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Successive Derivatives and Integer Sequences

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In memory of my sister Fedra Marina Jakimczuk (1970–2010)

Abstract

We consider the functions $1/f$ and h/f, and study some properties of the successive derivatives of these functions. We also study certain integer sequences connected with these successive derivatives.

1 Introduction.

In the next section we shall need the generalization of the chain rule of calculus to higher derivatives, that is, the Faá di Bruno's formula.

For the compact establishment of the Faá di Bruno's formula in the next section we need certain compact notation on partitions and multinomial coefficients. This compact notation is due to Vella [2]. We now explain this notation.

A partition π of a positive integer n is a representation of n as a sum of positive integers, $n = p_1 + p_2 + \cdots + p_m$, called summands or parts of the partition. The order of the summands is irrelevant. The number of partitions of n we shall denote $p(n)$. The number of parts of the partition π we shall denote $\ell(\pi)$. Consequently $\ell(\pi) = m$.

For example, let us consider the following partition π of 55,

$$1+1+1+1+1+1+2+2+2+6+6+7+7+7+10 = 55.$$

We have $\ell(\pi) = 15$.

A partition can be written more simply in the form $\pi = \{p_1, p_2, \ldots, p_m\}$. For example our former partition π of 55 can be written in the form

$$\pi = \{1, 1, 1, 1, 1, 1, 2, 2, 2, 6, 6, 7, 7, 7, 10\}.$$

For each i ($1 \le i \le n$), the number of times that i appears as a part of the partition π of n is denoted π_i and is called the multiplicity of the part i in π. For example, in our former partition π of 55 we have $\pi_1 = 6$, $\pi_2 = 3$ and $\pi_3 = 0$. Note that $\ell(\pi) = \sum_{i=1}^{n} \pi_i$.

The standard notation for partitions is $\pi = [1^{\pi_1}, 2^{\pi_2}, \dots, n^{\pi_n}]$ with parts of multiplicity zero omitted. For example, in standard notation, our partition π of 55 is

$$\pi = [1^6, 2^3, 6^2, 7^3, 10^1].$$

Note that $\{\pi_1, \pi_2, \dots, \pi_n\}$ is a partition of $\ell(\pi)$. We shall call to this partition the derived partition of π, and denote it $\delta(\pi)$. For example, the derived partition $\delta(\pi)$ of our partition π of 55 is the following partition of $\ell(\pi) = 15$.

$$\delta(\pi) = \{1, 2, 3, 3, 6\} = [1^1, 2^1, 3^2, 6^1].$$

Let us consider the partition $\pi = \{p_1, p_2, \dots, p_m\}$ of n. We shall use the notation $\pi! = \prod_{i=1}^{m}(p_i!)$ and we shall use the notation $\binom{n}{\pi}$ for the multinomial coefficient

$$\binom{n}{p_1, p_2, \dots, p_m} = \frac{n!}{\prod_{i=1}^{m}(p_i!)}.$$

That is,

$$\binom{n}{p_1, p_2, \dots, p_m} = \binom{n}{\pi} = \frac{n!}{\pi!}.$$

In the next section (using this compact notation) we establish the Faá di Bruno's formula and use this formula in the study of the successive derivatives of the function $\frac{1}{f(x)}$. Furthermore, we introduce an integer sequence A_n connected with these successive derivatives.

In the third section we study the successive derivatives of the function $\frac{h(x)}{f(x)}$ and generalize the quotient rule to n^{th} derivatives.

Finally, in the fourth section, we study some asymptotic properties of the integer sequence A_n.

2 Successive Derivatives of the Function $\frac{1}{f(x)}$.

In the following theorem we establish in a compact and useful way the Faá di Bruno's formula. This establishment is due to Vella [2].

Theorem 1. *Suppose $y = g(u)$ and $u = f(x)$ are differentiable n times. Then the composite function $y = (g \circ f)(x)$ is also differentiable n times and*

$$(g \circ f)^{(n)}(x) = \sum_{\pi \in \Omega_n} \frac{\binom{n}{\pi}}{\delta(\pi)!}.\ g^{(\ell(\pi))} \circ f(x).\prod_{i=1}^{n}[f^{(i)}(x)]^{\pi_i}, \tag{1}$$

where Ω_n denotes the set of all partitions of n.

If $f = f(x)$ we shall use the convention $f = f^{(0)}$.

Now, consider the function.

$$\frac{1}{f(x)} = \frac{1}{f} = \left(\frac{1}{f}\right)^{(0)}. \tag{2}$$

We have the following general theorem.

Theorem 2. *The successive derivatives of the function (2) satisfy the following formula,*

$$\left(\frac{1}{f}\right)^{(n)} = \frac{P_n}{f^{n+1}} \qquad (n \geq 0), \tag{3}$$

where P_n is a polynomial of integer coefficients in the variables $f, f^{(1)}, \ldots, f^{(n)}$.

If $n = 0$ then

$$P_0 = 1, \tag{4}$$

and if $n \geq 1$ then

$$\begin{aligned} P_n &= \sum_{\pi \in \Omega_n} (-1)^{\ell(\pi)} \binom{n}{\pi} \binom{\ell(\pi)}{\delta(\pi)} . f(x)^{n-\ell(\pi)} . \prod_{i=1}^{n} [f^{(i)}(x)]^{\pi_i} \\ &= \sum_{\pi \in \Omega_n} (-1)^{\ell(\pi)} \binom{n}{\pi} \binom{\ell(\pi)}{\delta(\pi)} . f^{n-\ell(\pi)} . \prod_{i=1}^{n} [f^{(i)}]^{\pi_i} \end{aligned} \tag{5}$$

Proof. Let $g(u) = \frac{1}{u} = u^{-1}$ be. Note that

$$g^{(n)}(u) = \frac{(-1)^n n!}{u^{n+1}}. \tag{6}$$

Substituting equation (6) into equation (1) we find that

$$\begin{aligned} \left(\frac{1}{f}\right)^{(n)} &= (g \circ f)^{(n)}(x) = \sum_{\pi \in \Omega(n)} \frac{\binom{n}{\pi}}{\delta(\pi)!} . \frac{(-1)^{\ell(\pi)} \ell(\pi)!}{f(x)^{\ell(\pi)+1}} . \prod_{i=1}^{n} [f^{(i)}(x)]^{\pi_i} \\ &= \sum_{\pi \in \Omega(n)} \frac{\binom{n}{\pi}}{\delta(\pi)!} . \frac{(-1)^{\ell(\pi)} \ell(\pi)!}{f(x)^{n+1}} . f(x)^{n-\ell(\pi)} . \prod_{i=1}^{n} [f^{(i)}(x)]^{\pi_i} \\ &= \sum_{\pi \in \Omega(n)} \binom{n}{\pi} \binom{\ell(\pi)}{\delta(\pi)} \frac{(-1)^{\ell(\pi)}}{f(x)^{n+1}} . f(x)^{n-\ell(\pi)} . \prod_{i=1}^{n} [f^{(i)}(x)]^{\pi_i} \\ &= \frac{\sum_{\pi \in \Omega_n} (-1)^{\ell(\pi)} \binom{n}{\pi} \binom{\ell(\pi)}{\delta(\pi)} . f(x)^{n-\ell(\pi)} . \prod_{i=1}^{n} [f^{(i)}(x)]^{\pi_i}}{f(x)^{n+1}} \\ &= \frac{P_n}{f^{n+1}}. \end{aligned} \tag{7}$$

$\square$

Using (5) we obtain the first polynomials P_n.

$$P_1 = -f^{(1)}(x) = -f^{(1)}, \tag{8}$$

$$P_2 = -f(x)\ f^{(2)}(x) + 2f^{(1)}(x)f^{(1)}(x) = -f\ f^{(2)} + 2f^{(1)}f^{(1)}, \tag{9}$$

$$P_3 = -fff^{(3)} + 6ff^{(1)}f^{(2)} - 6f^{(1)}f^{(1)}f^{(1)}, \tag{10}$$

$$\begin{aligned} P_4 &= -ffff^{(4)} + 8fff^{(1)}f^{(3)} + 6fff^{(2)}f^{(2)} - 36ff^{(1)}f^{(1)}f^{(2)} \\ &+ 24f^{(1)}f^{(1)}f^{(1)}f^{(1)}, \end{aligned} \tag{11}$$

$$\begin{aligned} P_5 &= -fffff^{(5)} + 10ffff^{(1)}f^{(4)} - 60fff^{(1)}f^{(1)}f^{(3)} + 20ffff^{(2)}f^{(3)} \\ &- 90fff^{(1)}f^{(2)}f^{(2)} + 240ff^{(1)}f^{(1)}f^{(1)}f^{(2)} - 120f^{(1)}f^{(1)}f^{(1)}f^{(1)}f^{(1)} \end{aligned} \tag{12}$$

In the following theorem we establish some general properties of the polynomials P_n.

Theorem 3. *The polynomial P_n $(n \geq 1)$ has the following properties.*

a) Each term (monomial) in the polynomial P_n has n factors and the sum of superscripts is n. That is, the monomials are of the form $f^{(i_1)}f^{(i_2)}\cdots f^{(i_n)}$ where $i_1 + i_2 + \cdots + i_n = n$ $(f = f^{(0)})$. Consequently we can establish a correspondence between each monomial and a partition of n. The number of terms (monomials) in the polynomial P_n is $p(n)$, that is, the number of partitions of n.

b) If n is even then the sum of the coefficients is 1. On the other hand, if n is odd the sum of the coefficients is -1. That is, the sum of the coefficients of the polynomial P_n is $(-1)^n$.

c) If n is even the monomials with an even number of f have positive coefficient and the monomials with a odd number of f have negative coefficient.

If n is odd the monomials with an even number of f have negative coefficient and the monomials with a odd number of f have positive coefficient.

d) If A_n is the sum of the absolute values of the coefficients in the polynomial P_n then the following formula holds $(A_0 = 1)$:

$$q(x) = \frac{1}{2 - e^x} = \sum_{k=0}^{\infty} \frac{A_k}{k!} x^k. \tag{13}$$

Therefore, A_k $(k \geq 0)$ is the k^{th} derivative of the function $q(x) = \frac{1}{2-e^x}$ at $x = 0$ $(A_k = q^{(k)}(0))$. The radius of convergence of (13) is $R = \log 2$. Furthermore,

$$A_n = \sum_{\pi \in \Omega_n} \binom{n}{\pi} \binom{\ell(\pi)}{\delta(\pi)} \qquad (n \geq 1). \tag{14}$$

e) The coefficient of the monomial $f \cdots ff^{(n)}$ is -1.

The coefficient of the monomial $f^{(1)} \cdots f^{(1)}$ is $(-1)^n n!$. Consequently $A_n \geq n!$ (see part (d)).

Proof. Proof of part (a). It is an immediate consequence of equation (5). Part (a) is proved.

Proof of part (b). We apply (7) in the case $f(x) = e^x$. Note that $f^{(i)}(0) = 1$ for all $i \geq 0$. Therefore the left side of (7) at $x = 0$ becomes

$$\left(\frac{1}{e^x}\right)^{(n)}(0) = \frac{d^n}{dx^n}(e^{-x})\mid_{x=0} = (-1)^n.$$

On the other hand the right side of (7) at $x = 0$ becomes

$$\begin{aligned}
& \frac{\sum_{\pi\in\Omega_n}(-1)^{\ell(\pi)}\binom{n}{\pi}\binom{\ell(\pi)}{\delta(\pi)}.f(0)^{n-\ell(\pi)}.\prod_{i=1}^{n}[f^{(i)}(0)]^{\pi_i}}{f(0)^{n+1}} \\
= & \frac{\sum_{\pi\in\Omega_n}(-1)^{\ell(\pi)}\binom{n}{\pi}\binom{\ell(\pi)}{\delta(\pi)}.1^{n-\ell(\pi)}.\prod_{i=1}^{n}(1)^{\pi_i}}{1^{n+1}} \\
= & \sum_{\pi\in\Omega_n}(-1)^{\ell(\pi)}\binom{n}{\pi}\binom{\ell(\pi)}{\delta(\pi)},
\end{aligned}$$

which is the sum of the coefficients of the polynomial P_n. Part (b) is proved.

Proof of part (c). It is an immediate consequence of equation (5). Part (c) is proved.

Proof of part (d). We apply (7) in the case $f(x) = 2 - e^x$. Note that $f(0) = 1$ and $f^{(i)}(0) = -1$ for all $i > 0$. Consequently, the right side of (7) at $x = 0$ becomes

$$\begin{aligned}
& \frac{\sum_{\pi\in\Omega_n}(-1)^{\ell(\pi)}\binom{n}{\pi}\binom{\ell(\pi)}{\delta(\pi)}.f(0)^{n-\ell(\pi)}.\prod_{i=1}^{n}[f^{(i)}(0)]^{\pi_i}}{f(0)^{n+1}} \\
= & \frac{\sum_{\pi\in\Omega_n}(-1)^{\ell(\pi)}\binom{n}{\pi}\binom{\ell(\pi)}{\delta(\pi)}.1^{n-\ell(\pi)}.\prod_{i=1}^{n}(-1)^{\pi_i}}{1^{n+1}} \\
= & \sum_{\pi\in\Omega_n}(-1)^{\ell(\pi)}\binom{n}{\pi}\binom{\ell(\pi)}{\delta(\pi)}.(-1)^{\sum_{i=1}^{n}\pi_i} \\
= & \sum_{\pi\in\Omega_n}(-1)^{\ell(\pi)}\binom{n}{\pi}\binom{\ell(\pi)}{\delta(\pi)}.(-1)^{\ell(\pi)} \\
= & \sum_{\pi\in\Omega_n}\binom{n}{\pi}\binom{\ell(\pi)}{\delta(\pi)}. \qquad (15)
\end{aligned}$$

Note that (15) (see the numerator of (7)) is the sum A_n of the absolute values of the coefficients in the polynomial P_n. On the other hand, the left side of (7) at $x = 0$ becomes

$$(g \circ f)^{(n)}(0) = q^{(n)}(0).$$

Therefore

$$q(x) = (g \circ f)(x) = \frac{1}{2 - e^x} = \sum_{k=0}^{\infty}\frac{A_k}{k!}x^k.$$

Note that the function of the complex variable $q(z) = \frac{1}{2-e^z}$ is analytical in the disk $|z| < \log 2$ and consequently the radius of convergence of (13) is $R = \log 2$. Note also that in terms of generating functions (see reference [3]) we have proved that $q(x) = \frac{1}{2-e^x}$ is the exponential generating function of the sequence A_n. Part (d) is proved.

Proof of part (e). It is an immediate consequence of equation (5). Part (e) is proved. $\square$

Note that the first values of the integer sequence A_n are (see either (4), (8), (9), (10), (11) and (12) or use directly equation (14)): $A_0 = 1$, $A_1 = 1$, $A_2 = 3$, $A_3 = 13$, $A_4 = 75$, $A_5 = 541$.

In the next section we shall prove the following recurrence combinatorial formula.

$$A_0 = 1,$$

$$A_n = \sum_{k=0}^{n-1} \binom{n}{k} A_k \qquad (n \geq 1). \tag{16}$$

A simple consequence of this formula is the inequality

$$A_{n+1} > A_n \qquad (n \geq 1).$$

3 Successive Derivatives of the Function $\frac{h(x)}{f(x)}$.

We shall need the well-known binomial formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k, \tag{17}$$

where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ and $0! = 1$.

We also shall need the following well-known result of Leibniz on the successive derivatives of the function $fg = f(x)g(x)$.

Lemma 4. *Let us consider the function fg. The following formula holds:*

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)} \qquad (n \geq 0), \tag{18}$$

where $f^{(0)} = f$ and $g^{(0)} = g$.

Theorem 5. *Let us consider the function*

$$\frac{h(x)}{f(x)} = \frac{h}{f}.$$

The following formula holds.

$$\left(\frac{h}{f}\right)^{(n)} = \frac{Q_n}{f^{n+1}} = \frac{\sum_{k=0}^{n} \binom{n}{k} h^{(n-k)} f^{n-k} P_k}{f^{n+1}} \qquad (n \geq 0). \tag{19}$$

Proof. We have

$$\frac{h}{f} = h\frac{1}{f}.$$

Lemma 4 and equation (3) give

$$\begin{aligned}\left(\frac{h}{f}\right)^{(n)} &= \sum_{k=0}^{n}\binom{n}{k}h^{(n-k)}\left(\frac{1}{f}\right)^{(k)} = \sum_{k=0}^{n}\binom{n}{k}h^{(n-k)}\frac{P_k}{f^{k+1}} \\ &= \frac{\sum_{k=0}^{n}\binom{n}{k}h^{(n-k)}f^{n-k}P_k}{f^{n+1}}.\end{aligned}$$

That is, equation (19). □

Using (19), (4), (8), (9) and (10) we obtain the first polynomials Q_n.

$$Q_0 = h,$$
$$Q_1 = h^{(1)}f - hf^{(1)},$$
$$Q_2 = h^{(2)}ff - 2h^{(1)}ff^{(1)} - hff^{(2)} + 2hf^{(1)}f^{(1)},$$

$$\begin{aligned}Q_3 &= h^{(3)}fff - 3h^{(2)}fff^{(1)} - 3h^{(1)}fff^{(2)} + 6h^{(1)}ff^{(1)}f^{(1)} - hfff^{(3)} \\ &+ 6hff^{(1)}f^{(2)} - 6hf^{(1)}f^{(1)}f^{(1)}.\end{aligned}$$

In the following theorem we obtain some information on the polynomial Q_n.

Theorem 6. *Let us consider the polynomial $Q(n)$ $(n \geq 0)$.*

a) The sum of the coefficients of the polynomial $Q(n)$ $(n \geq 1)$ is zero.

b) If C_n is the sum of the absolute values of the coefficients in the polynomial Q_n then the following formula holds:

$$C_n = \sum_{k=0}^{n}\binom{n}{k}A_k \qquad (n \geq 0). \tag{20}$$

c) If n is even the monomials with an even number of f have positive coefficient and the monomials with a odd number of f have negative coefficient.

If n is odd the monomials with an even number of f have negative coefficient and the monomials with a odd number of f have positive coefficient.

d) If C_n is the sum of the absolute values of the coefficients in the polynomial Q_n then the following formula holds:

$$p(x) = \frac{e^x}{2-e^x} = \sum_{k=0}^{\infty}\frac{C_k}{k!}x^k. \tag{21}$$

Therefore, C_k $(k \geq 0)$ is the k^{th} derivative of the function $p(x) = \frac{e^x}{2-e^x}$ at $x = 0$ $(C_k = p^{(k)}(0))$. The radius of convergence of (21) is $R = \log 2$.

e) We have

$$C_0 = A_0 = 1,$$
$$C_n = 2A_n \qquad (n \geq 1).$$

f) The following combinatorial formula holds:

$$A_n = \sum_{k=0}^{n-1} \binom{n}{k} A_k \qquad (n \geq 1).$$

g) Each term (monomial) in the polynomial Q_n has $(n+1)$ factors and the sum of superscripts is n. That is, the monomials are of the form

$$h^{(i_1)} f^{(i_2)} \cdots f^{(i_{(n+1)})},$$

where $i_1 + i_2 + \cdots + i_{(n+1)} = n$ and $h = h^{(0)}$, $f = f^{(0)}$.

The number of term (monomials) in the polynomial Q_n is $\sum_{k=0}^{n} p(k)$ where $p(k)$ is the number of partitions of k $(p(0) = 1)$.

Proof. Proof of part (a). The sum of the coefficients in the polynomial P_k is $(-1)^k$ (see part (b) of Theorem 3). Consequently the sum of the coefficients in the polynomial Q_n will be (see (19) and (17))

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = (1-1)^n = 0.$$

Part (a) is proved.

Proof of part (b). The sum of the absolute values of the coefficients in the polynomial P_k is A_k (see part (d) of Theorem 3). Consequently the sum of the absolute values of the coefficients in the polynomial Q_n will be (20) (see (19)). Part (b) is proved.

Proof of part (c). It is an immediate consequence of (19) and part (c) of Theorem 3. Part (c) is proved.

Proof of part (d). Let us consider the function $f(x) = f = e^x - 2$. We have $f(0) = -1$ and since $f^{(n)}(x) = f^{(n)} = e^x$ $(n \geq 1)$ we have $f^{(n)}(0) = 1$. On the other hand consider the function $h(x) = h = e^x$. We have $h(0) = 1$ and since $h^{(n)}(x) = h^{(n)} = e^x$ $(n \geq 1)$ we have $h^{(n)}(0) = 1$. Therefore part (c) and equation (19) imply that the function $p(x) = -\frac{h(x)}{f(x)} = -\frac{e^x}{e^x - 2}$ satisfies $p^{(n)}(0) = C_n$ $(n \geq 0)$. On the other hand the function of the complex variable $p(z) = -\frac{h(z)}{f(z)} = -\frac{e^z}{e^z - 2}$ is analytical in the disk $|z| < \log 2$ and consequently the radius of convergence of (21) is $R = \log 2$. Part (d) is proved.

Proof of part (e). We have (see part (d) of Theorem 3)

$$p(x) = \frac{e^x}{2 - e^x} = -1 + 2\left(\frac{1}{2 - e^x}\right) = -1 + 2q(x).$$

Consequently

$$C_0 = p(0) = q(0) = A_0 = 1.$$

On the other hand we have

$$p^{(n)}(x) = 2q^{(n)}(x) \qquad (n \geq 1).$$

Therefore

$$C_n = p^{(n)}(0) = 2q^{(n)}(0) = 2A_n \qquad (n \geq 1).$$

Part (e) is proved.

Proof of part (f). It is an immediate consequence of part (b) and part (e). Part (f) is proved.

Proof of part (g). It is an immediate consequence of (19) and part (a) of Theorem 3. Part (g) is proved. □

4 Asymptotic Results on the Sequence A_n.

Lemma 7. *Let us consider the power series (13), that is*

$$\sum_{k=0}^{\infty} \frac{A_k}{k!} x^k.$$

The ratio test for power series is valid, that is the following limit holds:

$$\lim_{k\to\infty} \frac{\frac{A_k}{k!}}{\frac{A_{k-1}}{(k-1)!}} = \lim_{k\to\infty} \frac{A_k}{kA_{k-1}} = \frac{1}{\log 2}. \tag{22}$$

Proof. We have

$$A_n = \sum_{k=0}^{n-1} \binom{n}{k} A_k = \sum_{k=0}^{n-2} \binom{n}{k} A_k + nA_{n-1}.$$

Consequently

$$\frac{A_n}{A_{n-1}} \geq n, \tag{23}$$

and

$$\frac{A_{n-1}}{A_n} \leq \frac{1}{n}. \tag{24}$$

Using (24) repeatedly we obtain

$$\frac{A_0}{A_{n-1}} = \frac{A_0}{A_1}\frac{A_1}{A_2}\cdots\frac{A_{n-2}}{A_{n-1}} \leq \frac{1}{1}\frac{1}{2}\cdots\frac{1}{(n-1)} = \frac{0!}{(n-1)!}$$

$$\frac{A_1}{A_{n-1}} = \frac{A_1}{A_2}\frac{A_2}{A_3}\cdots\frac{A_{n-2}}{A_{n-1}} \leq \frac{1}{2}\frac{1}{3}\cdots\frac{1}{(n-1)} = \frac{1!}{(n-1)!}$$

$$\frac{A_2}{A_{n-1}} = \frac{A_2}{A_3}\frac{A_3}{A_4}\cdots\frac{A_{n-2}}{A_{n-1}} \leq \frac{1}{3}\frac{1}{4}\cdots\frac{1}{(n-1)} = \frac{2!}{(n-1)!}$$

$$\vdots$$

That is,

$$\frac{A_k}{A_{n-1}} \leq \frac{k!}{(n-1)!} \qquad (0 \leq k \leq n-1).$$

This last equation and (16) give,

$$\begin{aligned} \frac{A_n}{A_{n-1}} &= \sum_{k=0}^{n-1} \binom{n}{k} \frac{A_k}{A_{n-1}} \leq \sum_{k=0}^{n-1} \frac{n!}{(n-k)!k!} \frac{k!}{(n-1)!} = n \sum_{k=0}^{n-1} \frac{1}{(n-k)!} \\ &= n \sum_{k=1}^{n} \frac{1}{k!} \leq n(e-1). \end{aligned} \tag{25}$$

Finally, (23) and (25) give

$$1 \leq \frac{A_n}{nA_{n-1}} \leq (e-1). \tag{26}$$

Suppose that the following inequality holds from a certain $n = n' + 1$.

$$\frac{A_n}{A_{n-1}} \geq hn. \tag{27}$$

That is,

$$\frac{A_{n-1}}{A_n} \leq \frac{1}{hn}. \tag{28}$$

We obtain,

$$\frac{A_k}{A_{n-1}} \leq \frac{k!}{(n-1)!} \left(\frac{1}{h}\right)^{n-k-1} \qquad (n' \leq k \leq n-1). \tag{29}$$

Equations (16) and (29) give,

$$\begin{aligned} \frac{A_n}{A_{n-1}} &= \sum_{k=0}^{n-1} \binom{n}{k} \frac{A_k}{A_{n-1}} = \sum_{k=0}^{n'-1} \binom{n}{k} \frac{A_k}{A_{n-1}} + \sum_{k=n'}^{n-1} \binom{n}{k} \frac{A_k}{A_{n-1}} \\ &\leq C + \sum_{k=n'}^{n-1} \frac{n!}{(n-k)!k!} \frac{k!}{(n-1)!} \left(\frac{1}{h}\right)^{n-k-1} = C + n \sum_{k=n'}^{n-1} \frac{\left(\frac{1}{h}\right)^{n-k-1}}{(n-k)!} \\ &= C + n \sum_{k=1}^{n-n'} \frac{\left(\frac{1}{h}\right)^{k-1}}{k!} \leq C + n \sum_{k=1}^{\infty} \frac{\left(\frac{1}{h}\right)^{k-1}}{k!} = C + n \frac{e^{\frac{1}{h}} - 1}{\frac{1}{h}}, \end{aligned} \tag{30}$$

where $C = \sum_{k=0}^{n'-1} \binom{n}{k} \frac{A_k}{A_{n-1}}$.

Equation (30) gives the inequality (from a certain n)

$$\frac{A_n}{nA_{n-1}} \leq \frac{e^{\frac{1}{h}} - 1}{\frac{1}{h}} + \epsilon \qquad (\epsilon > 0). \tag{31}$$

Suppose that the following inequality holds from a certain $n = n' + 1$.

$$\frac{A_n}{A_{n-1}} \leq pn. \tag{32}$$

That is,

$$\frac{A_{n-1}}{A_n} \geq \frac{1}{pn}. \tag{33}$$

We obtain

$$\frac{A_k}{A_{n-1}} \geq \frac{k!}{(n-1)!} \left(\frac{1}{p}\right)^{n-k-1} \qquad (n' \leq k \leq n-1). \tag{34}$$

Equations (16) and (34) give,

$$\begin{aligned}
\frac{A_n}{A_{n-1}} &= \sum_{k=0}^{n-1} \binom{n}{k} \frac{A_k}{A_{n-1}} = \sum_{k=0}^{n'-1} \binom{n}{k} \frac{A_k}{A_{n-1}} + \sum_{k=n'}^{n-1} \binom{n}{k} \frac{A_k}{A_{n-1}} \\
&\geq \sum_{k=n'}^{n-1} \frac{n!}{(n-k)!k!} \frac{k!}{(n-1)!} \left(\frac{1}{p}\right)^{n-k-1} = n \sum_{k=n'}^{n-1} \frac{\left(\frac{1}{p}\right)^{n-k-1}}{(n-k)!} \\
&= n \sum_{k=1}^{n-n'} \frac{\left(\frac{1}{p}\right)^{k-1}}{k!}.
\end{aligned} \tag{35}$$

Equation (35) gives (from a certain n)

$$\frac{A_n}{nA_{n-1}} \geq \sum_{k=1}^{n-n'} \frac{\left(\frac{1}{p}\right)^{k-1}}{k!} \geq \sum_{k=1}^{\infty} \frac{\left(\frac{1}{p}\right)^{k-1}}{k!} - \lambda = \frac{e^{\frac{1}{p}} - 1}{\frac{1}{p}} - \lambda \qquad (\lambda > 0).$$

That is,

$$\frac{A_n}{nA_{n-1}} \geq \frac{e^{\frac{1}{p}} - 1}{\frac{1}{p}} - \lambda \qquad (\lambda > 0). \tag{36}$$

Note that the function

$$f(x) = \frac{e^x - 1}{x} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!}$$

is positive and strictly increasing in the interval $(-\infty, \infty)$ since its derivative is positive in the interval $(-\infty, \infty)$. On the other hand its image is the interval $(0, \infty)$. Furthermore, $f(0) = 1$ and $f(1) = e - 1$.

Let us consider the inequality (26). That is,

$$h_1 = 1 = \frac{e^0 - 1}{0} \leq \frac{A_n}{nA_{n-1}} \leq \frac{e^1 - 1}{1} = e - 1 = p_1. \tag{37}$$

This inequality is our first inequality.

The inequality in the right side (that is, equation (32)) give us the inequality (see (36))

$$h_2 = \frac{e^{\frac{1}{p_1}} - 1}{\frac{1}{p_1}} - \lambda_2 \leq \frac{A_n}{nA_{n-1}}. \tag{38}$$

Note that since $p_1 > 0$ we have the inequality

$$\frac{e^{\frac{1}{p_1}} - 1}{\frac{1}{p_1}} > 1 = h_1.$$

Let λ_2 be the least number between

$$1/2$$

and

$$\frac{\frac{e^{\frac{1}{p_1}}-1}{\frac{1}{p_1}} - h_1}{2}.$$

Consequently

$$h_2 > h_1 = 1. \tag{39}$$

Inequality (38) (that is, equation (27)) give us (see equation (31)) the inequality

$$\frac{A_n}{nA_{n-1}} \leq \frac{e^{\frac{1}{h_2}} - 1}{\frac{1}{h_2}} + \epsilon_2 = p_2. \tag{40}$$

Note that since (see (39)) $h_2 > 1$ we have the inequality

$$\frac{e^{\frac{1}{h_2}} - 1}{\frac{1}{h_2}} < \frac{e^1 - 1}{1} = e - 1 = p_1.$$

Let ϵ_2 be the least number between

$$1/2$$

and

$$\frac{p_1 - \frac{e^{\frac{1}{h_2}}-1}{\frac{1}{h_2}}}{2}.$$

Consequently

$$p_2 < p_1 = e - 1. \tag{41}$$

On the other hand, inequalities (38) and (40) give us the following second inequality.

$$h_2 = \frac{e^{\frac{1}{p_1}} - 1}{\frac{1}{p_1}} - \lambda_2 \leq \frac{A_n}{nA_{n-1}} \leq \frac{e^{\frac{1}{h_2}} - 1}{\frac{1}{h_2}} + \epsilon_2 = p_2. \tag{42}$$

Inequality (40) (that is, equation (32)) give us the inequality (see (36))

$$h_3 = \frac{e^{\frac{1}{p_2}} - 1}{\frac{1}{p_2}} - \lambda_3 \leq \frac{A_n}{nA_{n-1}}. \tag{43}$$

Equation (41) implies that

$$\frac{e^{\frac{1}{p_2}}-1}{\frac{1}{p_2}} > \frac{e^{\frac{1}{p_1}}-1}{\frac{1}{p_1}}.$$

Let λ_3 be the least number between

$$1/3$$

and

$$\frac{\frac{e^{\frac{1}{p_2}}-1}{\frac{1}{p_2}} - \frac{e^{\frac{1}{p_1}}-1}{\frac{1}{p_1}}}{2}.$$

Consequently

$$h_3 > h_2 > h_1 = 1. \tag{44}$$

Inequality (43) (that is, equation (27)) give us (see equation (31)) the inequality

$$\frac{A_n}{nA_{n-1}} \leq \frac{e^{\frac{1}{h_3}}-1}{\frac{1}{h_3}} + \epsilon_3 = p_3. \tag{45}$$

Equation (44) implies that

$$\frac{e^{\frac{1}{h_3}}-1}{\frac{1}{h_3}} < \frac{e^{\frac{1}{h_2}}-1}{\frac{1}{h_2}}.$$

Let ϵ_3 be the least number between

$$1/3$$

and

$$\frac{\frac{e^{\frac{1}{h_2}}-1}{\frac{1}{h_2}} - \frac{e^{\frac{1}{h_3}}-1}{\frac{1}{h_3}}}{2}.$$

Consequently

$$p_3 < p_2 < p_1 = e - 1. \tag{46}$$

On the other hand, inequalities (43) and (45) give us the third inequality

$$h_3 = \frac{e^{\frac{1}{p_2}}-1}{\frac{1}{p_2}} - \lambda_3 \leq \frac{A_n}{nA_{n-1}} \leq \frac{e^{\frac{1}{h_3}}-1}{\frac{1}{h_3}} + \epsilon_3 = p_3. \tag{47}$$

In this form we build the following sequence of inequalities (see (37), (42), (47)).

$$h_1 = 1 = \frac{e^0-1}{0} \leq \frac{A_n}{nA_{n-1}} \leq \frac{e^1-1}{1} = e - 1 = p_1$$

$$h_2 = \frac{e^{\frac{1}{p_1}}-1}{\frac{1}{p_1}} - \lambda_2 \leq \frac{A_n}{nA_{n-1}} \leq \frac{e^{\frac{1}{h_2}}-1}{\frac{1}{h_2}} + \epsilon_2 = p_2$$

$$h_3 = \frac{e^{\frac{1}{p_2}}-1}{\frac{1}{p_2}} - \lambda_3 \leq \frac{A_n}{nA_{n-1}} \leq \frac{e^{\frac{1}{h_3}}-1}{\frac{1}{h_3}} + \epsilon_3 = p_3$$

$$h_4 = \frac{e^{\frac{1}{p_3}} - 1}{\frac{1}{p_3}} - \lambda_4 \leq \frac{A_n}{nA_{n-1}} \leq \frac{e^{\frac{1}{h_4}} - 1}{\frac{1}{h_4}} + \epsilon_4 = p_4$$

$$\vdots$$

In this sequence of inequalities we have the strictly increasing and bounded sequence h_n and the strictly decreasing and bounded sequence p_n. Consequently h_n has limit l_1 and p_n has limit l_2. Next, we shall prove that $l_1 = l_2 = l = \frac{1}{\log 2}$. Therefore we obtain the desired limit (22). That is,

$$\lim_{n\to\infty} \frac{A_n}{nA_{n-1}} = \frac{1}{\log 2}.$$

Note that the sequence λ_n has limit zero since $\lambda_n \leq \frac{1}{n}$ and the sequence ϵ_n has also limit zero since $\epsilon_n \leq \frac{1}{n}$

The sequence h_n satisfies (see the sequence of inequalities) the following recurrence relation

$$h_n = \frac{e^{\frac{1}{p_{n-1}}} - 1}{\frac{1}{p_{n-1}}} - \lambda_n = \frac{e^{\frac{1}{\frac{e^{\frac{1}{h_{n-1}}}-1}{\frac{1}{h_{n-1}}}+\epsilon_{n-1}}} - 1}{\frac{1}{\frac{e^{\frac{1}{h_{n-1}}}-1}{\frac{1}{h_{n-1}}}+\epsilon_{n-1}}} - \lambda_n.$$

Consequently if we take limits in both sides we obtain that l_1 satisfies the equation

$$l_1 = \frac{e^{\frac{1}{\frac{e^{\frac{1}{l_1}}-1}{\frac{1}{l_1}}}} - 1}{\frac{1}{\frac{e^{\frac{1}{l_1}}-1}{\frac{1}{l_1}}}}. \tag{48}$$

The sequence p_n satisfies (see the sequence of inequalities) the following recurrence relation

$$p_n = \frac{e^{\frac{1}{h_n}} - 1}{\frac{1}{h_n}} + \epsilon_n = \frac{e^{\frac{1}{\frac{e^{\frac{1}{p_{n-1}}}-1}{\frac{1}{p_{n-1}}}-\lambda_n}} - 1}{\frac{1}{\frac{e^{\frac{1}{p_{n-1}}}-1}{\frac{1}{p_{n-1}}}-\lambda_n}} + \epsilon_n.$$

Consequently if we take limits in both sides we obtain that l_2 satisfies the equation

$$l_2 = \frac{e^{\frac{1}{\frac{e^{\frac{1}{l_2}}-1}{\frac{1}{l_2}}}} - 1}{\frac{1}{\frac{e^{\frac{1}{l_2}}-1}{\frac{1}{l_2}}}}. \tag{49}$$

That is, l_1 and l_2 satisfy the same equation (see (48) and (49))

$$l = \frac{e^{\frac{\frac{1}{e^{\frac{1}{l}}-1}}{\frac{1}{l}}} - 1}{\frac{\frac{1}{e^{\frac{1}{l}}-1}}{\frac{1}{l}}}. \tag{50}$$

This equation has the solution

$$l = \frac{1}{\log 2}.$$

We shall prove that this solution is the unique positive solution to equation (50). Consequently $l_1 = l_2 = l = \frac{1}{\log 2}$.

Equation (50) is,

$$l = \frac{e^{\frac{1}{\left(e^{\frac{1}{l}}-1\right)l}} - 1}{\frac{1}{\left(e^{\frac{1}{l}}-1\right)l}}.$$

That is,

$$\frac{1}{e^{\frac{1}{l}} - 1} = e^{\frac{1}{\left(e^{\frac{1}{l}}-1\right)l}} - 1.$$

That is,

$$\left(e^{\frac{1}{l}} - 1\right)\left(e^{\frac{1}{\left(e^{\frac{1}{l}}-1\right)l}} - 1\right) = 1.$$

That is,

$$e^{\frac{1}{l}} e^{\frac{1}{\left(e^{\frac{1}{l}}-1\right)l}} - e^{\frac{1}{l}} - e^{\frac{1}{\left(e^{\frac{1}{l}}-1\right)l}} = 0.$$

That is,

$$e^{\frac{1}{\left(e^{\frac{1}{l}}-1\right)l}} - e^{-\frac{1}{l}} e^{\frac{1}{\left(e^{\frac{1}{l}}-1\right)l}} = 1.$$

That is,

$$e^{\frac{1}{\left(e^{\frac{1}{l}}-1\right)l}}\left(1 - e^{-\frac{1}{l}}\right) = 1.$$

Let us write $x = \frac{1}{l}$ and consider the function

$$f(x) = e^{\frac{x}{e^x - 1}}(1 - e^{-x}).$$

We have to prove that the unique positive solution to the equation

$$f(x) = e^{\frac{x}{e^x - 1}}(1 - e^{-x}) = 1$$

is $x = \log 2$. Note that $f(0) = 0$ and if $x > 0$ then $f(x) > 0$. On the other hand

$$\lim_{x\to\infty} f(x) = 1.$$

If $x > 0$ the derivative of $f(x)$ is

$$f'(x) = e^{\frac{x}{e^x-1}} \frac{-x+2-2e^{-x}}{e^x-1}.$$

Let us consider the function

$$g(x) = -x + 2 - 2e^{-x}.$$

We have

$$g(0) = 0,$$

$$g(\log 2) > 0,$$

$$\lim_{x\to\infty} g(x) = -\infty.$$

Its derivative is

$$g'(x) = 2e^{-x} - 1.$$

On the interval $[0, \log 2)$ we have $g'(x) > 0$.

In $x = \log 2$ we have $g'(x) = 0$.

On the interval $(\log 2, \infty]$ we have $g'(x) < 0$.

Therefore there exists $a > \log 2$ such that $g(a) = 0$. On the interval $(0, a)$ is $g(x) > 0$ and on the interval (a, ∞) is $g(x) < 0$.

Hence $f(x)$ is strictly increasing in the interval $[0, a)$ and $f(x)$ is strictly decreasing and greater than 1 in the interval $[a, \infty)$. Consequently the unique positive solution to (50) is $x = \log 2$. □

Theorem 8. *The following asymptotic formulae hold*

$$\frac{A_n}{A_{n-1}} \sim \frac{n}{\log 2}, \tag{51}$$

$$\lim_{n\to\infty} \frac{A_{n+1}}{A_n} = \infty, \tag{52}$$

$$(A_{n+1} - A_n) \sim A_{n+1}, \tag{53}$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{\frac{A_1}{A_0}\frac{A_2}{A_1}\cdots\frac{A_n}{A_{n-1}}}}{\frac{A_n}{A_{n-1}}} = \frac{1}{e}, \tag{54}$$

$$A_{n+1} \sim e A_n^{1+\frac{1}{n}}, \tag{55}$$

$$\log A_n = n \log n - (1 + \log\log 2)n + o(n), \tag{56}$$

$$\log A_n \sim n \log n, \quad (57)$$

$$A_n = \frac{n^n}{(\log 2)^n e^{(1+o(1))n}}. \quad (58)$$

Proof. Equation (51) is an immediate consequence of limit (22). Equations (54), (55), (56) and (58) can be proved from limit (22) (see reference [1, Theorem 1.3]). Equation (52) is an immediate consequence of equation (51). Equation (53) is an immediate consequence of equation (52). Equation (57) is an immediate consequence of equation (56). □

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