Journal of Integer Sequences, Vol. 14 (2011),
Article 11.5.8

Integer Sequences, Functions of Slow Increase, and the Bell Numbers

Rafael Jakimczuk
División Matemática
Universidad Nacional de Luján
Buenos Aires
Argentina
jakimczu@mail.unlu.edu.ar

In memory of my sister Fedra Marina Jakimczuk (1970–2010)

Abstract

In this article we first prove a general theorem on integer sequences A_n such that the following asymptotic formula holds,

$$\frac{A_n}{A_{n-1}} \sim Cn^{\alpha} f(n)^{\beta},$$

where $f(x)$ is a function of slow increase, $C > 0$, $\alpha > 0$ and β is a real number.

We also obtain some results on the Bell numbers B_n using well-known formulae. We compare the Bell numbers with a^n $(a > 0)$ and $(n!)^h$ $(0 < h \leq 1)$.

Finally, applying the general statements proved in the article we obtain the formula

$$B_{n+1} \sim e \ (B_n)^{1+\frac{1}{n}}.$$

1 Integer Sequences. A General Theorem.

We shall need the following well-known lemmas [12, pp. 332, 294].

Lemma 1. *If s_n is a sequence of positive numbers with limit s then the sequence*

$$\sqrt[n]{s_1 s_2 \cdots s_n}$$

has also limit s.

Lemma 2. *The following limit holds,*

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

We recall the definition of function of slow increase [7, Definition 1].

Definition 3. Let $f(x)$ be a function defined on interval $[a,\infty)$ such that $f(x) > 0$, $\lim_{x\to\infty} f(x) = \infty$ and with continuous derivative $f'(x) > 0$. The function $f(x)$ is of slow increase if and only if the following condition holds

$$\lim_{x\to\infty} \frac{f'(x)}{\frac{f(x)}{x}} = \lim_{x\to\infty} \frac{xf'(x)}{f(x)} = 0. \tag{1}$$

Typical functions of slow increase are $f(x) = \log x,\ f(x) = \log^2 x$ and $f(x) = \log\log x$.

Lemma 4. *If $f(x)$ is a function of slow increase on the interval $[b,\infty)$ then the following asymptotic formula holds*

$$\sqrt[n]{f(b)f(b+1)\cdots f(n)} \sim f(n), \tag{2}$$

where b is a positive integer.

Proof. Note that we always can suppose that $f(x) > 1$ on the interval $[b,\infty)$.

Since $\log f(x)$ is increasing and positive in the interval $[b,\infty)$ we find that

$$\begin{aligned}\sum_{i=b}^{n} \log f(i) = \sum_{i=b}^{n} (1 \cdot \log f(i)) \ &= \int_b^n \log f(x)\ dx + O(\log f(n)) = n \log f(n) \\ &+ \int_b^n \frac{xf'(x)}{f(x)}\ dx + O(\log f(n)).\end{aligned} \tag{3}$$

Note that the second equation in (3) is a sum of areas of rectangles of height $\log f(i)$ and base 1. Consequently the third equation in (3) is immediate.

L'Hôpital's rule gives (see (1))

$$\lim_{x\to\infty} \frac{\log f(x)}{x} = \lim_{x\to\infty} \frac{f'(x)}{f(x)} = 0.$$

Therefore

$$O(\log f(n)) = o(n). \tag{4}$$

If the integral $\int_b^x \frac{tf'(t)}{f(t)}\ dt$ converges we obtain

$$\lim_{x\to\infty} \frac{\int_b^x \frac{tf'(t)}{f(t)}\ dt}{x} = 0.$$

On the other hand, if the integral $\int_b^x \frac{tf'(t)}{f(t)}\ dt$ diverges we obtain from L'Hôpital's rule and (1) that

$$\lim_{x\to\infty} \frac{\int_b^x \frac{tf'(t)}{f(t)}\ dt}{x} = 0.$$

Therefore

$$\int_b^n \frac{xf'(x)}{f(x)}\ dx = o(n). \tag{5}$$

Equations (3), (4) and (5) give

$$\sum_{i=b}^{n} \log f(i) = n \log f(n) + o(n). \tag{6}$$

That is,

$$\frac{1}{n}\sum_{i=b}^{n} \log f(i) = \log f(n) + o(1).$$

That is (2). □

Theorem 5. *Let A_n $(n \geq 0)$ be a sequence of positive numbers (in particular integers) such that*

$$\frac{A_n}{A_{n-1}} \sim Cn^{\alpha} f(n)^{\beta}, \tag{7}$$

where $f(x)$ is a function of slow increase on the interval $[b, \infty)$, $C > 0$, $\alpha > 0$ and β is a real number. If $1 \leq n < b$ we put $f(n) = 1$.

The following formulae hold,

$$\frac{\sqrt[n]{\frac{A_1}{A_0}\frac{A_2}{A_1}\cdots\frac{A_n}{A_{n-1}}}}{\frac{A_n}{A_{n-1}}} \to \frac{1}{e^{\alpha}}, \tag{8}$$

$$A_{n+1} \sim e^{\alpha} A_n^{1+\frac{1}{n}}, \tag{9}$$

$$\log A_n = \alpha n \log n + \beta n \log f(n) + (-\alpha + \log C)n + o(n), \tag{10}$$

$$\log A_n \sim \alpha n \log n, \tag{11}$$

$$A_n = \frac{\left(Cn^{\alpha} f(n)^{\beta}\right)^n}{e^{(\alpha+o(1))n}}. \tag{12}$$

Proof. We have (see (7))

$$\frac{\frac{A_n}{A_{n-1}}}{Cn^{\alpha} f(n)^{\beta}} \to 1. \tag{13}$$

Consequently (13) and Lemma 1 give

$$\sqrt[n]{\prod_{k=1}^{n} \frac{\frac{A_k}{A_{k-1}}}{Ck^{\alpha} f(k)^{\beta}}} = \frac{\sqrt[n]{\prod_{k=1}^{n} \frac{A_k}{A_{k-1}}}}{\sqrt[n]{\prod_{k=1}^{n} Ck^{\alpha} f(k)^{\beta}}} \to 1.$$

That is

$$\sqrt[n]{A_n} \sim \sqrt[n]{\frac{A_1}{A_0}\frac{A_2}{A_1}\cdots\frac{A_n}{A_{n-1}}} \sim \sqrt[n]{\prod_{k=1}^{n} Ck^{\alpha} f(k)^{\beta}}. \tag{14}$$

Lemma 2 and Lemma 4 give

$$\sqrt[n]{\prod_{k=1}^{n} Ck^{\alpha} f(k)^{\beta}} = C\left(\sqrt[n]{n!}\right)^{\alpha}\left(\sqrt[n]{f(1)f(2)\cdots f(n)}\right)^{\beta} \sim C\frac{n^{\alpha}}{e^{\alpha}} f(n)^{\beta}. \tag{15}$$

Equations (14), (15) and (7) give

$$\sqrt[n]{A_n} \sim \sqrt[n]{\frac{A_1}{A_0}\frac{A_2}{A_1}\cdots\frac{A_n}{A_{n-1}}} \sim C\frac{n^{\alpha}}{e^{\alpha}} f(n)^{\beta} \sim \frac{1}{e^{\alpha}}\frac{A_n}{A_{n-1}}. \tag{16}$$

Equation (16) gives (8). Equation (16) and [7, Theorem 8] give

$$A_n^{\frac{1}{n}} \sim A_{n-1}^{\frac{1}{n-1}}. \tag{17}$$

Equations (17) and (16) give

$$A_n \sim e^{\alpha} A_{n-1}^{1+\frac{1}{n-1}}.$$

That is (9). Equation (16) gives

$$\frac{1}{n}\log A_n = \log\left(C\frac{n^{\alpha}}{e^{\alpha}} f(n)^{\beta}\right) + o(1).$$

That is (10). Equation (10) gives (11), since (from L'Hôpital's rule and (1))

$$\lim_{x\to\infty}\frac{\log f(x)}{\log x} = \lim_{x\to\infty}\frac{xf'(x)}{f(x)} = 0.$$

Finally, equation (12) is an immediate consequence of equation (10). □

Remark 6. Note that: (i) The following limit holds $Cn^{\alpha} f(n)^{\beta} \to \infty$ (see(7)).

If $\beta \geq 0$ the proof is trivial. If $\beta < 0$ use [7, Theorem 2 and Theorem 4]. Consequently we have

$$\lim_{n\to\infty}\frac{A_{n+1}}{A_n} = \infty.$$

(ii) This last limit implies the following formula $(A_{n+1} - A_n) \sim A_{n+1}$.

(iii) Equation (7) implies the more general relation,

$$\frac{A_n}{A_{n-m}} = \frac{A_n}{A_{n-1}}\frac{A_{n-1}}{A_{n-2}}\cdots\frac{A_{n-m+1}}{A_{n-m}} \sim C^m \prod_{k=n-m+1}^{n} k^{\alpha} f(k)^{\beta}.$$

2 Introduction to Bell Numbers.

The n-th Bell number B_n is the number of partitions of a set of n elements in disjoint subsets.

The Bell numbers satisfy the following recurrence relation [1, p. 216].

$$B_0 = 1,$$

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k. \tag{18}$$

The first Bell numbers are $B_0 = 1$, $B_1 = 1$, $B_2 = 2$, $B_3 = 5$, $B_4 = 15$, $B_5 = 52$, $B_6 = 203$, $B_7 = 877$, $B_8 = 4140$, $B_9 = 21147$, $B_{10} = 115975$.

N. G. de Bruijn [6, pp. 102–109] proved the following asymptotic formula,

$$\log B_n = n \log n - n \log \log n - n + o(n). \tag{19}$$

L. Lovász [10, Ex. 9(b), p. 17] proved the following asymptotic formula

$$B_n \sim n^{-\frac{1}{2}} \left(\lambda(n)\right)^{n+\frac{1}{2}} e^{\lambda(n)-n-1}, \tag{20}$$

where

$$\lambda(n) = \frac{n}{W(n)}. \tag{21}$$

The function $x = W(y)$ is the inverse function of $y = xe^x$ on the interval $(0, \infty)$. The function $x = W(y)$ is called Lambert W-function.

The following results are well-known [5]. We establish these results in the next lemma. For sake of completeness we give a proof of the lemma.

Lemma 7. *The function $x = W(y)$ is positive, strictly increasing on the interval $(0, \infty)$ and $\lim_{y\to\infty} W(y) = \infty$.*

The following formulae hold.

$$W(y) \sim \log y, \tag{22}$$

$$W'(y) = \frac{W(y)}{y(1 + W(y))}. \tag{23}$$

Proof. The first statement is trivial.

We have (definition of $x = W(y)$)

$$y = W(y)e^{W(y)}. \tag{24}$$

Consequently

$$1 = W'(y)e^{W(y)} + W(y)e^{W(y)}W'(y).$$

That is

$$W'(y) = \frac{1}{e^{W(y)}(1 + W(y))} = \frac{W(y)}{y(1 + W(y))}.$$

On the other hand (24) gives

$$\log y = \log W(y) + W(y).$$

Therefore

$$\frac{W(y)}{\log y} = 1 - \frac{\log W(y)}{\log y}. \tag{25}$$

Also (from L'Hôpital's rule and (23))

$$\lim_{y\to\infty} \frac{\log W(y)}{\log y} = \lim_{y\to\infty} \frac{yW'(y)}{W(y)} = \lim_{y\to\infty} \frac{1}{1+W(y)} = 0. \tag{26}$$

Finally, equations (25) and (26) give (22). □

Remark 8. Note that the Lambert W-function $W(y)$ is a function of slow increase since (see (1) and (23))

$$\lim_{y\to\infty} \frac{yW'(y)}{W(y)} = \lim_{y\to\infty} \frac{1}{1+W(y)} = 0.$$

3 Some Results on Bell Numbers.

The limit

$$\lim_{n\to\infty} \frac{B_n}{n!} = 0.$$

is well-known [9, p. 64]. In the following Theorem we include it for sake of completeness.

Theorem 9. *The following limits hold.*

$$\lim_{n\to\infty} \frac{B_n}{a^n} = \infty \qquad (a > 0), \tag{27}$$

$$\lim_{n\to\infty} \frac{B_n}{(n!)^h} = \infty \qquad (0 < h < 1), \tag{28}$$

$$\lim_{n\to\infty} \frac{B_n}{n!} = 0.$$

Proof. Equation (19) gives

$$\log\left(\frac{B_n}{a^n}\right) = \log B_n - n\log a = n\log n - n\log a + o(n\log n).$$

Therefore

$$\lim_{n\to\infty} \log\left(\frac{B_n}{a^n}\right) = \infty,$$

and consequently

$$\lim_{n\to\infty} \frac{B_n}{a^n} = \infty.$$

That is (27).

The well-known Stirling formula is

$$n! \sim \sqrt{2\pi}\frac{n^n\sqrt{n}}{e^n}.$$

Therefore

$$\log n! = n\log n - n + \frac{1}{2}\log n + \log\sqrt{2\pi} + o(1) = n\log n - n + o(n),$$

and

$$\log (n!)^h = hn\log n - hn + o(n).$$

Consequently (see (19))

$$\log\left(\frac{B_n}{(n!)^h}\right) = (1-h)n\log n - n\log\log n - n + hn + o(n). \tag{29}$$

If $0 < h < 1$ equation (29) gives

$$\lim_{n\to\infty}\log\left(\frac{B_n}{(n!)^h}\right) = \infty.$$

That is,

$$\lim_{n\to\infty}\frac{B_n}{(n!)^h} = \infty.$$

On the other hand if $h = 1$ equation (29) gives

$$\lim_{n\to\infty}\log\left(\frac{B_n}{n!}\right) = -\infty.$$

That is,

$$\lim_{n\to\infty}\frac{B_n}{n!} = 0.$$

□

M. Klazar [8, Proposition 2.6] and D. E. Knuth [9, eq. (30), p. 69] proved the following asymptotic formula

$$\frac{B_{n+1}}{B_n} \sim \frac{n}{\log n}.$$

This formula is derived as a consequence of the asymptotic formula obtained in the classical paper [11].

In the following Theorem we derive this formula from the Lovász's formula (20). We also use the well-known properties of the Lambert W-function established in Lemma 7.

Theorem 10. *The following asymptotic formula holds,*

$$\frac{B_{n+1}}{B_n} \sim \frac{n}{\log n}. \tag{30}$$

Proof. Substituting (21) into (20) we obtain

$$B_n \sim \frac{n^n}{W(n)^{n+\frac{1}{2}}} e^{\frac{n}{W(n)}-n-1}.$$

Consequently

$$\frac{B_{n+1}}{B_n} \sim \frac{n+1}{W(n+1)} \sqrt{\frac{W(n)}{W(n+1)}} \left(\frac{W(n)}{W(n+1)}\right)^n e^{\left(\frac{n+1}{W(n+1)}-\frac{n}{W(n)}\right)}. \tag{31}$$

Equation (22) gives

$$W(n+1) \sim W(n). \tag{32}$$

Equations (32) and (22) give

$$\frac{n+1}{W(n+1)} \sim \frac{n}{\log n}. \tag{33}$$

Equation (32) gives

$$\sqrt{\frac{W(n)}{W(n+1)}} \to 1. \tag{34}$$

Let us consider the function $\frac{y}{W(y)}$. The derivative of $\frac{y}{W(y)}$ is (see (23))

$$\frac{W(y) - yW'(y)}{W(y)^2} = \frac{W(y) - \frac{W(y)}{1+W(y)}}{W(y)^2} = \frac{1}{1+W(y)}.$$

Consequently we have (Lagrange's Theorem)

$$\frac{n+1}{W(n+1)} - \frac{n}{W(n)} = \frac{1}{1+W(n+\epsilon(n))} \to 0, \tag{35}$$

where $0 < \epsilon(n) < 1$.

We have

$$\left(\frac{W(n+1)}{W(n)}\right)^n = \exp\left(n\left(\log W(n+1) - \log W(n)\right)\right). \tag{36}$$

Let us consider the function $\log W(y)$. The derivative of $\log W(y)$ is (see (23))

$$\frac{W'(y)}{W(y)} = \frac{1}{y(1+W(y))}.$$

Consequently we have (Lagrange's Theorem)

$$n\left(\log W(n+1) - \log W(n)\right) = n\frac{1}{(n+\epsilon(n))(1+W(n+\epsilon(n)))}$$
$$= \quad \frac{1}{(1+\frac{\epsilon(n)}{n})(1+W(n+\epsilon(n)))} \to 0, \tag{37}$$

where $0 < \epsilon(n) < 1$.

Equations (36) and (37) give

$$\left(\frac{W(n)}{W(n+1)}\right)^n \to 1. \tag{38}$$

Finally, equations (31), (33), (34), (35) and (38) give (30). □

The asymptotic formula (30) implies that the Bell numbers satisfy condition (7). In this case $C = 1$, $\alpha = 1$, $\beta = -1$ and $f(n) = \log n$. Consequently we have the following Corollary.

Corollary 11. *The following formulae hold,*

$$\lim_{n\to\infty} \frac{B_{n+1}}{B_n} = \infty,$$

$$(B_{n+1} - B_n) \sim B_{n+1},$$

$$\frac{\sqrt[n]{\frac{B_1}{B_0}\frac{B_2}{B_1}\frac{B_3}{B_2}\cdots\frac{B_n}{B_{n-1}}}}{\frac{B_n}{B_{n-1}}} = \frac{\sqrt[n]{B_n}}{\frac{B_n}{B_{n-1}}} \to \frac{1}{e},$$

$$B_{n+1} \sim e\ \ (B_n)^{1+\frac{1}{n}}.$$

Proof. It is an immediate consequence of Theorem 5 and Remark 6. □

The following Theorem is well-known [4, Ex. 1(2), p. 291] [2, Corollary 5]. We give a short proof using equation (18).

Theorem 12. *The sequence $B_{n+1} - B_n$ is strictly increasing.*

Proof. We have (see (18))

$$B_{n+2} - B_{n+1} = \sum_{k=0}^{n} \binom{n+1}{k} B_k, \qquad B_{n+1} - B_n = \sum_{k=0}^{n-1} \binom{n}{k} B_k.$$

Consequently

$$(B_{n+2} - B_{n+1}) - (B_{n+1} - B_n) = \sum_{k=1}^{n-1} \left(\binom{n+1}{k} - \binom{n}{k}\right) B_k + (n+1)B_n > 0,$$

since

$$\binom{n+1}{k} > \binom{n}{k} \qquad (k = 1, \ldots, n).$$

□

In closing the article we give one more property of the Bell numbers but before prove the following general statement.

Theorem 13. *Let F_n be a strictly increasing sequence of positive integers such that*

$$\log F_n \sim Cn \log n \qquad (C > 0).$$

Let $\omega(x)$ be the number of F_n that do not exceed x. The following asymptotic formula holds.

$$\omega(x) \sim \frac{\log x}{C \log\log x}.$$

Proof. Let α_n be a strictly increasing sequence of positive numbers and let $\alpha(x)$ be the number of α_n that do not exceed x. It is well-known [3, p. 129] that

$$\alpha_n \sim Cn \log n \Leftrightarrow \alpha(x) \sim \frac{x}{C \log x}.$$

Now,

$$F_n \leq x \Leftrightarrow \alpha_n = \log F_n \leq \log x.$$

Consequently

$$\omega(x) = \alpha(\log x) \sim \frac{\log x}{C \log\log x}.$$

□

Example 14. If $F_n = B_n$ is the n-th Bell number and $\omega(x)$ is the number of Bell numbers that do not exceed x then (see (19))

$$\omega(x) \sim \frac{\log x}{\log\log x}.$$

4 Acknowledgements

The author would like to thank the anonymous referees for their valuable comments and suggestions for improving the original version of this manuscript. The author is also very grateful to Universidad Nacional de Luján.

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2000 *Mathematics Subject Classification*: Primary 11B99; Secondary 11B73.
Keywords: functions of slow increase, integer sequences, Bell numbers.

(Concerned with sequence A000110.)

Received December 9 2010; revised versions received March 28 2011; May 4 2011. Published in *Journal of Integer Sequences*, May 10 2011.

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