: $P^{(1)}(n) = P(n)$, $P^{(k+1)}(n) = \sum_{j=1}^{n} P^{(k)}(\gcd(j,n))$ $(k \ge 1)$. Then $P^{(k)} = \underbrace{\mu * \cdots * \mu}_{k} * E\tau_k$ and asymptotic formulae for $P^{(k)}$ can be given.

Let $n_1, \ldots, n_r$ be positive integers, where $r \ge 1$ and $m = \operatorname{lcm}[n_1, \ldots, n_r]$. The multivariate function

$$P(n_1, \ldots, n_r) = \frac{1}{m} \sum_{k=1}^{m} \gcd(k, n_1) \cdots \gcd(k, n_r) \tag{44}$$

was considered by Minami [30]. For $r = 1$ this reduces to P. One has, inserting $\gcd(k, n_i) = \sum_{d_i|\gcd(k,n_i)} \phi(d_i)$,

$$P(n_1, \ldots, n_r) = \sum_{d_1|n_1, \ldots, d_r|n_r} \frac{\phi(d_1)\cdots\phi(d_r)}{\operatorname{lcm}[d_1, \ldots, d_r]}, \tag{45}$$

formula not given in [30].

Schramm [38] investigated the discrete Fourier transform of functions of the form $f(\gcd(n,r))$, where f is an arbitrary arithmetic function. He considered also various special functions f and deduced interesting identities, for example,

$$\phi(r) = \sum_{k=1}^{r} \gcd(k, r) \exp(-2\pi i k/r), \tag{46}$$

$$\gcd(n, r) = \sum_{k=1}^{r} \exp(2\pi i k n/r) \sum_{d|r} c_d(k)/d, \tag{47}$$

valid for $n, r \ge 1$, where $c_d(k)$ denotes the Ramanujan sum.

The function α (cf. A057660) defined in the Introduction was considered also by Sándor and Kramer [37].

4 Proofs of Theorems 1 and 4

Proof of Theorem 1. i) First we show that

$$A^2(n) = \sum_{de=n} \tau^2(d)g(e), \tag{48}$$

where g is multiplicative and $g(p) = -4/p + 1/p^2$, $g(p^a) = 4(-1)^a/p$ for any prime p and $a \ge 2$.

By the multiplicativity of the involved functions it is enough to verify (48) for prime powers p^a ($a \ge 1$). We have

$$\sum_{de=p^a} \tau^2(d)g(e) = \sum_{j=1}^{a-1} \tau^2(p^{j-1})g(p^{a-j+1}) + \tau^2(p^{a-1})g(p) + \tau^2(p^a)$$

$$= \sum_{j=1}^{a-1} j^2(-1)^{a-j+1}\frac{4}{p} + a^2(-4/p + 1/p^2) + (a+1)^2$$

$$= (-1)^a \frac{4}{p} \sum_{j=1}^{a-1} (-1)^{j-1} j^2 + \frac{a^2}{p^2} - \frac{4a^2}{p} + (a+1)^2 = (a+1-a/p)^2 = A^2(p^a),$$

which follows by the elementary formula

$$\sum_{j=1}^{n} (-1)^{j-1} j^2 = (-1)^{n-1}\frac{n(n+1)}{2} \quad (n \ge 1).$$

Here the Dirichlet series of g is given by

$$G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \prod_p \left(1 + \frac{1}{p^{s+2}} - \frac{4}{p(p^s+1)}\right),$$

which is absolutely convergent for $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$. Therefore, for any $\varepsilon > 0$,

$$\sum_{n \le x} g(n) = \mathcal{O}(x^{\varepsilon}), \quad \sum_{n > x} \frac{g(n)}{n} = \mathcal{O}(x^{-1+\varepsilon}).$$

We need the next formula of Ramanujan, cf. Wilson [57],

$$\sum_{n \le x} \tau^2(n) = x(a\log^3 x + b\log^2 x + c\log x + d) + \mathcal{O}(x^{1/2+\varepsilon}), \tag{49}$$

where $a = 1/\pi^2, b, c, d$ are constants.

By (48) and (49) we obtain

$$\sum_{n \le x} A^2(n) = \sum_{d \le x} g(d) \sum_{e \le x/d} \tau^2(e)$$

$$= ax \sum_{d\le x} \frac{g(d)}{d} \log^3(x/d) + bx \sum_{d\le x} \frac{g(d)}{d} \log^2(x/d) + cx \sum_{d\le x} \frac{g(d)}{d} \log(x/d) + dx \sum_{d\le x} \frac{g(d)}{d}$$

$$+\mathcal{O}\left(x^{1/2+\varepsilon} \sum_{d\le x} \frac{|g(d)|}{d^{1/2+\varepsilon}} \right).$$

Now formula (6) follows by usual estimates with the constants

$$C_1 = aG(1), \quad C_2 = 3aG'(1) + bG(1), \quad C_3 = 3aG''(1) + 2bG'(1) + cG(1), \tag{50}$$

$$C_4 = aG'''(1) + bG''(1) + cG'(1) + dG(1),$$

where G', G'', G''' are the derivatives of G.

ii) Assume that $\alpha_4 < 1/2$. We use that in this case the error term for $\sum_{n\le x} \tau^2(n)$ in (49) is $\mathcal{O}(x^{1/2}\delta(x))$, as it was proved by Suryanarayana and Sitaramachandra Rao [43]. We obtain, applying that $x^\varepsilon\delta(x)$ is increasing, that the error term for $\sum_{n\le x} A^2(n)$ is

$$\ll \sum_{d\le x} |g(d)|(x/d)^{1/2}\delta(x/d) = \sum_{d\le x} |g(d)|(x/d)^{1/2-\varepsilon}(x/d)^\varepsilon\delta(x/d)$$

$$\ll x^{1/2-\varepsilon}(x^\varepsilon\delta(x)) \sum_{d\le x} \frac{|g(d)|}{d^{1/2-\varepsilon}} \ll x^{1/2}\delta(x).$$

iii) Assume RH. Then we apply that the error term of (49) is $\mathcal{O}(x^{(2-\alpha_4)/(5-4\alpha_4)}\eta(x))$, cf. [43, Lemma 2.4, Th. 3.2], where $\sum_{n\le x} \mu(n) \ll x^{1/2}\eta(x)$ according to the result of Soundararajan [42] quoted in the Introduction. Using that $\eta(x)$ is increasing, we obtain the given error term.

Proof of Theorem 4. The function H is multiplicative and for any prime power p^a $(a \ge 1)$,

$$H(p^a) = \frac{p^{2a}(p+1)}{p^{2a+1}+1}. \tag{51}$$

Now write

$$\frac{H(n)}{n} = \sum_{\substack{de=n\\(d,e)=1}} \frac{h(d)}{\phi(e)}$$

as the unitary convolution of the functions h and $1/\phi$, where h is multiplicative and for every prime power p^a $(a \ge 1)$,

$$\frac{H(p^a)}{p^a} = h(p^a) + \frac{1}{\phi(p^a)}, \quad h(p^a) = -\frac{p^{2a-1}+1}{p^{a-1}(p-1)(p^{2a+1}+1)},$$

where

$$|h(p^a)| < \frac{1}{p^a(p-1)^2}, \quad |h(n)| \le \frac{f(n)}{\phi(n)} \ (n \ge 1),$$

with $f(n) = \prod_{p|n}(p(p-1))^{-1}$.

We need the following known result, cf. for example Montgomery and Vaughan [31, p. 43],

$$\sum_{\substack{n\le x\\(n,k)=1}} \frac{1}{\phi(n)} = Ka(k)\left(\log x+\gamma+b(k)\right)+\mathcal{O}\left(2^{\omega(k)}\frac{\log x}{x}\right),$$

where γ is Euler's constant, $\omega(k)$ stands for the number of distinct prime divisors of k,

$$K=\frac{\zeta(2)\zeta(3)}{\zeta(6)},\ a(k)=\prod_{p|k}\left(1-\frac{p}{p^2-p+1}\right)\le\frac{\phi(k)}{k},$$

$$b(k)=\sum_{p|k}\frac{\log p}{p-1}-\sum_{p\nmid k}\frac{\log p}{p^2-p+1}\ll\frac{\psi(k)\log k}{\phi(k)},\quad\text{with}\quad \psi(k)=k\prod_{p|k}\left(1+\frac{1}{p}\right).$$

We obtain

$$\sum_{n\le x}\frac{H(n)}{n}=\sum_{d\le x}h(d)\sum_{\substack{e\le x/d\\ \gcd(e,d)=1}}\frac{1}{\phi(e)}=$$

$$=K\left((\log x+\gamma)\sum_{d\le x}h(d)a(d)+\sum_{d\le x}h(d)a(d)(b(d)-\log d)\right)+O\left(\frac{\log x}{x}\sum_{d\le x}d|h(d)|2^{\omega(d)}\right),$$

and we obtain the given result with the constants

$$C_5=K\sum_{n=1}^{\infty}h(n)a(n),\ C_6=K\gamma\sum_{n=1}^{\infty}h(n)a(n)+K\sum_{n=1}^{\infty}h(n)a(n)(b(n)-\log n),$$

these series being convergent taking into account the estimates of above. For the error terms,

$$\sum_{n>x}|h(n)|a(n)\le\sum_{n>x}\frac{f(n)}{n}<\sum_{n>x}\frac{f(n)}{n}(\frac{n}{x})^{1-\varepsilon}<x^{-1+\varepsilon}\sum_{n=1}^{\infty}\frac{f(n)}{n^{\varepsilon}}$$

$$=x^{-1+\varepsilon}\prod_p\left(1+\frac{1}{p(p-1)(p^{\varepsilon}-1)}\right)\ll x^{-1+\varepsilon},$$

in a similar way,

$$\sum_{n>x}a(n)|h(n)(b(n)-\log n)|\ll x^{-1+\varepsilon},$$

and

$$\sum_{n\le x}n|h(n)|2^{\omega(n)}\le\sum_{n\le x}(\prod_{p|n}1/(p-1)^2)2^{\omega(n)}(\frac{x}{n})^{\varepsilon}<x^{\varepsilon}\sum_{n=1}^{\infty}\frac{2^{\omega(n)}}{n^{\varepsilon}}(\prod_{p|n}1/(p-1)^2)$$

$$=x^{\varepsilon}\prod_p\left(1+\frac{2}{(p-1)^2(p^{\varepsilon}-1)}\right)\ll x^{\varepsilon},$$

ending the proof, which is similar to that of Tóth [52, Th. 6].

Proof of Corollary 5. Follows from Theorem 4 by partial summation.

5 Analogs of the gcd-sum function

5.1 Unitary analog

Recall, that a positive integer d is said to be a unitary divisor of n if $d \mid n$ and $\gcd(d, n/d) = 1$, notation $d \mid\mid n$. The unitary analogue of the function P is the function

$$P^*(n) = \sum_{k=1}^{n} (k, n)_*, \tag{52}$$

where $(k, n)_* := \max\{d \in \mathbb{N} : d \mid k, d \mid\mid n\}$, which was introduced by Tóth [47]. The function P^* (A145388) is also multiplicative and $P^*(p^a) = 2p^a - 1$ for every prime power p^a $(a \geq 1)$. It has also other properties, including asymptotic ones, which are close to the usual gcd-sum function.

Consider the function ϕ^* (the unitary Euler function, A047994) defined by

$$\phi^*(n) = \#\{k \in \mathbb{N} : 1 \leq k \leq n, (k, n)_* = 1\}, \tag{53}$$

which is multiplicative and $\phi^*(p^a) = p^a - 1$ for every prime power p^a $(a \geq 1)$. Then

$$P^*(n) = \sum_{d \mid\mid n} d\phi^*(n/d). \tag{54}$$

It was proved by Tóth [49] that

$$\sum_{n \leq x} P^*(n) = \frac{\alpha}{2\zeta(2)} x^2 \log x + \beta x^2 + \mathcal{O}(x^{3/2} \log x), \tag{55}$$

where $\alpha = \prod_p (1 - 1/(p+1)^2) \approx 0.775883$, cf. [16, p. 110] and β are constants.

Note that we also have

$$\limsup_{n \to \infty} \frac{\log(P^*(n)/n) \log\log n}{\log n} = \log 2, \tag{56}$$

the same result as for $P(n)$. This is not given in the literature. For the proof, which is similar to that of [53, Th. 1], take into account (24), where the limsup is attained for a sequence of square-free integers (more exactly for $n_k = \prod_{k/\log^2 k < p \leq k} p$, $k \to \infty$), see [10, Th. 4.1], and use that $P^*(n) \leq P(n)$ for every $n \geq 1$, with equality for any n square-free.

5.2 Bi-unitary analog

Let $(k, n)_{**} = \max\{d \in \mathbb{N} : d \mid\mid k, d \mid\mid n\}$ stand for the greatest common unitary divisor of k and n and

$$P^{**}(n) = \sum_{k=1}^{n} (k, n)_{**} \tag{57}$$

be the bi-unitary gcd-sum function, introduced by Haukkanen [20].

Haukkanen [20, Cor. 3.1] showed that

$$P^{**}(n) = \sum_{d||n} \phi^*(d)\phi(n/d, d), \tag{58}$$

where $\phi(x, n) = \#\{k \in \mathbb{N} : 1 \leq k \leq x, \gcd(k, n) = 1\}$ is the Legendre function.

Note that for every $n \geq 1$,

$$P^{**}(n) \leq P^*(n) \leq P(n). \tag{59}$$

The function P^{**} is not multiplicative and a combinatorial type formula for $P^{**}(n)$ was given by Tóth [54]. In that paper it was also proved, that

$$\sum_{n \leq x} P^{**}(n) = \frac{1}{2} B x^2 \log x + \mathcal{O}(x^2), \tag{60}$$

where

$$B = \prod_p \left(1 - \frac{3p-1}{p^2(p+1)}\right) = \zeta(2) \prod_p \left(1 - \frac{(2p-1)^2}{p^4}\right). \tag{61}$$

5.3 Analog involving exponential divisors

The next analog is concerning exponential divisors. Let $n > 1$ be an integer of canonical form $n = \prod_{i=1}^r p_i^{a_i}$. The integer d is called an exponential divisor of n if $d = \prod_{i=1}^r p_i^{c_i}$, where $c_i \mid a_i$ for every $1 \leq i \leq r$, notation: $d \mid_e n$. By convention $1 \mid_e 1$. Note that 1 is not an exponential divisor of $n > 1$, the smallest exponential divisor of $n > 1$ is its square-free kernel $\kappa(n) = \prod_{i=1}^r p_i$.

Two integers $n, m > 1$ have common exponential divisors iff they have the same prime factors and in this case, i.e., for $n = \prod_{i=1}^r p_i^{a_i}$, $m = \prod_{i=1}^r p_i^{b_i}$, $a_i, b_i \geq 1$ $(1 \leq i \leq r)$, the greatest common exponential divisor of n and m is

$$(n, m)_{(e)} = \prod_{i=1}^r p_i^{(a_i, b_i)}. \tag{62}$$

Here $(1,1)_{(e)} = 1$ by convention and $(1, m)_{(e)}$ does not exist for $m > 1$.

Let $P^{(e)}(n)$ be given by

$$P^{(e)}(n) = \sum_{\substack{1 \leq k \leq n \\ \kappa(k) = \kappa(n)}} (k, n)_{(e)}, \tag{63}$$

introduced by Tóth [51]. The function $P^{(e)}$ is multiplicative and for every prime power p^a,

$$P^{(e)}(p^a) = \sum_{1 \leq j \leq a} p^{(j,a)} = \sum_{d|a} p^d \phi(a/d), \tag{64}$$

here $P^{(e)}(p) = p$, $P^{(e)}(p^2) = p + p^2$, $P^{(e)}(p^3) = 2p + p^3$, $P^{(e)}(p^4) = 2p + p^2 + p^4$, etc.

We have, see [51, Th. 3],

$$\sum_{n\le x} P^{(e)}(n) = Ex^2 + \mathcal{O}(x(\log x)^{5/3}), \tag{65}$$

where the constant E is given by

$$E = \frac{1}{2}\prod_p \left(1 + \sum_{a=2}^{\infty} \frac{P^{(e)}(p^a) - pP^{(e)}(p^{a-1})}{p^{2a}}\right). \tag{66}$$

Pétermann [34, Th. 2] showed that for $s \in \mathbb{C}$, $\operatorname{Re} s > 2$,

$$\sum_{n=1}^{\infty} \frac{P^{(e)}(n)}{n^s} = \frac{\zeta(s-1)\zeta(2s-1)}{\zeta(3s-2)} W(s), \tag{67}$$

where $W(s)$ is absolutely convergent for $\operatorname{Re} s > 3/4$ and that the error term of (65) is $\Omega_{\pm}(x \log\log x)$.

Concerning the maximal order of the function $P^{(e)}$ we have by [51, Th. 4],

$$\limsup_{n\to\infty} \frac{P^{(e)}(n)}{n \log\log n} = \frac{6}{\pi^2} e^{\gamma}, \tag{68}$$

where γ is Euler's constant.

5.4 Analog involving regular integers (mod n)

Next we give another analog. Let $n > 1$ be an integer with prime factorization $n = p_1^{\nu_1} \cdots p_r^{\nu_r}$. An integer k is called regular (mod n) if there exists an integer x such that $k^2 x \equiv k \pmod{n}$. It can be shown that $k \ge 1$ is regular (mod n) if and only if for every $i \in \{1, \ldots, r\}$ either $p_i \nmid k$ or $p_i^{\nu_i} \mid k$. Also, $k \ge 1$ is regular (mod n) if and only if $\gcd(k, n)$ is a unitary divisor of n. These and other characterizations of regular integers are given by Tóth [52].

Let $\mathrm{Reg}_n = \{k : 1 \le k \le n \text{ and } k \text{ is regular} \pmod{n}\}$. Tóth [53] introduced the function

$$\widetilde{P}(n) = \sum_{k \in \mathrm{Reg}_n} \gcd(k, n) \tag{69}$$

(A176345) and showed the following properties. For every $n \ge 1$,

$$\widetilde{P}(n) = \sum_{d \| n} d\, \phi(n/d), \tag{70}$$

hence $\widetilde{P}$ is a multiplicative function and

$$\widetilde{P}(n) = n \prod_{p \mid n} \left(2 - \frac{1}{p}\right). \tag{71}$$

Also, the minimal order of $\widetilde{P}(n)$ is $3n/2$ and the maximal order of $\log(\widetilde{P}(n)/n)$ is $\log 2 \log n / \log\log n$. We have

$$\sum_{n\leq x} \widetilde{P}(n) = \frac{x^2}{2\zeta(2)}(K_1 \log x + K_2) + \mathcal{O}(x^{3/2}\delta(x)), \tag{72}$$

where K_1 and K_2 are certain constants and $\delta(x)$ is given by (8).

If RH is true, then the error term of (72) is $\mathcal{O}(x^{(7-5\theta)/(5-4\theta)}\omega(x))$, where

$$\omega(x) = \exp(c \log x(\log\log x)^{-1}) \tag{73}$$

with a positive constant c and θ the exponent in the Dirichlet divisor problem (20). For $\theta \approx 0.3149$ one has $(7-5\theta)/(5-4\theta) \approx 1.4505$.

Zhang and Zhai [58] showed that for $s \in \mathbb{C}$, $\operatorname{Re} s > 2$,

$$\sum_{n=1}^{\infty} \frac{\widetilde{P}(n)}{n^s} = \frac{\zeta^2(s-1)}{\zeta(2s-2)} H(s), \tag{74}$$

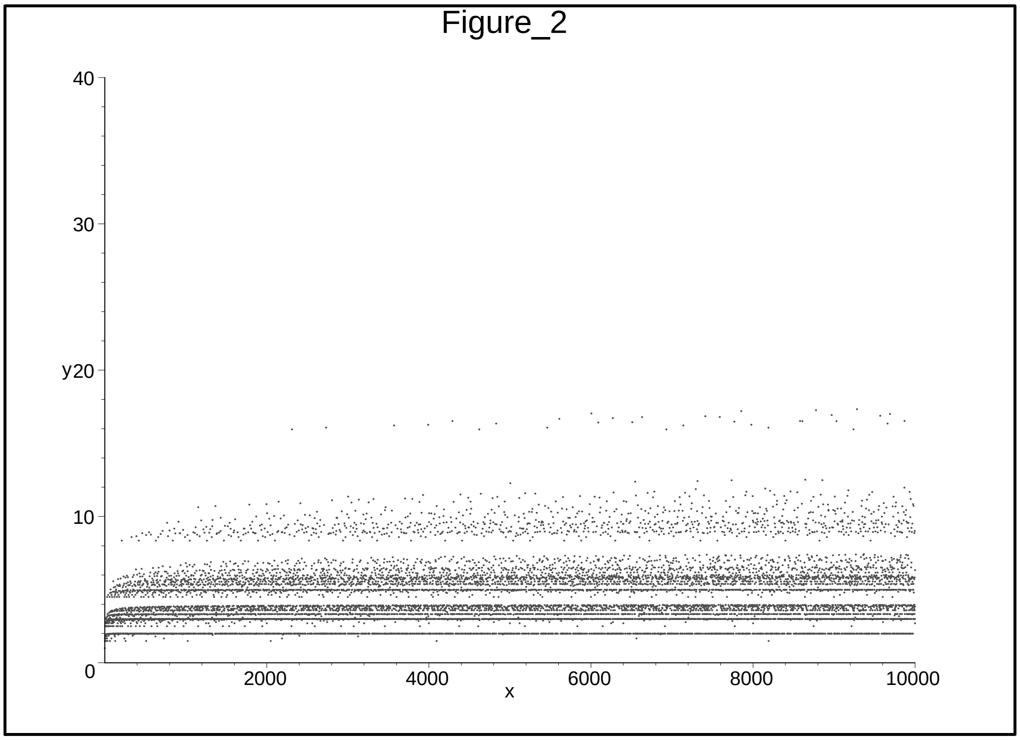
where $H(s) = \prod_p \left(1 - \frac{1}{p(p^{s-1}+1)}\right)$ is absolutely convergent for $\operatorname{Re} s > 1$, the error term of (72) is connected to the square-free divisor problem and it is $\mathcal{O}(x^{15/11+\varepsilon})$, where $15/11 \approx 1.3636$, assuming RH.

De Koninck and Kátai [24] showed that for $\widetilde{A}(n) = \widetilde{P}(n)/n$,

$$\sum_{p\leq x} \widetilde{A}(p-1) = L'x + \mathcal{O}(x(\log\log x)^{-1}), \tag{75}$$

where L' is a constant, result which is similar to (26).

Figure 2 is a plot of the function $\widetilde{A}(n)$ for $1 \leq n \leq 10\,000$, produced using Maple.



5.5 Analog concerning subsets of the set $\{1, 2, \ldots, n\}$

For a nonempty subset A of $\{1, 2, \ldots, n\}$ let $\gcd(A)$ denote the gcd of the elements of A. Consider the gcd-sum type functions $P_{\mathcal{S}}$ and $P_{\mathcal{S},k}$ defined by

$$P_{\mathcal{S}}(n) = \sum_{A} \gcd(\gcd(A), n)), \quad P_{\mathcal{S},k}(n) = \sum_{\#A=k} \gcd(\gcd(A), n)), \tag{76}$$

where the sums are over all nonempty subsets A of $\{1, 2, \ldots, n\}$ and over all subsets A of $\{1, 2, \ldots, n\}$ having k elements ($k \geq 1$ fixed), respectively. For $k = 1$ this reduces to the function P.

These are special cases of more general gcd-sum type functions investigated by Tóth [55]. We have, cf. [55, eq. (34),(37),(38)],

$$P_{\mathcal{S}}(n) = \sum_{d|n} \phi(d) 2^{n/d} - n \quad (n \geq 1), \tag{77}$$

$$P_{\mathcal{S},k}(n) = \sum_{d|n} \phi(d) \binom{n/d}{k} \quad (n \geq 1), \tag{78}$$

$$\sum_{n \leq x} P_{\mathcal{S},k}(n) = \frac{\zeta(k)}{(k+1)!\zeta(k+1)} x^{k+1} + \mathcal{O}(\psi_k(x)) \quad (k \geq 2), \tag{79}$$

where $\psi_k(x) = x^k$ for $k \geq 3$ and $\psi_2(x) = x^2 \log x$.

6 Open problems

Here I list some open problems concerning the functions discussed above.

1. Determine the integers $n \geq 1$ such that $n \mid P(n)$, that is $\sum_{d|n} \frac{\phi(d)}{d}$ is an integer.

The first few values are: $1, 4, 15, 16, 27, 48, 60, 64, 108, 144, 240, 256, 325, 432, 729, 891, 960$. This is sequence A066862 in Sloane's Encyclopedia. From (14) it is clear that $n = \prod_{i=1}^{r} p_i^{a_i p_i}$ are solutions for any distinct primes p_i and any $a_i \geq 1$.

For square-free values $n = \prod_{i=1}^{r} p_i$ this is equivalent to $\prod_{i=1}^{r} \left(2 - \frac{1}{p_i}\right)$ be an integer. It can be shown that the only square-free solutions having at most three distinct prime factors are $n = 1$ and $n = 15$.

I conjecture that there are no other square-free solutions. I have verified this for the integers $n < 10^6$.

2. Determine the integers $n \geq 1$ such that $n \mid \widetilde{P}(n)$. This holds iff $\prod_{p|n} \left(2 - \frac{1}{p}\right)$ is an integer. If the previous conjecture is true, then the only integers solutions to the present problem are $n = 1$ and $n = 15$.

3. Investigate the equation $\widetilde{P}(n) = \widetilde{P}(n+1)$.

The first few solutions are: $45, 225, 1125, 2025, 3645, 140625, 164025, 257174$. According to de Koninck and Kátai [24] this equation has 37 solutions $< 10^{10}$ and 21 of them are of form $n = 3^a 5^b$.

4. What is the minimal order of $P(n)$ ($P^*(n)$)?

5. Derive asymptotic formulae for $\sum_{n \leq x} (f(n))^k$, where f is one of the functions P, P^*, $P^{(e)}$, $\widetilde{P}$ and $k > 2$.

6. Investigate asymptotic properties of the iterates $f(f(n))$, where f is one of the functions P, P^*, $P^{(e)}$, $\widetilde{P}$.

7 Added in proof

The author thanks the anonymous referee for helpful suggestions and for the following results concerning $\sum_{n \leq x} 1/f(n)$, where f is any of the functions P, P^*, $P^{(e)}$, $\widetilde{P}$. The problem of finding such asymptotic formulae was included originally in the previous section.

Theorem 6.

$$\sum_{n \leq x} \frac{1}{P(n)} = K (\log x)^{1/2} + \mathcal{O}((\log x)^{-1/2}), \tag{80}$$

$$\sum_{n \leq x} \frac{1}{P^*(n)} = K^* (\log x)^{1/2} + \mathcal{O}((\log x)^{-1/2}), \tag{81}$$

$$\sum_{n \leq x} \frac{1}{\widetilde{P}(n)} = \widetilde{K} (\log x)^{1/2} + \mathcal{O}((\log x)^{-1/2}), \tag{82}$$

$$\sum_{n \leq x} \frac{1}{P^{(e)}(n)} = K^{(e)} \log x + \mathcal{O}(1), \tag{83}$$

where

$$K = \frac{2}{\sqrt{\pi}} \prod_p \left(1-\frac{1}{p}\right)^{1/2} \left(1+\sum_{a=1}^{\infty} \frac{1}{P(p^a)}\right), \tag{84}$$

$$K^* = \frac{2}{\sqrt{\pi}} \prod_p \left(1-\frac{1}{p}\right)^{1/2} \left(1+\sum_{a=1}^{\infty} \frac{1}{2p^a-1}\right), \tag{85}$$

$$\widetilde{K} = \frac{2}{\sqrt{\pi}} \prod_p \left(1-\frac{1}{p}\right)^{1/2} \left(1+\frac{1}{p-1}-\frac{1}{2p-1}\right) \approx 1.46851, \tag{86}$$

$$K^{(e)} = \prod_p \left(1-\frac{1}{p}\right) \left(1+\sum_{a=1}^{\infty} \frac{1}{P^{(e)}(p^a)}\right). \tag{87}$$

For the proof we apply the following useful theorem, which is a particular case of a more general result, proved by Martin [28].

Theorem [28, Proposition A. 3]. *Let f be any nonnegative multiplicative function such that $f(n) \ll n^{\alpha}$ for some $\alpha < 1/2$ and satisfying*

$$\sum_{p\le x} \frac{f(p)\log p}{p} = \kappa \log x + \mathcal{O}_f(1) \quad (x \ge 2),$$

where $\kappa = \kappa_f > 0$. Then we have uniformly for all $x \ge 2$,

$$\sum_{n\le x} \frac{f(n)}{n} = \mathcal{M}_{f,\kappa} (\log x)^{\kappa} + \mathcal{O}_f((\log x)^{\kappa-1}), \tag{88}$$

where

$$\mathcal{M}_{f,\kappa} = \frac{1}{\Gamma(\kappa+1)} \prod_p \left(1-\frac{1}{p}\right)^{\kappa} \left(1+\sum_{a=1}^{\infty} \frac{f(p^a)}{p^a}\right).$$

Let $f(n) = n/P(n)$, which is multiplicative and by (14), $f(p^a) = \dfrac{p}{(a+1)p-a}$ for any prime power p^a $(a \ge 1)$. Hence $f(p^a) \le \dfrac{2}{a+1}$ and $f(n) \le \dfrac{2^{\omega(n)}}{\tau(n)} \le 1$ for any $n \ge 1$. Since

$$\sum_{p\le x} \frac{f(p)\log p}{p} = \sum_{p\le x} \frac{\log p}{p} \left(\frac{1}{2}+\frac{1}{4p-2}\right) = \frac{1}{2}\log x + \mathcal{O}(1),$$

the cited theorem gives the formula (80), by choosing $\kappa = 1/2$, where $\Gamma(3/2) = \sqrt{\pi}/2$.

By similar arguments we obtain formulae (81), (82) and (83).

References

[1] K. Alladi, On generalized Euler functions and related totients, in *New Concepts in Arithmetic Functions*, Matscience Report 83, Madras, 1975.

[2] G. E. Andrews, *Number Theory*, W. B. Saunders, 1971.

[3] M. Balazard and A. De Roton, Notes de lecture de l'article "Partial sums of the Möbius function" de Kannan Soundararajan (French), Preprint, 2008. Available at http://front.math.ucdavis.edu/0810.3587.

[4] A. Bege, *Old and New Arithmetic Functions* (Hungarian), Scientia, Kolozsvár [Cluj-Napoca], 2006.

[5] O. Bordellès, A note on the average order of the gcd-sum function, *J. Integer Sequences* **10** (2007), Article 07.3.3.

[6] O. Bordellès, Mean values of generalized gcd-sum and lcm-sum functions, *J. Integer Sequences* **10** (2007), Article 07.9.2.

[7] O. Bordellès, The composition of the gcd and certain arithmetic functions, *J. Integer Sequences* **13** (2010), Article 10.7.1.

[8] K. A. Broughan, The gcd-sum function, *J. Integer Sequences* **4** (2001), Article 01.2.2.

[9] K. A. Broughan, The average order of the Dirichlet series of the gcd-sum function, *J. Integer Sequences* **10** (2007), Article 07.4.2.

[10] J. Chidambaraswamy and R. Sitaramachandrarao, Asymptotic results for a class of arithmetical functions, *Monatsh. Math.* **99** (1985), 19–27.

[11] E. Cohen, Arithmetical functions of a greatest common divisor, I., *Proc. Amer. Math. Soc.* **11** (1960), 164–171.

[12] E. Cohen, Arithmetical functions of a greatest common divisor, II. An alternative approach, *Boll. Un. Mat. Ital.* **17** (1962), 349–356.

[13] E. Cohen, Arithmetical functions of a greatest common divisor, III. Cesàro's divisor problem, *Proc. Glasgow Math. Assoc.* **5** (1961-1962), 67–75.

[14] P. Diaconis and P. Erdős, On the distribution of the greatest common divisor, in *A festschrift for Herman Rubin, IMS Lecture Notes Monogr. Ser., Inst. Math. Statist.*, **45**, (2004), 56-61 (original version: Technical Report No. 12, Department of Statistics, Stanford University, Stanford, 1977).

[15] L. E. Dickson, *History of the Theory of Numbers*, vol. 1, Chelsea, New York, 1952.

[16] S. R. Finch, *Mathematical Constants*, Cambridge University Press, 2003.

[17] J. von zur Gathen, A. Knopfmacher, F. Luca, L. G. Lucht, and I. E. Shparlinski, Average order in cyclic groups, *J. Théor. Nombres Bordeaux* **16** (2004), 107–123. Available at http://www.numdam.org/item?id=JTNB_2004__16_1_107_0.

[18] H. W. Gould and T. Shonhiwa, Functions of GCD's and LCM's, *Indian J. Math.* **39** (1997), 11–35.

[19] P. Haukkanen, Rational arithmetical functions of order $(2,1)$ with respect to regular convolutions, *Portugal. Math.* **56** (1999), 329–343.

[20] P. Haukkanen, On a gcd-sum function, *Aequationes Math.* **76** (2008), 168–178.

[21] M. N. Huxley, Exponential sums and lattice points III., *Proc. London Math. Soc.* **87** (2003), 591–609.

[22] A. Ivic̀, E. Krätzel, M. Kühleitner, and W. G. Nowak, Lattice points in large regions and related arithmetic functions: recent developments in a very classic topic, Elementare und analytische Zahlentheorie, *Schr. Wiss. Ges. Johann Wolfgang Goethe Univ. Frankfurt am Main*, **20**, 89–128, Franz Steiner Verlag Stuttgart, 2006.

[23] P. Kesava Menon, On the sum $\sum (a-1,n)[(a,n)=1]$, *J. Indian Math. Soc.* **29** (1965), 155–163.

[24] J.-M. de Koninck and I. Kátai, Some remarks on a paper of L. Toth, *J. Integer Sequences* **13** (2010), Article 10.1.2.

[25] H. G. Kopetzky, Ein asymptotischer Ausdruck für eine zahlentheoretische Funktion, *Monatsh. Math.* **84** (1977), 213–217.

[26] Kee-Wai Lau, Problem 6615, *Amer. Math. Monthly* **96** (1989), 847, solution in **98** (1991), 562–565.

[27] A. D. Loveless, General gcd-product function, *Integers* **6** (2006), Paper A19, 13 pp, corrigendum Paper A39.

[28] G. Martin, An asymptotic formula for the number of smooth values of a polynomial, *J. Number Theory* **93** (2002), 108–182.

[29] P. J. McCarthy, *Introduction to Arithmetical Functions*, Universitext, Springer, 1986.

[30] N. Minami, On the random variable $\mathbb{N} \ni l \mapsto \gcd(l,n_1)\gcd(l,n_2)\cdots\gcd(l,n_k) \in \mathbb{N}$, Preprint, 2009. Available at http://arxiv.org/abs/0907.0918.

[31] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I. Classical Theory*, Cambridge University Press, 2007.

[32] I. Niven and H. S. Zuckerman, *An Introduction to the Theory of Numbers*, 3th edition, John Wiley & Sons, 1972.

[33] Y.-F. S. Pétermann, On an estimate of Walfisz and Saltykov for an error term related to the Euler function, *J. Théor. Nombres Bordeaux* **10** (1998), 203–236.

[34] Y.-F. S. Pétermann, Arithmetical functions involving exponential divisors: Note on two papers by L. Tóth, *Annales Univ. Sci. Budapest., Sect. Comp.*, to appear.

[35] S. S. Pillai, On an arithmetic function, *J. Annamalai Univ.* **2** (1933), 243–248.

[36] J. Sándor and B. Crstici, *Handbook of Number Theory II.*, Kluwer Academic Publishers, 2004.

[37] J. Sándor and A.-V. Kramer, Über eine zahlentheoretische Funktion, *Math. Morav.* **3** (1999), 53–62.

[38] W. Schramm, The Fourier transform of functions of the greatest common divisor, *Integers* **8** (2008), #A50.

[39] J. Shallit, Problem E 2821, *Amer. Math. Monthly* **87** (1980), 220. Solution in **88** (1981), 444–445.

[40] R. Sivaramakrishnan, On three extensions of Pillai's arithmetic function $\beta(n)$, *Math. Student* **39** (1971), 187–190.

[41] T. Shonhiwa and H. W. Gould, A generalization of Cesàro's function and other results, *Indian J. Math.* **39** (1997), 183–194.

[42] K. Soundararajan, Partial sums of the Möbius function, *J. Reine Angew. Math.* **631** (2009), 141–152.

[43] D. Suryanarayana and R. Sitaramachandra Rao, On an asymptotic formula of Ramanujan, *Math. Scand.* **32** (1973), 258–264.

[44] E. Teuffel, Aufgabe 599, *Elem. Math.* **25** (1970), 65–66.

[45] Y. Tanigawa and W. Zhai, On the gcd-sum function, *J. Integer Sequences* **11** (2008), Article 08.2.3.

[46] E. C. Titchmarsh, *The Theory of the Riemann Zeta Function*, Second edition, Edited and with a preface by D. R. Heath-Brown, Clarendon Press, Oxford University Press, New York, 1986.

[47] L. Tóth, The unitary analogue of Pillai's arithmetical function, *Collect. Math.* **40** (1989), 19–30.

[48] L. Tóth, Some remarks on a generalization of Euler's function, *Semin. Arghiriade* **23** (1990).

[49] L. Tóth, The unitary analogue of Pillai's arithmetical function II., *Notes Number Theory Discrete Math.* **2** (1996), no. 2, 40–46. Available at `http://www.ttk.pte.hu/matek/ltoth/Toth_Pillai2_1996.pdf`.

[50] L. Tóth, A generalization of Pillai's arithmetical function involving regular convolutions, Proceedings of the 13th Czech and Slovak International Conference on Number Theory (Ostravice, 1997), *Acta Math. Inform. Univ. Ostraviensis* **6** (1998), 203–217. Available at http://dml.cz/dmlcz/120534.

[51] L. Tóth, On certain arithmetic functions involving exponential divisors, *Annales Univ. Sci. Budapest., Sect. Comp.* **24** (2004), 285–294. Available at http://front.math.ucdavis.edu/0610.5274.

[52] L. Tóth, Regular integers (mod n), *Annales Univ. Sci. Budapest., Sect. Comp.* **29** (2008), 263–275. Available at http://front.math.ucdavis.edu/0710.1936.

[53] L. Tóth, A gcd-sum function over regular integers (mod n), *J. Integer Sequences* **12** (2009), Article 09.2.5.

[54] L. Tóth, On the bi-unitary analogues of Euler's arithmetical function and the gcd-sum function, *J. Integer Sequences* **12** (2009), Article 09.5.2.

[55] L. Tóth, On the number of certain relatively prime subsets of $\{1, 2, \ldots, n\}$, *Integers*, to appear.

[56] A. C. Vasu, On a certain arithmetic function, *Math. Student* **34** (1966), 93–95.

[57] B. M. Wilson, Proofs of some formulae enunciated by Ramanujan, *Proc. London Math. Soc.* (2) **21** (1922), 235–255.

[58] D. Zhang and W. Zhai, Mean values of a gcd-sum function over regular integers modulo n, *J. Integer Sequences* **13** (2010), Article 10.4.7.

2000 *Mathematics Subject Classification*: Primary 11A25; Secondary 11N37.
Keywords: gcd-sum function, Euler's arithmetical function, divisor function, unitary divisor, exponential divisor, regular integer (mod n), average order.

(Concerned with sequence A001880, A018804, A047994, A057660, A066862, A067911, A145388, and A176345.)

Received April 29 2010; revised version received July 20 2010. Published in *Journal of Integer Sequences*, August 2 2010.

Return to Journal of Integer Sequences home page.