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A Note on Narayana Triangles and Related Polynomials, Riordan Arrays, and MIMO Capacity Calculations

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Abstract

We study the Narayana triangles and related families of polynomials. We link this study to Riordan arrays and Hankel transforms arising from a special case of capacity calculation related to MIMO communication systems. A link is established between a channel capacity calculation and a series reversion.

1 Introduction

The Narayana numbers, which are closely related to the ubiquitous Catalan numbers, have an important and growing literature. Their applications are varied. In this note, we look at the mathematics of one application in the area of MIMO (multiple input, multiple output) wireless communication. For our purposes, it is useful to distinguish between three different "Narayana triangles" and their associated "Narayana polynomials". These triangles are documented separately in Sloane's *Encyclopedia* [32] along with other variants. We will find it useful in this note to use the language of Riordan arrays [30] for later sections. In the next section we provide a quick introduction to the Riordan group. We also use the notion of "Deleham array" [3], which is explained in Appendix B. Our approach to Deleham arrays is based on continued fractions [39]. We shall be interested in the Hankel transform [23, 22, 29] of a number of integer sequences that we shall encounter. We recall that if a_n is a given sequence, then the sequence with general term given by the determinant $|a_{i+j}|_{0 \le i,j \le n}$ is called the Hankel transform of a_n. We shall mention well-known orthogonal polynomials in this note. General references are [8, 18]. Links between orthogonal polynomials and Riordan arrays have been studied in [1, 2]. Techniques to calculate Hankel transforms using associated orthogonal polynomials will follow methods to be found in [10, 29].

For an integer sequence a_n, that is, an element of $\mathbb{Z}^{\mathbb{N}}$, the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ is called the *ordinary generating function* or g.f. of the sequence. a_n is thus the coefficient of x^n in this series. We denote this by $a_n = [x^n]f(x)$. For instance, $F_n = [x^n]\frac{x}{1-x-x^2}$ is the n-th Fibonacci number A000045, while $C_n = [x^n]\frac{1-\sqrt{1-4x}}{2x}$ is the n-th Catalan number A000108. We use the notation $0^n = [x^n]1$ for the sequence $1, 0, 0, 0, \ldots$, A000007. Thus $0^n = [n = 0] = \delta_{n,0} = \binom{0}{n}$. Here, we have used the Iverson bracket notation [19], defined by $[\mathcal{P}] = 1$ if the proposition $\mathcal{P}$ is true, and $[\mathcal{P}] = 0$ if $\mathcal{P}$ is false.

For a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $f(0) = 0$ we define the reversion or compositional inverse of f to be the power series $\bar{f}(x)$ such that $f(\bar{f}(x)) = x$. We sometimes write $\bar{f} = \text{Rev} f$.

2 Riordan arrays

The *Riordan group* [30, 34], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1 x + g_2 x^2 + \ldots$ and $f(x) = f_1 x + f_2 x^2 + \ldots$ where $f_1 \neq 0$ [34]. The associated matrix is the matrix whose i-th column is generated by $g(x)f(x)^i$ (the first column being indexed by 0). The matrix corresponding to the pair f, g is denoted by (g, f) or $\mathcal{R}(g, f)$. The group law is then given by

$$(g, f) \cdot (h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of f. Also called the reversion of f, we will use the notation $\bar{f} = \text{Rev}(f)$ as well.

A Riordan array of the form $(g(x), x)$, where $g(x)$ is the generating function of the sequence a_n, is called the *sequence array* of the sequence a_n. Its general term is a_{n-k}. Such arrays are also called *Appell* arrays as they form the elements of the Appell subgroup.

If $\mathbf{M}$ is the matrix (g, f), and $\mathbf{a} = (a_0, a_1, \ldots)'$ is an integer sequence with ordinary generating function $\mathcal{A}(x)$, then the sequence $\mathbf{Ma}$ has ordinary generating function $g(x)\mathcal{A}(f(x))$. The (infinite) matrix (g, f) can thus be considered to act on the ring of integer sequences $\mathbf{Z}^{\mathbf{N}}$ by multiplication, where a sequence is regarded as a (infinite) column vector. We can extend this action to the ring of power series $\mathbf{Z}[[x]]$ by

$$(g, f) : \mathcal{A}(x) \longrightarrow (g, f) \cdot \mathcal{A}(x) = g(x)\mathcal{A}(f(x)).$$

Example 1. The binomial matrix $\mathbf{B}$ is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group. It has general element $\binom{n}{k}$. More generally, $\mathbf{B}^m$ is the element $(\frac{1}{1-mx}, \frac{x}{1-mx})$ of the Riordan group, with general term $\binom{n}{k}m^{n-k}$. It is easy to show that the inverse $\mathbf{B}^{-m}$ of $\mathbf{B}^m$ is given by $(\frac{1}{1+mx}, \frac{x}{1+mx})$.

The row sums of the matrix (g, f) have generating function

$$(g, f) \cdot \frac{1}{1-x} = \frac{g(x)}{1-f(x)}$$

while the diagonal sums of (g, f) have generating function $g(x)/(1 - xf(x))$.

Many interesting examples of Riordan arrays can be found in Neil Sloane's On-Line Encyclopedia of Integer Sequences, [32, 33]. Sequences are frequently referred to by their OEIS number. For instance, the matrix $\mathbf{B}$ is A007318.

3 The Narayana Triangles and their generating functions

In this section, we define four separate though related "Narayana triangles", and we describe their (bi-variate) generating functions.

The number triangle $\mathbf{N}_0$ with general term

$$N_0(n, k) = \frac{1}{n + 0^n}\binom{n}{k}\binom{n}{k+1} \tag{1}$$

has [40] generating function $\phi_0(x, y)$ which satisfies the equation

$$xy\phi_0^2 + (x + xy - 1)\phi_0 + x = 0.$$

Solving for $\phi_0(x, y)$ yields

$$\phi_0(x, y) = \frac{1 - x(1 + y) - \sqrt{1 - 2x(1 + y) + x^2(1 - y)^2}}{2xy}. \tag{2}$$

This triangle begins

$$\mathbf{N}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 6 & 6 & 1 & 0 & 0 & \dots \\ 1 & 10 & 20 & 10 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The triangle $\mathbf{N}_1$ with general term

$$N_1(n,k) = 0^{n+k} + \frac{1}{n+0^n}\binom{n}{k}\binom{n}{k+1} = \frac{1}{k+1}\binom{n-1}{k}\binom{n}{k} \tag{3}$$

which begins

$$\mathbf{N}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 6 & 6 & 1 & 0 & 0 & \ldots \\ 1 & 10 & 20 & 10 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

clearly has generating function

$$\phi_1(x,y) = 1 + \phi_0(x,y) = \frac{1 - x(1-y) - \sqrt{1-2x(1+y)+x^2(1-y)^2}}{2xy}. \tag{4}$$

The triangle $\mathbf{N}_2$, which is the reversal of $\mathbf{N}_1$, has general term

$$\begin{aligned} N_2(n,k) &= [k \le n] N_1(n, n-k) = 0^{n+k} + \frac{1}{n+0^{nk}}\binom{n}{k}\binom{n}{k-1} && (5) \\ &= \frac{1}{n-k+1}\binom{n-1}{n-k}\binom{n}{k}, && (6) \end{aligned}$$

and begins

$$\mathbf{N}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 3 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 6 & 6 & 1 & 0 & \ldots \\ 0 & 1 & 10 & 20 & 10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This triangle has generating function

$$\phi_2(x,y) = 1 + y\phi_0(x,y) = \frac{1 + x(1-y) - \sqrt{1-2x(1+y)+x^2(1-y)^2}}{2x}. \tag{7}$$

Finally the “Pascal-like” variant $\mathbf{N}_3$ with general term

$$N_3(n,k) = N_0(n+1,k) = \frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1} \tag{8}$$

which begins

$$\mathbf{N}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 6 & 6 & 1 & 0 & 0 & \ldots \\ 1 & 10 & 20 & 10 & 1 & 0 & \ldots \\ 1 & 15 & 50 & 50 & 15 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

has generating function

$$\phi_3(x,y) = \frac{\phi_0(x,y)}{x} = \frac{1-x(1+y)-\sqrt{1-2x(1+y)+x^2(1-y)^2}}{2x^2y}. \tag{9}$$

Triangle	A-number	Generating function
$\mathbf{N}_1$	A131198	$\phi_1(x,y) = \frac{1-x(1-y)-\sqrt{1-2x(1+y)+x^2(1-y)^2}}{2xy}$
$\mathbf{N}_2$	A090181	$\phi_2(x,y) = \frac{1+x(1-y)-\sqrt{1-2x(1+y)+x^2(1-y)^2}}{2x}$
$\mathbf{N}_3$	A001263	$\phi_3(x,y) = \frac{1-x(1+y)-\sqrt{1-2x(1+y)+x^2(1-y)^2}}{2x^2y}$

4 Narayana Triangles and series reversion

Using the generating functions obtained in the last section, we can relate the Narayana triangles to the process of reverting sequences.

Proposition 2. *We have*

$$\phi_1(x,y) = \frac{1}{x}Rev_x\frac{x(1-xy)}{1-x(y-1)}. \tag{10}$$

Proof. We calculate the reversion of the expression

$$\frac{x(1-xy)}{1-x(y-1)},$$

considered as a function in x, with parameter y. This amounts to solving the equation

$$\frac{u(1-uy)}{1-u(y-1)} = x$$

for the unknown u. We obtain

$$u = x\phi_1(x,y).$$

Thus we have

$$\phi_1(x,y) = \frac{1}{x}\text{Rev}_x\frac{x(1-xy)}{1-x(y-1)},$$

as required. □

In like manner, we have

Proposition 3.

$$\phi_2(x,y) = \frac{1}{x} Rev_x \frac{x(1-x)}{1-x(1-y)}. \tag{11}$$

Proof. We calculate the x-reversion of the expression $\frac{x(1-x)}{1-x(1-y)}$. Thus we wish to solve for u, where

$$\frac{u(1-u)}{1-u(1-y)} = x.$$

We obtain

$$u = x\phi_2(x,y).$$

Thus

$$\phi_2(x,y) = \frac{1}{x}\mathrm{Rev}_x \frac{x(1-x)}{1-x(1-y)}.$$

□

In similar fashion, we can establish that

Proposition 4.

$$\phi_3(x,y) = \frac{1}{x} Rev_x \frac{x}{1+(1+y)x+yx^2}. \tag{12}$$

We note that the Narayana triangles are not Riordan arrays.

5 The Narayana Triangles and continued fractions

In this section, we develop continued fraction versions for each of the generating functions ϕ_1, ϕ_2, ϕ_3. In this case of ϕ_3, we give two distinct (but equivalent) continued fraction expressions.

Proposition 5. *[4, Section 3.5] We have the following continued fraction expansion of $\phi_1(x,y)$:*

$$\phi_1(x,y) = \cfrac{1}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cdots}}}}}. \tag{13}$$

Proof. It is easy to see that $\phi_1(x,y)$ obeys the equation [4]

$$xy\phi_1^2 - (xy-x+1)\phi_1 + 1 = 0. \tag{14}$$

Thus

$$\phi_1(1-x-xy\phi_1) = 1 - xy\phi_1$$

and hence

$$\begin{aligned}\phi_1 &= \frac{1-xy\phi_1}{1-xy\phi_1-x} \\ &= \frac{1}{1-\frac{x}{1-xy\phi_1}}.\end{aligned}$$

We thus obtain the result that $\phi_1(x,y)$ can be expressed as the continued fraction

$$\phi_1(x,y) = \cfrac{1}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cdots}}}}}.$$

□

Corollary 6. *We have*

$$\mathbf{N}_1 = [1,0,1,0,1,0,1,\ldots] \quad \Delta \quad [0,1,0,1,0,1,\ldots].$$

Proposition 7. *We have the following continued fraction expansion of $\phi_2(x,y)$:*

$$\phi_2(x,y) = \cfrac{1}{1-\cfrac{xy}{1-\cfrac{x}{1-\cfrac{xy}{1-\cfrac{x}{1-\cdots}}}}}. \tag{15}$$

Proof. It is easy to establish that

$$x\phi_2^2 - (1+x-xy)\phi_2 + 1 = 0 \tag{16}$$

from which we deduce

$$\phi_2(1-x\phi_2-xy) = 1-x\phi_2$$

and hence

$$\begin{aligned}\phi_2 &= \frac{1-x\phi_2}{1-x\phi_2-xy} \\ &= \frac{1}{1-\frac{xy}{1-x\phi_2}}.\end{aligned}$$

The result follows from this. □

Corollary 8. *We have*

$$\mathbf{N}_2 = [0,1,0,1,0,1,\ldots] \quad \Delta \quad [1,0,1,0,1,0,1,\ldots].$$

In order to find an expression for ϕ_3, we first note that

$$\phi_3 = \frac{\phi_1 - 1}{x} \Rightarrow \phi_1 = 1 + x\phi_3.$$

Substituting into Eq. (14) and simplifying, we find that

$$\phi_3(1 - xy - x^2y\phi_3) = 1 + x\phi_3 \tag{17}$$

and hence

$$\begin{aligned}
\phi_3 &= \frac{1 + x\phi_3}{1 - xy(1 + x\phi_3)} \\
&= \frac{1}{-xy + \frac{1}{1+x\phi_3}} \\
&= \frac{1}{-xy + \frac{1}{\phi_1}}.
\end{aligned}$$

But

$$\frac{1}{\phi_1} = 1 - \cfrac{x}{1 - \cfrac{xy}{1 - \cfrac{x}{1 - \cdots}}}.$$

Hence we obtain:

Proposition 9. *We have the following continued fraction expansion of $\phi_3(x,y)$:*

$$\phi_3(x,y) = \cfrac{1}{1 - xy - \cfrac{x}{1 - \cfrac{xy}{1 - \cfrac{x}{1 - \cdots}}}}. \tag{18}$$

Corollary 10. *We have*

$$\mathbf{N}_3 = [0,1,0,1,0,1,\ldots] \quad \Delta^{(1)} \quad [1,0,1,0,1,0,1,\ldots].$$

We end this section by expressing the g.f. of $\mathbf{N}_3$ in another way.

Proposition 11. *The generating function of $\mathbf{N}_3$ has the following continued fraction expression*

$$\phi_3(x,y) = \cfrac{1}{1 - x - xy - \cfrac{x^2y}{1 - x - xy - \cfrac{x^2y}{1 - x - xy - \cfrac{x^2y}{1 - \cdots}}}}.$$

Proof. This can be seen by solving the equation

$$u = \frac{1}{1 - x - xy - x^2 yu}$$

and comparing the solution $u(x, y)$ with $\phi_3(x, y)$. □

6 Narayana polynomials and moment sequences

To each of the above triangles, there is a family of "Narayana" polynomials [35, 37], where the triangles take on the role of coefficient arrays. Thus we get the polynomials

$$\begin{aligned}
\mathcal{N}_{1,n}(y) &= \sum_{k=0}^{n} N_1(n,k)y^k \\
\mathcal{N}_{2,n}(y) &= \sum_{k=0}^{n} N_2(n,k)y^k \\
\mathcal{N}_{3,n}(y) &= \sum_{k=0}^{n} N_3(n,k)y^k.
\end{aligned}$$

Note that since $\mathbf{N}_2$ is the reversal of $\mathbf{N}_1$, we have

$$\mathcal{N}_{2,n}(y) = \sum_{k=0}^{n} N_2(n,k)y^k = \sum_{k=0}^{n} N_1(n,k)y^{n-k}.$$

Example 12. The first terms of the sequence $(\mathcal{N}_{1,n}(y))_{n\geq 0}$ are:

$$1, 1, 1+y, 1+3y+y^2, 1+6y+6y^2+y^3, 1+10y+20y^2+10y^3+y^4, \ldots$$

Using the results of section **3**, we see that

$$\begin{aligned}
\mathcal{N}_{1,n}(y) &= [x^{n+1}]\mathrm{Rev}_x \frac{x(1-xy)}{1-(y-1)x} \\
\mathcal{N}_{2,n}(y) &= [x^{n+1}]\mathrm{Rev}_x \frac{x(1-x)}{1-(1-y)x} \\
\mathcal{N}_{3,n}(y) &= [x^{n+1}]\mathrm{Rev}_x \frac{x}{1+(1+y)x+yx^2}.
\end{aligned}$$

Values of these polynomials are often of significant combinatorial interest. Sample values for these polynomials are tabulated below.

y	$\mathcal{N}_{1,0}(y), \mathcal{N}_{1,1}(y), \mathcal{N}_{1,2}(y), \ldots$	A-number	Name
1	$1, 1, 2, 5, 14, 42, \ldots$	A000108	Catalan numbers
2	$1, 1, 3, 11, 45, 197 \ldots$	A001003	little Schröder numbers
3	$1, 1, 4, 19, 100, 562, \ldots$	A007564	
4	$1, 1, 5, 29, 185, 1257, \ldots$	A059231	

y	$\mathcal{N}_{2,0}(y), \mathcal{N}_{2,1}(y), \mathcal{N}_{2,2}(y), \ldots$	A-number	Name
1	$1, 1, 2, 5, 14, 42, \ldots$	A000108	Catalan numbers
2	$1, 2, 6, 22, 90, 394, \ldots$	A006318	Large Schröder numbers
3	$1, 3, 12, 57, 300, 1686, \ldots$	A047891	
4	$1, 4, 20, 116, 740, 5028, \ldots$	A082298	

y	$\mathcal{N}_{3,0}(y), \mathcal{N}_{3,1}(y), \mathcal{N}_{3,2}(y), \ldots$	A-number	Name
1	$1, 2, 5, 14, 42, 132, \ldots$	A000108(n+1)	shifted Catalan numbers
2	$1, 3, 11, 45, 197, 903, \ldots$	A001003(n+1)	shifted little Schröder numbers
3	$1, 4, 19, 100, 562, 3304, \ldots$	A007564(n+1)	
4	$1, 5, 29, 185, 1257, 8925, \ldots$	A059231(n+1)	

We can derive a moment representation for these polynomials using the generating functions above and the Stieltjes transform. We obtain the following:

Proposition 13. *The families of polynomials* $(\mathcal{N}_{1,n}(y))_{n\geq 0}$, $(\mathcal{N}_{2,n}(y))_{n\geq 0}$, $(\mathcal{N}_{3,n}(y))_{n\geq 0}$, *are each a family of moments corresponding to an associated family of orthogonal functions.*

Proof. Using the established generating functions $\phi_1(x,y)$, $\phi_2(x,y)$ and $\phi_3(x,y)$, and the Stieltjes-Perron transform (see Appendix C), we can establish the following moment representations, for the densities shown.

$$\begin{aligned}
\mathcal{N}_{1,n}(y) &= \frac{y-1}{y}0^n + \frac{1}{2\pi}\int_{y-2\sqrt{y}+1}^{y+2\sqrt{y}+1} x^n \frac{\sqrt{-x^2+2x(1+y)-(1-y)^2}}{2y}dx,\\
\mathcal{N}_{2,n}(y) &= \frac{1}{2\pi}\int_{y-2\sqrt{y}+1}^{y+2\sqrt{y}+1} x^n \frac{\sqrt{-x^2+2x(1+y)-(1-y)^2}}{x}dx,\\
\mathcal{N}_{3,n}(y) &= \frac{1}{2\pi}\int_{y-2\sqrt{y}+1}^{y+2\sqrt{y}+1} x^n \frac{\sqrt{-x^2+2x(1+y)-(1-y)^2}}{y}dx.
\end{aligned}$$

The associated orthogonal polynomials are determined by the densities shown. □

Using the theory developed in [1, 2], we can exhibit these families of polynomials as the first columns of three related Riordan arrays. More precisely, we have

Proposition 14. *The elements of the three families of polynomials* $(\mathcal{N}_{1,n}(y))_{n\geq 0}$, $(\mathcal{N}_{2,n}(y))_{n\geq 0}$, $(\mathcal{N}_{3,n}(y))_{n\geq 0}$ *are given by the first column of the inverse Riordan arrays given by* $\left(\frac{1}{1+x}, \frac{x}{(1+x)(1+yx)}\right)$, $\left(\frac{1}{1+yx}, \frac{x}{(1+x)(1+yx)}\right)$, *and* $\left(\frac{1}{(1+x)(1+yx)}, \frac{x}{(1+x)(1+yx)}\right)$, *respectively. These Riordan arrays are the coefficient arrays of the corresponding families of orthogonal polynomials. Thus*

$$\begin{aligned}
&\mathcal{N}_{1,n}(y) \quad \textit{is given by the first column of} \quad \left(\frac{1}{1+x}, \frac{x}{(1+x)(1+yx)}\right)^{-1},\\
&\mathcal{N}_{2,n}(y) \quad \textit{is given by the first column of} \quad \left(\frac{1}{1+yx}, \frac{x}{(1+x)(1+yx)}\right)^{-1}\\
&\mathcal{N}_{3,n}(y) \quad \textit{is given by the first column of} \quad \left(\frac{1}{(1+x)(1+yx)}, \frac{x}{(1+x)(1+yx)}\right)^{-1}.
\end{aligned}$$

Proof. We look at the case of $\mathcal{N}_{1,n}$, as the other cases are proved in similar manner. Thus we let

$$\left(\frac{1}{1+x}, \frac{x}{(1+x)(1+yx)}\right) = (g, f).$$

We wish then to show that

$$\phi_1(x,y) = \frac{1}{g(\text{Rev}_x f(x,y))}.$$

For $f(x,y) = \frac{x}{(1+x)(1+yx)}$, we find that

$$\text{Rev}_x f(x,y) = \frac{1 - x(1+y) - \sqrt{1 - 2x(1+y) + x^2(1-y)^2}}{2xy}.$$

Then since $g(x) = \frac{1}{1+x}$, we find that

$$\frac{1}{g(\text{Rev}_x f(x,y))} = 1 + \text{Rev}_x f(x,y) = 1 + \frac{1 - x(1+y) - \sqrt{1 - 2x(1+y) + x^2(1-y)^2}}{2xy} = \phi_1(x,y)$$

as required. Now

$$\begin{aligned} \left(\frac{1}{1+x}, \frac{x}{(1+x)(1+yx)}\right) &= \left(\frac{1+yx}{(1+x)(1+yx)}, \frac{x}{(1+x)(1+yx)}\right) \\ &= \left(\frac{1+yx}{1+(1+y)x+yx^2}, \frac{x}{1+(1+y)x+yx^2}\right), \end{aligned}$$

and hence [2] $\left(\frac{1}{1+x}, \frac{x}{(1+x)(1+yx)}\right)$ is the coefficient array of a family of orthogonal polynomials. $\square$

7 An investigation inspired by a MIMO application of the Narayana numbers

The role of the Catalan numbers and more recently the Narayana polynomials in the elucidation of the behaviour of certain families of random matrices, along with applications to areas such as MIMO wireless communication, is an active field of research. See for instance [15, 16, 20, 25, 26, 31, 38]. Other areas where Narayana polynomials and their generalizations find applications include that of associahedra [6, 17, 28] and secondary RNA structures [14].

The investigations in this section and those that follow are inspired by MIMO (multiple input, multiple output) data-communication applications in [25] and [38]. The reader is referred to Appendix A for the link with MIMO capacity calculations. We let

$$G_\beta(z) = -\frac{1}{2} + \frac{\beta - 1}{2z} + \sqrt{\frac{(1-\beta)^2}{4z^2} + \frac{1}{4} - \frac{1+\beta}{2z}}.$$

In terms of wireless transmission,

$$\beta = \frac{T}{R}$$

where we have T transmit antennas and R receive antennas (see Appendix A). In this section β can be treated as a parameter. Then the function

$$g_\beta(x) = -\frac{1}{x}G_\beta(\frac{1}{x})$$

satisfies

$$g_\beta(x) = \frac{1+(1-\beta)x - \sqrt{1-2x(1+\beta)+(1-\beta)^2x^2}}{2x}$$

and generates the sequence

$$1, \beta, \beta(\beta+1), \beta(\beta^2+3\beta+1), \beta(\beta^3+6\beta^2+6\beta+1), \ldots$$

In other words, $g_\beta(x)$ is the generating function of the sequence

$$a_n^{(\beta)} = \sum_{k=0}^{n} N_2(n,k)\beta^k = \mathcal{N}_{2,n}(\beta).$$

Thus

$$g_\beta(x) = \phi_2(x,\beta).$$

We have the following moment representation:

$$\begin{aligned}
a_n^{(\beta)} &= \frac{1}{2\pi}\int_{1+\beta-2\sqrt{\beta}}^{1+\beta+2\sqrt{\beta}} x^n \frac{\sqrt{-x^2+2x(1+\beta)-(1-\beta)^2}}{x}dx \\
&= \frac{1}{2\pi}\int_{(1-\sqrt{\beta})^2}^{(1+\sqrt{\beta})^2} x^n \frac{\sqrt{((1-\sqrt{\beta})^2-x)(x-(1+\sqrt{\beta})^2)}}{x}dx \\
&= \frac{\sqrt{\beta}}{\pi}\int_{(1-\sqrt{\beta})^2}^{(1+\sqrt{\beta})^2} x^n \frac{\sqrt{1-\left(\frac{1+\beta-x}{2\sqrt{\beta}}\right)^2}}{x}dx \\
&= \frac{\sqrt{\beta}}{\pi}\int_{(1-\sqrt{\beta})^2}^{(1+\sqrt{\beta})^2} x^n \frac{w_U\left(\frac{1+\beta-x}{2\sqrt{\beta}}\right)}{x}dx
\end{aligned}$$

where $w_U(x) = \sqrt{1-x^2}$ is the weight function for the Chebyshev polynomials of the second kind. This is an example of the well-known Marčenko-Pastur [24] distribution.

8 Riordan arrays, orthogonal polynomials and $\mathbf{N}_2$

We now note that $xg_\beta(x)$ is in fact the series reversion of the function

$$\frac{x(1-x)}{1+(\beta-1)x}.$$

This simple form leads us to investigate the nature of the coefficient array of the orthogonal polynomials $P_n^{(\beta)}(x)$ associated to the weight function

$$w(x)=\frac{1}{2\pi}\frac{\sqrt{-x^2+2x(1+\beta)-(1-\beta)^2}}{x}=\frac{1}{2\pi}\frac{\sqrt{4\beta-(x-1-\beta)^2}}{x}dx$$

for which the elements

$$a_n^{(\beta)}=\sum_{k=0}^{n} N_2(n,k)\beta^k$$

are the moments. Put otherwise, these are the family of orthogonal polynomials associated to the Narayana polynomials $\mathcal{N}_{2,n}$. These polynomials can be expressed in terms of the Hankel determinants associated to the sequence $a_n^{(\beta)}$. We find that the coefficient array of the polynomials $P_n^{(\beta)}(x)$ is given by the Riordan array

$$\left(\frac{1}{1+\beta x}, \frac{x}{1+(1+\beta)x+\beta x^2}\right)$$

whose inverse is given by

$$\mathbf{L}=\left(g_\beta(x), \frac{g_\beta(x)-1}{\beta}\right)=\left(\phi_2(x,\beta), \frac{\phi_2(x,\beta)-1}{\beta}\right).$$

The Jacobi-Stieltjes array [11, 12, 27] for $\mathbf{L}$ is found [1, 2] to be

$$\left(\begin{array}{ccccccc}
\beta & 1 & 0 & 0 & 0 & 0 & \dots \\
\beta & \beta+1 & 1 & 0 & 0 & 0 & \dots \\
0 & \beta & \beta+1 & 1 & 0 & 0 & \dots \\
0 & 0 & \beta & \beta+1 & 1 & 0 & \dots \\
0 & 0 & 0 & \beta & \beta+1 & 1 & \dots \\
0 & 0 & 0 & 0 & \beta & \beta+1 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

indicating that the Hankel transform of the sequence $a_n^{(\beta)}$ is $\beta^{\binom{n+1}{2}}$, and that

$$g_\beta(x)=\cfrac{1}{1-\beta x-\cfrac{\beta x^2}{1-(\beta+1)x-\cfrac{\beta x^2}{1-(\beta+1)x-\cfrac{\beta x^2}{1-\cdots}}}}.$$

We note that the coefficient array $\mathbf{L}^{-1}$ can be factorized as follows:

$$\mathbf{L}^{-1}=\left(\frac{1}{1+\beta x}, \frac{x}{1+(1+\beta)x+\beta x^2}\right)=\left(1, \frac{x}{1+x}\right)\cdot\left(\frac{1-x}{1+(\beta-1)x}, \frac{x(1-x)}{1+(\beta-1)x}\right). \quad (19)$$

Hence

$$\begin{aligned}\mathbf{L} &= \left(\frac{1-x}{1+(\beta-1)x}, \frac{x(1-x)}{1+(\beta-1)x}\right)^{-1} \cdot \left(1, \frac{x}{1+x}\right)^{-1} \\ &= (g_\beta(x), xg_\beta(x)) \cdot \left(1, \frac{x}{1-x}\right).\end{aligned}$$

The general term of the matrix

$$\left(\frac{1-x}{1+(\beta-1)x}, \frac{x(1-x)}{1+(\beta-1)x}\right)^{-1} = (g_\beta(x), xg_\beta(x))$$

is given by

$$\frac{k+1}{n+1}\sum_{j=0}^{n-k}\binom{n+1}{k+j+1}\binom{n+j}{j}(\beta-1)^{n-k-j} = \sum_{j=0}^{n-k}\frac{k+1}{k+j+1}\binom{n}{k+j}\binom{n+j}{j}(\beta-1)^{n-k-j}.$$

For instance, when $\beta = 1$, which is the case of the matrix $(1-x, x(1-x))^{-1}$, we get the expression

$$\frac{k+1}{n+1}\binom{2n-k}{n-k}$$

for the general term. Now the general term of the matrix $\left(1, \frac{1}{1-x}\right)$ is given by

$$\binom{n-1}{k-1} + 0^n(-1)^k.$$

Hence the general term of $\mathbf{L}$ is given by

$$\sum_{j=0}^{n}\sum_{i=0}^{n-j}\frac{j+1}{i+j+1}\binom{n}{i+j}\binom{n+i}{i}(\beta-1)^{n-j-i}(\binom{j-1}{k-1} + 0^j(-1)^k). \tag{20}$$

It is interesting to note that

$$\mathbf{L}^{-1} = \left(\frac{1+x}{(1+x)(1+\beta x)}, \frac{x}{(1+x)(1+\beta x)}\right). \tag{21}$$

We can use the factorization in Eq. (19) to express the orthogonal polynomials $P_n^{(\beta)}(x)$ in terms of the Chebyshev polynomials of the second kind $U_n(x)$. Thus we recognize [2] that the Riordan array

$$\left(\frac{1}{1+(1+\beta)x+\beta x^2}, \frac{x}{1+(1+\beta)x+\beta x^2}\right)$$

is the coefficient array of the modified Chebyshev polynomials of the second kind $\beta^{\frac{n}{2}}U_n(\frac{x-(\beta+1)}{2\sqrt{\beta}})$. Hence by the factorization in Eq. (19) we obtain

$$P_n^{(\beta)}(x) = \beta^{\frac{n}{2}}U_n\left(\frac{x-(\beta+1)}{2\sqrt{\beta}}\right) + \beta^{\frac{n-1}{2}}U_{n-1}\left(\frac{x-(\beta+1)}{2\sqrt{\beta}}\right). \tag{22}$$

We state this as a proposition.

Proposition 15. *The family $\{P_n(x)\}$ of orthogonal polynomials for which the Narayana polynomials $a_n^{(\beta)} = \mathcal{N}_{2,n}(x)$ are moments is given by*

$$P_n^{(\beta)}(x) = \beta^{\frac{n}{2}} U_n\left(\frac{x-(\beta+1)}{2\sqrt{\beta}}\right) + \beta^{\frac{n-1}{2}} U_{n-1}\left(\frac{x-(\beta+1)}{2\sqrt{\beta}}\right).$$

9 On the Hankel transform of the row sums of L

We recall that

$$\mathbf{L} = \left(g_\beta(x), \frac{g_\beta(x)-1}{\beta}\right)$$

is the matrix whose first column is given by terms of the Narayana polynomial sequence $\mathcal{N}_{2,n}(\beta)$. We now wish to calculate the Hankel transform of the row sums s_n^β of the matrix $\mathbf{L}$. The generating function of these row sums is given by

$$g_s(x) = \frac{(\beta+1)\sqrt{1-2x(\beta+1)+(1-\beta)^2x^2}+(\beta-1)(x(\beta+1)+1)}{2(1-2x(1+\beta))}.$$

We infer from this (using Stieltjes-Perron) that the row sum elements $s_n^{(\beta)}$ are the moments for the following weight function :

$$w_s(x) = \frac{1}{2\pi}\frac{\sqrt{-x^2+2x(1+\beta)-(\beta-1)^2}(\beta+1)}{x(2(1+\beta)-x)}$$

with support on the interval $[1+\beta-2\sqrt{\beta}, 1+\beta+2\sqrt{\beta}]$. From this we can determine that the Hankel transform of $s_n^{(\beta)}$ is given by

$$(\beta+1)^n\beta^{\binom{n}{2}}.$$

In fact, if we let

$$\mathbf{H}_s = \mathbf{L}_s\mathbf{D}_s\mathbf{L}_s^t$$

be the LDU decomposition [27] of the Hankel matrix associated with $s_n^{(\beta)}$, then we have

$$\mathbf{L}_s = \left(g_s(x), \frac{1-(1+\beta)x-\sqrt{1-2x(1+\beta)+(1-\beta)^2x^2}}{2\beta x}\right).$$

Equivalently,

$$\mathbf{L}_s^{-1} = \left(\frac{1-x^2}{1+(1+\beta)x+\beta x^2}, \frac{x}{1+(1+\beta)x+\beta x^2}\right)$$

is the coefficient array of the orthogonal polynomials associated to the sequence $s_n^{(\beta)}$. This is so since the Stieltjes-Jacobi matrix associated to $s_n^{(\beta)}$ takes the form

$$\begin{pmatrix} \beta+1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \beta+1 & \beta+1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & \beta & \beta+1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & \beta & \beta+1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \beta & \beta+1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \beta & \beta+1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

10 The Hankel transform of the row sums of $(g_\beta(x), xg_\beta(x))$

We have seen that

$$\mathbf{L} = (g_\beta(x), xg_\beta(x)) \cdot \left(1, \frac{x}{1-x}\right).$$

In this section, we look at the Hankel transform of the row sums of the factor $(g_\beta(x), xg_\beta(x))$ of $\mathbf{L}$. The row sums in question have generating function

$$\frac{g_\beta(x)}{1-xg_\beta(x)} = \frac{1-(1+\beta)x-\sqrt{1-2(1+\beta)x+(1-\beta)^2x^2}}{2\beta x^2}.$$

From this we can infer that the row sums of $(g_\beta(x), xg_\beta(x))$ are the moments for the weight function

$$w(x) = \frac{1}{2\pi}\frac{\sqrt{-x^2+2(1+\beta)x-(1-\beta)^2}}{\beta}.$$

This then allows us to prove that the Hankel transform sought is $\beta^{\binom{n+1}{2}}$.

11 Formulas for the row sums of $(g_\beta(x), xg_\beta(x))$

We can characterize the row sums of $(g_\beta(x), xg_\beta(x))$ by observing that

$$\frac{xg_\beta(x)}{1-xg_\beta(x)} = \frac{1-(1+\beta)x-\sqrt{1-2(1+\beta)x+(1-\beta)^2x^2}}{2\beta x}$$

is the series reversion of the function

$$\frac{x}{1+(1+\beta)x+\beta x^2}.$$

Hence the row sums are given by the $(n+1)$-st term of the series reversion of $\frac{x}{1+(1+\beta)x+\beta x^2}$. Thus the row sums of $(g_\beta(x), xg_\beta(x))$ are precisely

$$\mathcal{N}_{3,n}(\beta) = \sum_{k=0}^{n} N_3(n,k)\beta^k.$$

This may also be expressed by

$$\sum_{k=0}^{n} \frac{k+1}{n+1} \sum_{j=0}^{n-k} \binom{n+1}{k+j+1} \binom{n+j}{j} (\beta-1)^{n-k-j}$$

or by

$$\sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{k+1}{k+j+1} \binom{n}{k+j} \binom{n+j}{j} (\beta-1)^{n-k-j}.$$

We have seen that the general term of the matrix $\mathbf{L}$ is given by Eq. (20) and hence the general term $s_n^{(\beta)}$ of the row sums of $\mathbf{L}$ is given by

$$s_n^{(\beta)} = \sum_{k=0}^{n} \sum_{j=0}^{n} \sum_{i=0}^{n-j} \frac{j+1}{j+i+1} \binom{n}{i+j} \binom{n+i}{i} (\beta-1)^{n-j-i} (\binom{j-1}{k-1} + 0^j (-1)^k).$$

12 Narayana polynomials and hypergeometric functions

We recall that the hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; x)$ is defined by

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{x^k}{k!},$$

where

$$(\alpha)_k = \prod_{j=0}^{k-1} (\alpha + j).$$

For $n \in \mathbb{Z}$, we have $(n)_k = (-1)^k k! \binom{-n}{k}$. Thus we have

$$\begin{aligned} {}_2F_1(-n, -n-1; 2; x) &= \sum_{k=0}^{\infty} \frac{(-n)_k (-n-1)_k}{(2)_k} \frac{x^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k k! \binom{n}{k} (-1)^k k! \binom{n+1}{k}}{(n+1)!} \frac{x^k}{k!} \\ &= \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} \binom{n+1}{k} x^k \\ &= \sum_{k=0}^{n} N_3(n,k) x^k \\ &= \mathcal{N}_{3,n}(x). \end{aligned}$$

In two recent articles [5, 21], a link between $\mathcal{N}_{3,n}(x)$ and the Jacobi polynomials has been established. This is that

$$\mathcal{N}_{3,n}(x) = \frac{1}{n+1} (1-x)^n P_n^{(1,1)} \left(\frac{1+x}{1-x} \right). \tag{23}$$

Now

$$P_n^{\alpha,\beta}(x) = \binom{n+\alpha}{n} {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; \frac{1}{2}(1-x)),$$

and so

$$\mathcal{N}_{3,n}(x) = (1-x)^n \, {}_2F_1\left(-n, n+3; 2; \frac{x}{x-1}\right). \tag{24}$$

We can modify Eqn. (23) to obtain the following expression for $\mathcal{N}_{1,n}(x)$:

$$\mathcal{N}_{1,n}(x) = \frac{1 - x \cdot 0^n}{n + 0^n}(1-x)^{n-1} P_{n+0^n-1}^{(1,1)}\left(\frac{1+x}{1-x}\right).$$

Straight-forward calculation also establishes that

$$N_2(n,k) = [x^{n-k}] \, {}_2F_1(k+1, k; 2; x).$$

Note also that the triangle with general term

$$T(n,k) = [x^{n-k}] \, {}_2F_1(k+1, k; 1; x)$$

is the triangle (see A103371) with general term

$$T(n,k) = 0^{n+k} + \binom{n-1}{k-1}\binom{n}{k},$$

which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 3 & 6 & 1 & 0 & 0 & \dots \\ 0 & 4 & 18 & 12 & 1 & 0 & \dots \\ 0 & 5 & 40 & 60 & 20 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This matrix and $\mathbf{N}_2$ are related as follows:

$$N_2(n,k) = \frac{T(n,k)}{(n-k+1)} = \frac{0^{n+k} + \binom{n-1}{k-1}\binom{n}{k}}{n-k+1}.$$

For the matrix $\mathbf{N}_1$, we have the following:

$$\mathcal{N}_{1,n}(x) = {}_2F_1(-n, -n+1, 2, x).$$

Since $\mathbf{N}_2$ is the reversal of $\mathbf{N}_1$, we obtain

$$\mathcal{N}_{2,n}(x) = x^n \, {}_2F_1(-n, -n+1, 2, 1/x).$$

We can also note the following. We have seen (section 5) that $\mathbf{N}_3$ has generating function

$$\cfrac{1}{1-x-xy-\cfrac{x^2y}{1-x-xy-\cfrac{x^2y}{1-x-xy-\cfrac{x^2y}{1-\cdots}}}}.$$

$\mathbf{N}_3$ is thus seen [3] to be the binomial transform of the array with generating function

$$\cfrac{1}{1-xy-\cfrac{x^2y}{1-xy-\cfrac{x^2y}{1-xy-\cfrac{x^2y}{1-\cdots}}}}.$$

This array begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 6 & 1 & 0 & \dots \\ 0 & 0 & 0 & 10 & 10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

This is A107131, which is the coefficient array of the polynomials given by

$$x^n\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};2;\frac{4}{x}\right).$$

It also counts the number of Motzkin paths of length n with k steps $U=(1,1)$ or $H=(1,0)$. The general term of this array is

$$[k \le n]\binom{n}{2n-2k}C_{n-k}.$$

Thus

$$N_3(n,k)=\sum_{j=0}^{n}\binom{n}{j}\binom{j}{2(j-k)}C_{j-k}. \tag{25}$$

Since the row sums of $\mathbf{N}_3$ yield the shifted Catalan numbers, we arrive at the identity

$$C_{n+1}=\sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{j}\binom{j}{2(j-k)}C_{j-k}. \tag{26}$$

13 Appendix A - calculation of MIMO capacity

We follow [20] to derive an expression for MIMO capacity in a special case. This is a form of transmission technology which increases the transmission channel capacity by taking advantage of the multipath nature of transmission when many antennas transmit to many receivers. Thus we assume that we have R receive antennas and T transmit antennas, modeled by

$$\mathbf{r} = \mathbf{H}\mathbf{s} + \mathbf{n}$$

where $\mathbf{r}$ is the receive signal vector, $\mathbf{s}$ is the source signal vector, $\mathbf{n}$ is an additive white Gaussian noise (AWGN) vector, which is a realization of a complex normal distribution $N(\mathbf{0}, \sigma^2\mathbf{I}_R)$, and the channel is represented by the complex matrix $\mathbf{H} \in \mathbb{C}^{R\times T}$. We have the eigenvalue decomposition

$$\mathbf{H}^H\mathbf{H} = \frac{1}{T}\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H.$$

We assume $T < R$. Then the capacity of the uncorrelated MIMO channels is given by [25]

$$\begin{aligned}
C_{MIMO} &= \frac{1}{R}\log_2\det(\mathbf{I}_T + \mathbf{H}^H(\sigma^2\mathbf{I}_R)^{-1}\mathbf{H}) \\
&= \frac{1}{R}\log_2\det(\mathbf{I}_T + \frac{1}{\sigma^2}\mathbf{H}^H\mathbf{H}) \\
&= \frac{1}{R}\log_2\det(\mathbf{I}_T + \frac{1}{\sigma^2}\frac{1}{T}\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H) \\
&= \frac{1}{R}\log_2\det(\mathbf{I}_T + \frac{1}{\sigma^2 T}\mathbf{\Lambda}) \\
&= \frac{T}{R}\frac{1}{T}\sum_{i=1}^{T}\log_2(1 + \frac{1}{\sigma^2 T}\lambda_i) \\
&= \frac{\beta}{\ln 2}\frac{1}{T}\sum_{i=1}^{T}\ln(1 + \frac{1}{\sigma^2 T}\lambda_i)
\end{aligned}$$

where we have set

$$\beta = \frac{T}{R}.$$

Now

$$\begin{aligned}
\ln(1+x) &= \ln(1+x_0) + \sum_{k=1}^{N}(-1)^{k-1}\frac{(x-x_0)^k}{k(1+x_0)^k}, \qquad |x-x_0| < 1 \\
&= \ln(1+x_0) + \sum_{k=1}^{N}\frac{(-1)^{k-1}}{k(1+x_0)^k}\sum_{j=0}^{k}\binom{k}{j}x^j(-1)^{k-j}x_0^{k-j} \\
&= \ln(1+x_0) + \sum_{k=1}^{n}\sum_{j=0}^{k}\binom{k}{j}(-1)^{j-1}\frac{x_0^{k-j}}{k(1+x_0)^k}x^j \\
&= \sum_{k=0}^{N}p_k x^k,
\end{aligned}$$

where it is appropriate to take $x_0 = \frac{1}{\sigma^2}$. We thus obtain

$$\begin{aligned} C_{MIMO} &= \frac{\beta}{\ln 2} \frac{1}{T} \sum_{i=1}^{T} \sum_{k=0}^{N} p_k \left(\frac{\lambda_i}{\sigma^2 T} \right)^k \\ &= \frac{\beta}{\ln 2} \sum_{k=0}^{N} \frac{p_k}{(\sigma^2 T)^k} \left(\frac{1}{T} \sum_{i=1}^{T} \lambda_i^k \right) \\ &= \frac{\beta}{\ln 2} \sum_{k=0}^{N} \frac{p_k}{(\sigma^2 T)^k} m_k \\ &= \frac{\beta}{\ln 2} \sum_{k=0}^{N} \frac{p_k}{(\sigma^2 T)^k} \sum_{j=0}^{k} N_2(k,j) \beta^j. \end{aligned}$$

Here, we have replaced the expression $m_k = \frac{1}{T} \sum_{i=1}^{T} \lambda_i^k$ by the k-th moment of the limiting eigenvalue distribution function, which following [31] can to be shown to have Stieltjes transform

$$G_\beta(z) = -\frac{1}{2} + \frac{\beta - 1}{2z} + \sqrt{\frac{(1-\beta)^2}{4z^2} + \frac{1}{4} - \frac{1+\beta}{2z}}.$$

Thus by our preceding results (see Sections **3** and **6**) we arrive at the new expression

$$C_{MIMO} = \frac{\beta}{\ln 2} \sum_{k=0}^{N} \frac{p_k}{(\sigma^2 T)^k} [x^{k+1}] \mathrm{Rev}_x \left[\frac{x(1-x)}{1-(1-\beta)x} \right]. \tag{27}$$

14 Appendix B - the Deleham construction

For the purposes of this note, we define the *Deleham construction* [3] as follows. Given two sequences r_n and s_n,we use the notation

$$r \quad \Delta \quad s = [r_0, r_1, r_2, \ldots] \quad \Delta \quad [s_0, s_1, s_2, \ldots]$$

to denote the number triangle whose bi-variate generating function is given by

$$\cfrac{1}{1 - \cfrac{(r_0 x + s_0 xy)}{1 - \cfrac{(r_1 x + s_1 xy)}{1 - \cfrac{(r_2 x + s_2 xy)}{1 - \cdots}}}}.$$

We furthermore define

$$r \quad \Delta^{(1)} \quad s = [r_0, r_1, r_2, \ldots] \quad \Delta^{(1)} \quad [s_0, s_1, s_2, \ldots]$$

to denote the number triangle whose bi-variate generating function is given by

$$\cfrac{1}{1-(r_0x+s_0xy)-\cfrac{(r_1x+s_1xy)}{1-\cfrac{(r_2x+s_2xy)}{1-\cdots}}}.$$

See A084938 for the original definition.

15 Appendix C - The Stieltjes transform of a measure

The *Stieltjes transform* of a measure μ on $\mathbb{R}$ is a function G_μ defined on $\mathbb{C}\setminus\mathbb{R}$ by

$$G_\mu(z)=\int_{\mathbb{R}}\frac{1}{z-t}\mu(t).$$

If f is a bounded continuous function on $\mathbb{R}$, we have

$$\int_{\mathbb{R}} f(x)\mu(x)=-\lim_{y\to 0^+}\int_{\mathbb{R}} f(x)\Im G_\mu(x+iy)dx.$$

If μ has compact support, then G_μ is holomorphic at infinity and for large z,

$$G_\mu(z)=\sum_{n=0}^{\infty}\frac{a_n}{z^{n+1}},$$

where $a_n=\int_{\mathbb{R}} t^n\mu(t)$ are the moments of the measure. If $\mu(t)=d\psi(t)=\psi'(t)dt$ then (Stieltjes-Perron)

$$\psi(t)-\psi(t_0)=-\frac{1}{\pi}\lim_{y\to 0^+}\int_{t_0}^{t}\Im G_\mu(x+iy)dx.$$

If now $g(x)$ is the generating function of a sequence a_n, with $g(x)=\sum_{n=0}^{\infty}a_nx^n$, then we can define

$$G(z)=\frac{1}{z}g\left(\frac{1}{z}\right)=\sum_{n=0}^{\infty}\frac{a_n}{z^{n+1}}.$$

By this means, under the right circumstances we can retrieve the density function for the measure that defines the elements a_n as moments.

Example 16. We let $g(z)=\frac{1-\sqrt{1-4z}}{2z}$ be the g.f. of the Catalan numbers. Then

$$G(z)=\frac{1}{z}g\left(\frac{1}{z}\right)=\frac{1}{2}\left(1-\sqrt{\frac{x-4}{x}}\right).$$

Then

$$\Im G_\mu(x+iy)=-\frac{\sqrt{2}\sqrt{\sqrt{x^2+y^2}\sqrt{x^2-8x+y^2+16}-x^2+4x-y^2}}{4\sqrt{x^2+y^2}},$$

and so we obtain

$$\begin{aligned}
\psi'(x) &= -\frac{1}{\pi}\lim_{y\to 0^+}\left\{-\frac{\sqrt{2}\sqrt{\sqrt{x^2+y^2}\sqrt{x^2-8x+y^2+16}-x^2+4x-y^2}}{4\sqrt{x^2+y^2}}\right\} \\
&= \frac{1}{2\pi}\frac{\sqrt{x(4-x)}}{x}.
\end{aligned}$$

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