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Mean Values of a Gcd-Sum Function Over Regular Integers Modulo n

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Abstract

In this paper we study the mean value of a gcd-sum function over regular integers modulo n. In particular, we improve the previous result under the Riemann hypothesis (RH). We also study the short interval problem for it without assuming RH.

1 Introduction

In general, an element k of a ring R is said to be (von Neumann) regular if there is an $x \in R$ such that $k = kxk$. Let $n > 1$ be an integer with prime factorization $n = p_1^{\nu_1} \cdots p_r^{\nu_r}$. An integer k is called *regular* (mod n) if there exists an integer x such that $k^2 x \equiv k \pmod{n}$, i.e., the residue class of k is a regular element (in the sense of J. von Neumann) of the ring $\mathbb{Z}_n$ of residue classes (mod n).

Let $\mathrm{Reg}_n = \{k : 1 \le k \le n \text{ and } k \text{ is regular } (\mathrm{mod}\ n)\}$. Tóth [11] first defined the gcd-sum function over regular integers modulo n by the relation

$$\tilde{P}(n) = \sum_{k \in \mathrm{Reg}_n} \gcd(k, n), \tag{1}$$

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where $\gcd(a, b)$ denotes the greatest common divisor of a and b. It is sequence A176345 in Sloane's Encyclopedia. This is analogous to the gcd-function, called also Pillai's arithmetical function,

$$P(n) = \sum_{k=1}^{n} \gcd(k, n),$$

which has been studied recently by several authors, see [2, 3, 4, 5, 6, 9, 12]; it is Sloane's sequence A018804. Tóth [11] proved that $\tilde{P}(n)$ is multiplicative and for every $n \geq 1$,

$$\tilde{P}(n) = n \prod_{p|n} (2 - \frac{1}{p}). \tag{2}$$

He also obtained the following asymptotic formula

$$\sum_{n \leq x} \tilde{P}(n) = \frac{x^2}{2\zeta(2)} (K_1 \log x + K_2) + O(x^{3/2}\delta(x)), \tag{3}$$

where the function $\delta(x)$ and constants K_1 and K_2 are given by

$$\delta(x) = \exp(-A(\log x)^{3/5}(\log\log x)^{-1/5}),$$

$$K_1 = \sum_{n=1}^{\infty} \frac{\mu(n)}{n\psi(n)} = \prod_{p} \left(1 - \frac{1}{p(p+1)}\right), \tag{4}$$

$$K_2 = K_1 \left(2\gamma - \frac{1}{2} - \frac{2\zeta'(2)}{\zeta(2)}\right) - \sum_{n=1}^{\infty} \frac{\mu(n)(\log n - \alpha(n) + 2\beta(n))}{n\psi(n)}, \tag{5}$$

where $\psi(n) = n \prod_{p|n}(1 + \frac{1}{p})$ denotes the Dedekind function, and

$$\alpha(n) = \sum_{p|n} \frac{\log p}{p-1}, \quad \beta(n) = \sum_{p|n} \frac{\log p}{p^2 - 1}.$$

It is very difficult to improve the exponent $\frac{3}{2}$ in the error term of (3) unless we have substantial progress in the study of the zero free region of $\zeta(s)$. Therefore it is reasonable to get better improvements by assuming the truth of the Riemann hypothesis (RH). Let $d(n)$ denote the Dirichlet divisor function and

$$\Delta(x) := \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1). \tag{6}$$

Dirichlet first proved that $\Delta(x) = O(x^{1/2})$. The exponent $1/2$ was improved by many authors. The latest result reads

$$\Delta(x) \ll x^{\theta+\epsilon}, \qquad \theta = 131/416, \tag{7}$$

due to Huxley [7]. Tóth [11] proved that if RH is true, then the error term of (3) can be replaced by $O(x^{(7-5\theta)/(5-4\theta)}\exp(B\log x(\log\log x)^{-1}))$. For $\theta = 131/416$ one has $(7-5\theta)/(5-4\theta) \approx 1.4505$.

In this paper, we will use the Dirichlet convolution method to study the mean value of $\tilde{P}(n)$, and we find that the estimate of $\sum_{n\le x}\tilde{P}(n)$ is closely related to the square-free divisor problem. Let $d^{(2)}(n)$ denote the number of square-free divisors of n. Note that $d^{(2)}(n) = 2^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime factors of n. Let

$$D^{(2)}(x) = \sum_{n\le x} d^{(2)}(n).$$

It was shown by Mertens [8] that

$$D^{(2)}(x) = \frac{1}{\zeta(2)}x\log x + \left(\frac{2\gamma-1}{\zeta(2)} - \frac{2\zeta'(2)}{\zeta^2(2)}\right)x + \Delta^{(2)}(x), \tag{8}$$

where $\Delta^{(2)}(x) = O(x^{1/2}\log x)$. The exponent $\frac{1}{2}$ is also difficult to be improved, because it is related to the zero distribution of $\zeta(s)$. One way of making progress is to assume the Riemann hypothesis (RH). Many authors investigated this problem, and the best result under the Riemann hypothesis is

$$\Delta^{(2)}(x) \ll x^{\lambda+\epsilon}, \tag{9}$$

where $\lambda = 4/11$, due to Baker [1].

In this paper, we shall prove the following results.

Theorem 1. *For any real numbers $x \ge 1$ and $\epsilon > 0$, if*

$$\Delta^{(2)}(x) \ll x^{\lambda+\epsilon},$$

then we have

$$\sum_{n\le x}\tilde{P}(n) = \frac{x^2}{2\zeta(2)}(K_1\log x + K_2) + O(x^{1+\lambda+\epsilon}), \tag{10}$$

where K_1, K_2 are defined by (4) and (5).

Corollary 2. *If RH is true, then*

$$\sum_{n\le x}\tilde{P}(n) = \frac{x^2}{2\zeta(2)}(K_1\log x + K_2) + O(x^{15/11+\epsilon}). \tag{11}$$

Remark. Note that $15/11 \approx 1.3636$, which improves the previous result.

In order to avoid assuming the truth of the Riemann hypothesis, we study the short interval problem for it.

Theorem 3. *For*

$$x^{\theta+3\epsilon} \leq y \leq x,$$

we have

$$\sum_{x<n\leq x+y} \tilde{P}(n) = \frac{1}{2\zeta(2)} \int_x^{x+y} u\left(2K_1 \log u + K_1 + 2K_2\right) du + O(yx^{1-\epsilon} + x^{1+\theta+2\epsilon}). \quad (12)$$

where θ is defined by (7).

Corollary 4. *For*

$$x^{131/416+3\epsilon} \leq y \leq x,$$

we have

$$\sum_{x<n\leq x+y} \tilde{P}(n) = \frac{1}{2\zeta(2)} \int_x^{x+y} u\left(2K_1 \log u + K_1 + 2K_2\right) du + O(yx^{1-\epsilon} + x^{\frac{547}{416}+2\epsilon}). \quad (13)$$

Notation. Throughout the paper ϵ always denotes a fixed but sufficiently small positive constant. We write $f(x) \ll g(x)$, or $f(x) = O(g(x))$, to mean that $|f(x)| \leq Cg(x)$. For any fixed integers $1 \leq a \leq b$,we consider the divisor function

$$d(a, b; n) = \sum_{n=m^a k^b} 1.$$

2 Proof of Theorem 1

Let s be complex numbers with $\Re s > 1$. We consider the mean value of the arithmetic function $\tilde{P}^*(n) = \frac{\tilde{P}(n)}{n}$. Define

$$F(s) := \sum_{n=1}^{\infty} \frac{\tilde{P}^*(n)}{n^s}. \quad (14)$$

By Euler product representation we have

$$\begin{aligned}
F(s) &= \prod_p \left(1 + \frac{2p-1}{p^{s+1}} + \frac{2p^2-p}{p^{2s+2}} + \frac{2p^3-p^2}{p^{3s+3}} + \cdots\right) \\
&= \zeta(s) \prod_p \left(1 - \frac{1}{p^s}\right)\left(1 + \frac{2}{p^s} - \frac{1}{p^{s+1}} + \frac{2}{p^{2s}} - \frac{1}{p^{2s+1}} + \cdots\right) \\
&= \zeta(s) \prod_p \left(1 + \frac{1}{p^s} - \frac{1}{p^{s+1}}\right) \\
&= \frac{\zeta^2(s)}{\zeta(2s)} \prod_p \left(1 - \frac{1}{p^s}\right) \prod_p \left(1 - \frac{1}{p^{2s}}\right)^{-1} \left(1 + \frac{1}{p^s} - \frac{1}{p^{s+1}}\right) \\
&= \frac{\zeta^2(s)}{\zeta(2s)} G(s),
\end{aligned}$$

where

$$G(s) = \prod_p \left(1 - \frac{1}{p^{s+1} + p}\right). \tag{15}$$

From the above formula, it is easy to see that $G(s)$ can be expanded to a Dirichlet series, which is absolutely convergent for $\Re s > 0$. Write

$$G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \tag{16}$$

then we can easily get

$$g(n) \ll n^{\epsilon}, \qquad \sum_{n \le x} |g(n)| = O(x^{\epsilon}). \tag{17}$$

Notice that

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{m=1}^{\infty} \frac{d^{(2)}(m)}{m^s}. \tag{18}$$

By the Dirichlet convolution, we have

$$\sum_{n \le x} \tilde{P}^*(n) = \sum_{m\ell \le x} d^{(2)}(m) g(\ell) = \sum_{\ell \le x} g(\ell) \sum_{m \le x/\ell} d^{(2)}(m),$$

and formula (8) applied to the inner sum gives

$$\sum_{n \le x} \tilde{P}^*(n) = \sum_{\ell \le x} g(\ell) \left\{ \frac{x}{\zeta(2)\ell} \left(\log(\frac{x}{\ell}) + 2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)} \right) + O\left((\frac{x}{\ell})^{\lambda+\epsilon} \right) \right\}$$

$$= \frac{x}{\zeta(2)} \left\{ \left(\log x + 2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)} \right) \sum_{\ell \le x} \frac{g(\ell)}{\ell} - \sum_{\ell \le x} \frac{g(\ell)\log \ell}{\ell} \right\} + O\left(x^{\lambda+\epsilon} \sum_{\ell \le x} \frac{|g(\ell)|}{\ell^{\lambda+\epsilon}} \right).$$

$$= \frac{x}{\zeta(2)} \left\{ \left(\log x + 2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)} \right) \sum_{\ell=1}^{\infty} \frac{g(\ell)}{\ell} - \sum_{\ell=1}^{\infty} \frac{g(\ell)\log \ell}{\ell} + O(x^{-1+\epsilon}) \right\} + O\left(x^{\lambda+\epsilon}\right),$$

if we notice by (17) that both of the infinite series $\sum\limits_{\ell=1}^{\infty} \frac{g(\ell)}{\ell}$ and $\sum\limits_{\ell=1}^{\infty} \frac{g(\ell)\log \ell}{\ell}$ are absolutely convergent.

From (15), (16) and the definitions of K_1, K_2, we have

$$\sum_{\ell=1}^{\infty} \frac{g(\ell)}{\ell} = G(1) = \prod_p \left(1 - \frac{1}{p^2 + p}\right) = K_1, \tag{19}$$

$$\sum_{\ell=1}^{\infty} \frac{g(\ell)\log \ell}{\ell} = \sum_{n=1}^{\infty} \frac{\mu(n)(\log n - \alpha(n) + 2\beta(n))}{n\psi(n)} \tag{20}$$

$$= K_1\left(2\gamma - \frac{1}{2} - \frac{2\zeta'(2)}{\zeta(2)}\right) - K_2.$$

Then

$$\sum_{n\le x}\tilde{P}^*(n) = \frac{x}{\zeta(2)}\left((\log x - \frac{1}{2})K_1 + K_2\right) + O(x^{\lambda+\epsilon}). \tag{21}$$

From the definitions of $\tilde{P}^*(n)$and Abel's summation formula, we can easily get

$$\begin{aligned}\sum_{n\le x}\tilde{P}(n) \quad &= \sum_{n\le x}\tilde{P}^*(n)n = \int_1^x td\left(\sum_{n\le t}\tilde{P}^*(n)\right)\\ &= \frac{x^2}{2\zeta(2)}\left(K_1\log x + K_2\right) + O(x^{1+\lambda+\epsilon}).\end{aligned}$$

Corollary 2 follows by taking $\lambda = 4/11$.

3 Proof of Theorem 3

From the proof of Theorem 1, we have

$$F(s) = \sum_{n=1}^{\infty}\frac{\tilde{P}^*(n)}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}G(s). \tag{22}$$

Let

$$\zeta^2(s)G(s) = \sum_{n=1}^{\infty}\frac{h(n)}{n^s}, \quad \Re s > 1. \tag{23}$$

Then we have

Lemma 5. *For any real numbers $x \ge 1$ and $\epsilon > 0$, we have*

$$\sum_{n\le x}h(n) = x\left(\left(\log x - \frac{1}{2} + \frac{2\zeta'(2)}{\zeta(2)}\right)K_1 + K_2\right) + O(x^{\theta+\epsilon}), \tag{24}$$

where θ is defined in (7).

Proof. Recall that

$$\sum_{n=1}^{\infty}\frac{g(n)}{n^s} = G(s), \quad g(n) \ll n^{\epsilon}.$$

Then we have

$$h(n) = \sum_{n=m\ell}d(m)g(\ell), \quad h(n) \ll n^{\epsilon}. \tag{25}$$

Thus from (6),(7) we get

$$\begin{aligned}
\sum_{n\le x} h(n) &= \sum_{m\ell\le x} d(m)g(\ell) = \sum_{\ell\le x} g(\ell)\sum_{m\le\frac{x}{\ell}} d(m)\\
&= \sum_{\ell\le x} g(\ell)\left\{\frac{x}{\ell}\left(\log(\frac{x}{\ell})+2\gamma-1\right)+O\left((\frac{x}{\ell})^{\theta+\epsilon}\right)\right\}\\
&= x\left\{(\log x+2\gamma-1)\sum_{\ell\le x}\frac{g(\ell)}{\ell}-\sum_{\ell\le x}\frac{g(\ell)\log\ell}{\ell}\right\}+O\left(x^{\theta+\epsilon}\sum_{\ell\le x}\frac{|g(\ell)|}{\ell^{\theta+\epsilon}}\right)\\
&= x\left\{(\log x+2\gamma-1)\sum_{\ell=1}^{\infty}\frac{g(\ell)}{\ell}-\sum_{\ell=1}^{\infty}\frac{g(\ell)\log\ell}{\ell}+O(x^{-1+\epsilon})\right\}+O\left(x^{\theta+\epsilon}\right)
\end{aligned}$$

Then Lemma 5 follows from the above formula and (19), (20).

□

Lemma 6. *For any real numbers $x\ge 1$ and $x<u\le 2x$, we have*

$$\sum_{x<n\le u}\tilde{P}^*(n) = M(u)-M(x)+E(u,x), \tag{26}$$

where

$$M(x)=\frac{x}{\zeta(2)}\left((\log x-\frac{1}{2})K_1+K_2\right)$$

is the main term of $\sum_{n\le x}\tilde{P}^*(n)$, and

$$E(u,x)\ll (u-x)x^{-\epsilon}+x^{\theta+2\epsilon}.$$

Proof. From (22) and (23), we have

$$\tilde{P}^*(n)=\sum_{n=\ell m^2} h(\ell)\mu(m).$$

Then

$$\sum_{x<n\le u}\tilde{P}^*(n)=\sum_{x<\ell m^2\le u} h(\ell)\mu(m)=\sum_1+\sum_2, \tag{27}$$

where

$$\sum_1=\sum_{m\le x^{2\epsilon}}\mu(m)\sum_{\frac{x}{m^2}<\ell\le\frac{u}{m^2}} h(\ell),$$

$$\sum_2=\sum_{\substack{x<\ell m^2\le u\\ m>x^{2\epsilon}}} h(\ell)\mu(m).$$

By Lemma 5 we have

$$\sum_1=\sum_{m\le x^{2\epsilon}}\mu(m)\left(H(\frac{u}{m^2})-H(\frac{x}{m^2})+O(\frac{x}{m^2})^{\theta+\epsilon}\right) \tag{28}$$

$$= \sum_{m\le x^{2\epsilon}} \mu(m) \left(H(\frac{u}{m^2}) - H(\frac{x}{m^2}) \right) + O(x^{\theta+2\epsilon}),$$

where

$$H(x) := ax\log x + bx$$

is the main term of $\sum_{n\le x} h(n)$, and $a = K_1, \quad b = \left(\frac{2\zeta'(2)}{\zeta(2)} - \frac{1}{2}\right) K_1 + K_2$. Then

$$\sum_{m\le x^{2\epsilon}} \mu(m) \left(H(\frac{u}{m^2}) - H(\frac{x}{m^2}) \right)$$

$$= \sum_{m\le x^{2\epsilon}} \mu(m) \left(\frac{H(u) - H(x)}{m^2} + \frac{2(ax - au)}{m^2} \log m \right)$$

$$= (H(u) - H(x)) \sum_{m\le x^{2\epsilon}} \frac{\mu(m)}{m^2} + 2(ax - au) \sum_{m\le x^{2\epsilon}} \frac{\mu(m)\log m}{m^2}$$

$$= (H(u) - H(x)) \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} + 2(ax - au) \sum_{m=1}^{\infty} \frac{\mu(m)\log m}{m^2} + O\left((u-x)x^{-2\epsilon}\right).$$

It is well known that

$$\frac{1}{\zeta(s)} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^s}, \qquad \Re s > 1,$$

which gives by differentiation

$$\frac{\zeta'(s)}{\zeta^2(s)} = \sum_{m=1}^{\infty} \frac{\mu(m)\log m}{m^s},$$

and hence

$$\sum_1 = \frac{H(u) - H(x)}{\zeta(2)} + 2(ax - au)\frac{\zeta'}{\zeta^2}(2) + O(x^{\theta+2\epsilon} + (u-x)x^{-2\epsilon})$$

$$= M(u) - M(x) + O\left(x^{\theta+2\epsilon} + (u-x)x^{-2\epsilon}\right), \tag{29}$$

where

$$M(x) = \frac{x}{\zeta(2)} \left((\log x - \frac{1}{2})K_1 + K_2 \right).$$

For $\sum_2$, if we notice that $h(n) \ll n^\epsilon$, then

$$\sum_2 \ll x^\epsilon \sum_{\substack{x<\ell m^2\le u \\ m > x^{2\epsilon}}} 1 := x^\epsilon \sum_3, \tag{30}$$

where

$$\sum_3 = \sum_{\substack{x<\ell m^2\le u\\ m>x^{2\epsilon}}} 1 = \sum_{x<\ell m^2\le u} 1 - \sum_{\substack{x<\ell m^2\le u\\ m\le x^{2\epsilon}}} 1 \tag{31}$$

$$= \sum_{x<n\le u} d(1,2;n) - \sum_{\substack{x<\ell m^2\le u\\ m\le x^{2\epsilon}}} 1 = \sum_{31} - \sum_{32},$$

say. From Richert [10] we have

$$\sum_{n\le x} d(1,2;n) = \zeta(2)x + \zeta(1/2)x^{1/2} + O(x^{2/9}\log x).$$

Then

$$\sum_{31} = \zeta(2)(u-x) + O\left((u-x)x^{-1/2} + x^{2/9}\log x\right). \tag{32}$$

For $\sum_{32}$ we have

$$\sum_{32} = \sum_{m\le x^{2\epsilon}} \sum_{\frac{x}{m^2}<\ell\le\frac{u}{m^2}} 1 = \sum_{m\le x^{2\epsilon}} \left(\frac{u-x}{m^2} + O(1)\right) \tag{33}$$

$$= \zeta(2)(u-x) + O\left((u-x)x^{-2\epsilon} + x^{2\epsilon}\right).$$

Then from (31)–(33) we have

$$\sum_3 \ll (u-x)x^{-2\epsilon} + x^{2/9}\log x, \tag{34}$$

and hence

$$\sum_2 \ll (u-x)x^{-\epsilon} + x^{2/9+\epsilon}. \tag{35}$$

Lemma 6 follows from (27), (29) and (35). □

Now we prove Theorem 3. From the definitions of $\tilde{P}^*(n)$ and Abel's summation formula, we have

$$\sum_{x<n\le x+y} \tilde{P}(n) = \sum_{x<n\le x+y} \tilde{P}^*(n)n = \int_x^{x+y} ud\left(\sum_{x<n\le u} \tilde{P}^*(n)\right),$$

and Lemma 6 applied to the sum in the right side gives

$$\sum_{x<n\le x+y} \tilde{P}(n) = \int_1 + \int_2, \tag{36}$$

where

$$\begin{aligned}\int_1 &= \int_x^{x+y} ud\left(M(u)-M(x)\right),\\ \int_2 &= \int_x^{x+y} ud\left(E(u,x)\right).\end{aligned}$$

In view of the definition of $M(x)$ in Lemma 6, we obtain

$$\int_1 = \int_x^{x+y} uM'(u)du = \frac{1}{2\zeta(2)}\int_x^{x+y} u\left(2K_1\log u + K_1 + 2K_2\right)du. \tag{37}$$

For $\int_2$, we integrate it by parts, to get

$$\begin{aligned}\int_2 &= \int_x^{x+y} ud\left(E(u,x)\right)\\ &= (x+y)E(x+y;x) - \int_x^{x+y} E(u,x)du.\end{aligned}$$

By Lemma 6 we get

$$E(u,x) \ll (u-x)x^{-\epsilon} + x^{\theta+2\epsilon}.$$

Therefore

$$\int_2 \ll x(yx^{-\epsilon}+x^{\theta+2\epsilon}) + \int_x^{x+y}\left((u-x)x^{-\epsilon}+x^{\theta+2\epsilon}\right)du \tag{38}$$

$$\begin{aligned}&\ll yx^{1-\epsilon} + x^{1+\theta+2\epsilon} + y^2x^{-\epsilon} + yx^{\theta+2\epsilon}\\ &\ll yx^{1-\epsilon} + x^{1+\theta+2\epsilon},\end{aligned}$$

if we notice that $y \leq x$.

Now Theorem 3 follows from (36)–(38). If we take $\theta = 131/416$, then we can get Corollary 4.

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