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Combinatorial Results for Semigroups of Orientation-Preserving Partial Transformations

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Abstract

Let $X_n = \{1, 2, \ldots, n\}$. On a partial transformation $\alpha : \mathrm{Dom}\,\alpha \subseteq X_n \to \mathrm{Im}\ \alpha \subseteq X_n$ of X_n the following parameters are defined: the *breadth or width* of α is $\mid \mathrm{Dom}\ \alpha \mid$, the *height* of α is $\mid \mathrm{Im}\ \alpha \mid$, and the *right (resp., left) waist* of α is $\max(\mathrm{Im}\ \alpha)$ (resp., $\min(\mathrm{Im}\ \alpha)$). We compute the cardinalities of some equivalences defined by equalities of these parameters on $\mathcal{OP}_n$, the semigroup of orientation-preserving full transformations of X_n, $\mathcal{POP}_n$ the semigroup of orientation-preserving partial transformations of X_n, $\mathcal{OR}_n$ the semigroup of orientation-preserving/reversing full transformations of X_n, and $\mathcal{POR}_n$ the semigroup of orientation-preserving/reversing partial transformations of X_n, and their partial one-to-one analogue semigroups, $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$.

1 Introduction and Preliminaries

Let $X_n = \{1, 2, \ldots, n\}$. A (partial) transformation $\alpha : \mathrm{Dom}\ \alpha \subseteq X_n \to \mathrm{Im}\ \alpha \subseteq X_n$ is said to be *full* or *total* if $\mathrm{Im}\ \alpha = X_n$; otherwise it is called *strictly* partial. The *breadth or width* of α is denoted and defined by $b(\alpha) =\mid \mathrm{Dom}\ \alpha \mid$. The *height* or *rank* of α is denoted and defined by $h(\alpha) =\mid \mathrm{Im}\ \alpha \mid$. The *right (resp., left) waist* of α is denoted and defined by $w^+(\alpha) = \max(\mathrm{Im}\ \alpha)$ (resp., $w^-(\alpha) = \min(\mathrm{Dom}\ \alpha)$). Of course, other parameters have been defined and many more could still be defined but we shall restrict ourselves to only these in this paper. It is worth noting that to define the left (right) waist of a transformation the base set X_n must be totally ordered. The main objects of study in this paper are $\mathcal{OP}_n$, the semigroup of orientation-preserving full transformations of X_n, $\mathcal{POP}_n$ the semigroup of orientation-preserving partial transformations of X_n, $\mathcal{OR}_n$ the semigroup of orientation-preserving/reversing full transformations of X_n, and $\mathcal{POR}_n$ the semigroup of orientation-preserving/reversing partial transformations of X_n, and their partial one-to-one analogue semigroups, $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$. For basic and standard concepts in transformation semigroup theory the reader may consult one of Howie [10], Higgins [7] or Ganyushkin and Mazorchuk [6].

Enumerative results of an essentially combinatorial nature in the study of semigroups of transformations are now numerous and interesting to warrant sections and plenty of exercises in [6] and survey articles [9, 5, 19, 20]. These enumeration problems lead to many numbers in Sloane's encyclopaedia of integer sequences [18] but there is also the likelihood of finding others that may not yet be recorded in [18]. Motivated by Higgins [8], Laradji and Umar wrote a series of papers [11, 12, 13, 14, 15] dealing exclusively with combinatorial questions.

Let S be a set of partial transformations on X_n. Next, let

$$F(n;r,p,k) = \mid \{\alpha \in S : b(\alpha) = r \wedge h(\alpha) = p \wedge w^+(\alpha) = k\} \mid,$$

and let $P = \{r,p,k\}$ be the set of counters for the breadth, height, and right waist of a transformation. Then any 3-parameter combinatorial function can be expressed as $F(n;a_1,a_2)$, where $\{a_1,a_2\} \subset P$. For example,

$$F(n;r,p) = \mid \{\alpha \in S : b(\alpha) = r \wedge h(\alpha) = p\} \mid .$$

Similarly, any 2-parameter combinatorial function can be expressed as $F(n;a_1)$. It is not difficult to see that

$$|S| = \sum_{a_1} F(n;a_1),\ \ F(n;a_1) = \sum_{a_2} F(n;a_1,a_2).$$

We note also that certain special cases of these combinatorial functions, when two or more parameters are equal or when these parameters take extreme values are worth pointing out, see for example, [13, 14].

In Section 2 we consider the semigroups of orientation-preserving full and partial transformations $\mathcal{OP}_n$ and $\mathcal{POP}_n$, while in Section 3 we consider $\mathcal{OR}_n$ and $\mathcal{POR}_n$, the semigroups of orientation-preserving/reversing full and partial transformations of X_n, respectively. And finally, in Section 4 we consider the partial one-to-one analogues of the earlier considered classes of semigroups, $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$. We conclude this section with a list of results that will be needed in our proofs.

Lemma 1. *Let α be a partial transformation of $X_n = \{1,2,\ldots,n\}$. Let $r = b(\alpha)$, $p = h(\alpha)$ and $k = w^+(\alpha)$. Then we have the following:*

1. $n \geq r \geq p \geq 0$;
2. $n \geq k \geq p \geq 0$;
3. $r = 1 \implies p = 1$;

4. $k = 1 \implies p = 1$;

5. $r = 0 \Leftrightarrow p = 0 \Leftrightarrow k = 0$.

Lemma 2. *For all natural number n we have*

$$\sum_{i=1}^{n}(i-1)\binom{n}{i} = (n-2)2^{n-1}+1.$$

Proof.

$$\sum_{i=1}^{n}(i-1)\binom{n}{i} = n\sum_{i=1}^{n}\binom{n-1}{i-1} - \sum_{i=1}^{n}\binom{n}{i} = n2^{n-1} - (2^n - 1) = (n-2)2^{n-1}+1.$$

□

Lemma 3. *(Vandemonde's Convolution Identity, [17, (3a), p. 8]). For all natural numbers m, n and p we have*

$$\sum_{k=0}^{n}\binom{n}{m-k}\binom{p}{k} = \binom{n+p}{m}.$$

Lemma 4. *[13, Lemma 1.3] For all natural numbers n and p we have*

$$\sum_{k=p}^{n}\binom{k-1}{p-1} = \binom{n}{p}.$$

2 Orientation-Preserving Partial Transformations

Let $a = (a_1, a_2, \ldots, a_t)$ be a sequence of t $(t > 0)$ elements from the chain X_n. We say that a is *cyclic* if there exists no more than one index $i \in \{1, 2, \ldots, t\}$ such that $a_i > a_{i+1}$, where a_{t+1} denotes a_1. For a partial transformation α of X_n, suppose that $\operatorname{Dom}\alpha = \{a_1, a_2, \ldots, a_t\}$, with $t \geq 0$ and $a_1 < a_2 < \ldots < a_t$. We say that α is *orientation-preserving* if $(a_1\alpha, a_2\alpha, \ldots, a_t\alpha)$ is cyclic. The semigroups of orientation-preserving full and partial transformations of X_n will be denoted by $\mathcal{OP}_n$ and $\mathcal{POP}_n$, respectively. Moreover, note that for all $\alpha \in \mathcal{POP}_n$ and $y \in \operatorname{Im}\alpha$, $y\alpha^{-1}$ is convex with respect to $\operatorname{Dom}\alpha$ and the circular order. Further note that to partition $\operatorname{Dom}\alpha$ into p nonempty convex subsets, we insert p symbols between the $r = |\operatorname{Dom}\alpha|$ spaces. Then we have the principal result of this section.

Proposition 5. *Let $S = \mathcal{POP}_n$. Then*

$$F(n;r,p,k) = \begin{cases} \binom{n}{r}\binom{k-1}{p-1}\binom{r}{p}p, & n \geq r, k \geq p > 1; \\ \binom{n}{r}, & r = 1 \text{ or } k = 1 \text{ or } p = 1. \end{cases}$$

Proof. First, note that we can choose the r elements of $\operatorname{Dom}\alpha$ from X_n in $\binom{n}{r}$ ways, and the p elements of $\operatorname{Im}\alpha$ from $\{1, 2, \ldots, k\}$ in $\binom{k-1}{p-1}$ ways, since k must be one of the choices. Next, we partition the r chosen elements of the domain into p-convex subsets in $\binom{r}{p}$ ways. It is clear that that there are exactly p ways of tying these p-subsets to the p-images. Hence the result follows. □

Let $a \in P = \{r, p, k\}$. Then for all $i \in \{0, 1, \ldots, n\}$ if $a = i$ we shall denote this by a_i.

Corollary 6. *Let $S = \mathcal{POP}_n$. Then*

$$F(n; r, p) = \begin{cases} \binom{n}{r}\binom{n}{p}\binom{r}{p}p, & n \geq r \geq p > 1; \\ n\binom{n}{r}, & r = 1 \, or \, p = 1. \end{cases}$$

Proof.

$$\begin{aligned} F(n; r, p) &= \sum_{k=p}^{n} F(n; r, p, k) = \sum_{k=p}^{n} \binom{n}{r}\binom{k-1}{p-1}\binom{r}{p}p \ (r \geq p \geq 2) \\ &= p\binom{n}{r}\binom{r}{p} \sum_{k=p\geq 2}^{n} \binom{k-1}{p-1} = \binom{n}{r}\binom{r}{p}\binom{n}{p}p \ \ (by\, Lemma\, 4). \end{aligned}$$

Moreover, it is not difficult to see that $F(n; r_1, p_1) = n^2$ and $F(n; r, p_1) = n\binom{n}{r}$. Hence the result follows. □

Corollary 7. *Let $S = \mathcal{POP}_n$. Then*

$$F(n; r, k) = r\binom{n}{r}\binom{r+k-2}{r-1} - (r-1)\binom{n}{r},$$

for $n \geq r, k \geq 1$.

Proof.

$$\begin{aligned} F(n; r, k) &= \sum_{p=1}^{k} F(n; r, p, k) = \binom{n}{r} + \sum_{p=2}^{n} \binom{n}{r}\binom{k-1}{p-1}\binom{r}{p}p \ (r, k \geq 2) \\ &= \binom{n}{r} + \sum_{p=1}^{n} \binom{n}{r}\binom{k-1}{p-1}\binom{r}{p}p - r\binom{n}{r} \\ &= r\binom{n}{r} \sum_{p=1}^{n} \binom{k-1}{p-1}\binom{r-1}{p-1} - (r-1)\binom{n}{r} \\ &= r\binom{n}{r}\binom{r+k-2}{r-1} - (r-1)\binom{n}{r} \ \ (by\, Lemma\, 3). \end{aligned}$$

Moreover, it is not difficult to see that $F(n; r_1, k) = n$ and $F(n; r, k_1) = \binom{n}{r}$. Hence the result follows. □

Corollary 8. *Let $S = \mathcal{POP}_n$. Then*

$$F(n;p,k) = \begin{cases} p2^{n-p}\binom{n}{p}\binom{k-1}{p-1}, & n \geq k \geq p > 1; \\ 2^n - 1, & k = 1 \text{ or } p = 1. \end{cases}$$

Proof.

$$\begin{aligned} F(n;p,k) &= \sum_{r=p}^{k} F(n;r,p,k) = \sum_{r=2}^{n} \binom{n}{r}\binom{k-1}{p-1}\binom{r}{p}p \ (k \geq p \geq 2) \\ &= p\binom{k-1}{p-1}\sum_{r=2}^{n}\binom{n}{r}\binom{r}{p} = p\binom{k-1}{p-1}\binom{n}{p}\sum_{r=2}^{n}\binom{n-p}{r-p} \\ &= p2^{n-p}\binom{k-1}{p-1}\binom{n}{p}. \end{aligned}$$

Moreover, it is not difficult to see that $F(n;p_1,k) = F(n;p,k_1) = 2^n - 1$. Hence the result follows. □

Corollary 9. *Let $S = \mathcal{POP}_n$. Then*

$$F(n;r) = \begin{cases} r\binom{n}{r}\binom{n+r-1}{n-1} - n(r-1)\binom{n}{r}, & n \geq r \geq 1; \\ 1, & r = 0. \end{cases}$$

Proof.

$$\begin{aligned} F(n;r) &= \sum_{p=1}^{r} F(n;r,p) = n\binom{n}{r} + \sum_{p=2}^{r}\binom{n}{r}\binom{r}{p}\binom{n}{p}p \ (r \geq 2) \\ &= n\binom{n}{r} + \binom{n}{r}\sum_{p=1}^{r}\binom{r}{p}\binom{n}{p}p - rn\binom{n}{r} \\ &= r\binom{n}{r}\sum_{p=1}^{r}\binom{r-1}{p-1}\binom{n}{p} - n(r-1)\binom{n}{r} \\ &= r\binom{n}{r}\binom{n+r-1}{r} - n(r-1)\binom{n}{r} \ \ (\textit{by Lemma } 3). \end{aligned}$$

Moreover, it is not difficult to see that $F(n;r_1) = n^2$ and $F(n;r_0) = 1$. Hence the result follows. Alternatively, we can get the same result from $F(n;r) = \sum_{k=1}^{n} F(n;r,k)$. □

Corollary 10. *Let $S = \mathcal{POP}_n$. Then*

$$F(n;p) = \begin{cases} p2^{n-p}\binom{n}{p}^2, & n \geq p > 1; \\ n(2^n - 1), & p = 1; \\ 1, & p = 0. \end{cases}$$

Proof.

$$\begin{aligned}
F(n;p) &= \sum_{r=p}^{n} F(n;r,p) = \sum_{r=p}^{n} \binom{n}{r}\binom{r}{p}\binom{n}{p} p \ (p \geq 2) \\
&= p\binom{n}{p}\sum_{r=p}^{n}\binom{n}{r}\binom{r}{p} = p\binom{n}{p}^2 \sum_{r=p}^{n}\binom{n-p}{r-p} \\
&= p2^{n-p}\binom{n}{p}^2.
\end{aligned}$$

It is not difficult to see that $F(n;p_1) = n(2^n - 1)$ and $F(n;p_0) = 1$. Hence the result follows. Alternatively, we can get the same result from $F(n;p) = \sum_{k=p}^{n} F(n;p,k)$. □

Corollary 11. *Let* $S = \mathcal{POP}_n$. *Then*

$$F(n;k) = \begin{cases} n\sum_{r=1}^{n}\binom{n-1}{r-1}\binom{r+k-2}{r-1} - (n-2)2^{n-1} - 1, & n \geq k \geq 1; \\ 1, & k = 0. \end{cases}$$

Proof.

$$\begin{aligned}
F(n;k) &= \sum_{r=1}^{n} F(n;r,k) = n + \sum_{r=2}^{n}[r\binom{n}{r}\binom{r+k-2}{r-1} - (r-1)\binom{n}{r}] \ (k \geq 2) \\
&= n + n\sum_{r=2}^{n}\binom{n-1}{r-1}\binom{r+k-2}{r-1} - ((n-2)2^{n-1} + 1) \ \ (\textit{by Lemma } 2) \\
&= n\sum_{r=1}^{n}\binom{n-1}{r-1}\binom{r+k-2}{r-1} - (n-2)2^{n-1} - 1.
\end{aligned}$$

It is not difficult to see that $F(n;k_1) = n(2^n - 1)$ and for convenience we set $F(n;k_0) = 1$. Hence the result follows. Alternatively, we get

$$\begin{aligned}
F(n;k) &= \sum_{p=1}^{k} F(n;p,k) = (2^n - 1) + \sum_{p=2}^{k} p\binom{k-1}{p-1}\binom{n}{p}2^{n-p} \ (k \geq 2) \\
&= (2^n - 1) + n\sum_{p=2}^{k}\binom{k-1}{p-1}\binom{n-1}{p-1}2^{n-p} \\
&= (2^n - 1) + n\sum_{p=1}^{k}\binom{k-1}{p-1}\binom{n-1}{p-1}2^{n-p} - n2^{n-1} \\
&= n\sum_{p=1}^{k}\binom{k-1}{p-1}\binom{n-1}{p-1}2^{n-p} - (n-2)2^{n-1} - 1.
\end{aligned}$$

□

From the proof of Corollary 11 we deduce the following non-trivial identity.

Proposition 12. *For all natural numbers n and k we have*

$$\sum_{r=1}^{n}\binom{n-1}{r-1}\binom{r+k-2}{r-1}=\sum_{r=1}^{k}\binom{k-1}{r-1}\binom{n-1}{r-1}2^{n-r}.$$

We can now deduce the order of $\mathcal{POP}_n$. The first expression is [4, Proposition 1.9].

Corollary 13. *For all natural number n we have*

$$\begin{aligned}\mid \mathcal{POP}_n \mid &= 1+n(2^n-1)+\sum_{p=2}^{n}p2^{n-p}\binom{n}{p}^2\\ &= 1+n\sum_{r=1}^{n}\binom{n-1}{r-1}\binom{n+r-1}{n-1}-n(n-2)2^{n-1}-n.\end{aligned}$$

Corollary 14. *Let $S=\mathcal{OP}_n$. Then*

$$\begin{cases}\binom{k-1}{p-1}\binom{n}{p}p, & n\geq k\geq p>1;\\ 1, & k=1 \text{ or } p=1.\end{cases}$$

Proof. The result follows by the substitution $r=n$ in Proposition 5. □

Corollary 15. *[1, p. 198] Let $S=\mathcal{OP}_n$. Then*

$$F(n;p)=\begin{cases}p\binom{n}{p}^2, & n\geq p>1;\\ n, & p=1.\end{cases}$$

Corollary 16. *Let $S=\mathcal{OP}_n$. Then*

$$F(n;k)=\begin{cases}n\binom{n+k-2}{k-1}-(n-1), & n\geq k>1;\\ 1, & k=1.\end{cases}$$

Corollary 17. *[16, Theorem 4.3] & [1, p. 194] For any natural number n we have*

$$\mid \mathcal{OP}_n \mid = \frac{n}{2}\binom{2n}{n}-n(n-1).$$

Remark 18. The triangles of numbers $F_{POP}(n;r)$, $F_{POP}(n;p)$, $F_{POP}(n;k)$, $F_{OP}(n;p)$ and $F_{OP}(n;k)$ are as at the time of submitting this paper not in Sloane [18]. However, note that $F_{POP}(n,p_1)$ is [18, A066524] and $F_{OP}(n,k_1)$ is [18, A002061].

$n\backslash r$	0	1	2	3	4	5	6	$\sum F(n;r) = \mid \mathcal{POP}_n \mid$
0	1							1
1	1	1						2
2	1	4	4					9
3	1	9	27	24				61
4	1	16	96	208	128			449
5	1	25	250	950	1325	610		3161
6	1	36	540	3120	7290	7416	2742	21145

Table 2.1

$n\backslash p$	0	1	2	3	4	5	6	$\sum F(n;p) = \mid \mathcal{POP}_n \mid$
0	1							1
1	1	1						2
2	1	6	2					9
3	1	21	36	3				61
4	1	60	288	96	4			449
5	1	155	1600	1200	200	5		3161
6	1	378	7200	9600	3600	360	6	21145

Table 2.2

$n\backslash k$	0	1	2	3	4	5	6	$\sum F(n;k) = \mid \mathcal{POP}_n \mid$
0	1							1
1	1	1						2
2	1	3	5					9
3	1	7	19	34				61
4	1	15	63	135	235			449
5	1	31	191	471	911	1556		3161
6	1	63	543	1503	3183	5883	9969	21145

Table 2.3

$n\backslash p$	1	2	3	4	5	6	7	$\sum F(n;p) = \mid \mathcal{OP}_n \mid$
1	1							1
2	2	2						4
3	3	18	3					24
4	4	72	48	4				128
5	5	200	300	100	5			610
6	6	450	1200	900	180	6		2742
7	7	882	3675	4900	2205	294	7	11970

Table 2.4

$n\backslash k$	1	2	3	4	5	6	7	$\sum F(n;k) = \mid \mathcal{OP}_n \mid$
1	1							1
2	1	3						4
3	1	7	16					24
4	1	13	37	77				128
5	1	21	71	171	346			610
6	1	31	121	331	751	1507		2742
7	1	43	190	582	1464	3228	6462	11970

Table 2.5

3 Orientation-Preserving or Reversing Partial Transformations

Let $a = (a_1, a_2, \ldots, a_t)$ be a sequence of t $(t > 0)$ elements from the chain X_n. We say that a is *anti-cyclic* if there exists no more than one index $i \in \{1, 2, \ldots, t\}$ such that $a_i < a_{i+1}$, where a_{t+1} denotes a_1. For a partial transformation α of X_n, suppose that $\operatorname{Dom}\alpha = \{a_1, a_2, \ldots, a_t\}$, with $t \geq 0$ and $a_1 < a_2 < \ldots < a_t$. We say that α is *orientation-reversing* if $(a_1\alpha, a_2\alpha, \ldots, a_t\alpha)$ is anti-cyclic. The semigroups of orientation-preserving or reversing full and partial transformations of X_n will be denoted by $\mathcal{OR}_n$ and $\mathcal{POR}_n$, respectively.

Remark 19. For $p = 1, 2$ every orientation-preserving transformation is also orientation-reversing but distinct otherwise [1, Lemma 1.1]. However, there is a bijection between the set of orientation-preserving transformations and that of orientation-reversing transformations [1, Lemma 5.1].

The proofs of all the results in this section are similar to the proofs of the corresponding results in Section 2, taking into account Remark 19.

Proposition 20. *Let $S = \mathcal{POR}_n$. Then*

$$F(n; r, p, k) = \begin{cases} 2\binom{n}{r}\binom{k-1}{p-1}\binom{r}{p}p, & n \geq r, k \geq p > 2; \\ 2(k-1)\binom{n}{r}\binom{r}{2}, & n \geq r, k \geq p = 2; \\ \binom{n}{r}, & r = 1 \, or \, k = 1 \, or \, p = 1. \end{cases}$$

Corollary 21. *Let $S = \mathcal{POR}_n$. Then*

$$F(n; r, p) = \begin{cases} 2\binom{n}{r}\binom{n}{p}\binom{r}{p}p, & r \geq p > 2; \\ 2\binom{n}{r}\binom{r}{2}\binom{n}{2}, & n \geq r \geq p = 2; \\ n\binom{n}{r}, & r = 1 \, or \, p = 1. \end{cases}$$

Corollary 22. *Let $S = \mathcal{POR}_n$. Then*

$$F(n; r, k) = \begin{cases} \binom{n}{r}[2r\binom{r+k-2}{k-1} - (2r-1) - 2(k-1)\binom{r}{2}], & r, k > 2; \\ (2k-1)\binom{n}{2}, & k \geq r = 2; \\ \binom{n}{r} + 2\binom{n}{r}\binom{r}{2}, & r \geq k = 2; \\ \binom{n}{r}, & r = 1 \, or \, k = 1. \end{cases}$$

Corollary 23. *Let $S = \mathcal{POR}_n$. Then*

$$F(n; p, k) = \begin{cases} p2^{n-p+1}\binom{k-1}{p-1}\binom{n}{p}p, & k \geq p > 2; \\ 2^{n-1}\binom{n}{2} + 2^n - 1, & k = 2; \\ (k-1)2^{n-1}\binom{n}{2}, & p = 2; \\ 2^n - 1, & k = 1 \, or \, p = 1. \end{cases}$$

Corollary 24. *Let $S = \mathcal{POR}_n$. Then*

$$F(n; r) = 2r\binom{n}{r}\binom{n+r-1}{n-1} - n(2r-1)\binom{n}{r} - 2\binom{n}{r}\binom{r}{2}\binom{n}{2}, \; for \, r \geq 1.$$

Corollary 25. *Let $S = \mathcal{POR}_n$. Then*

$$F(n;p) = \begin{cases} p2^{n-p+1}\binom{n}{p}^2, & p > 2; \\ 2^{n-1}\binom{n}{2}^2, & p = 2; \\ n(2^n - 1), & p = 1. \end{cases}$$

Corollary 26. *Let $S = \mathcal{POR}_n$. Then*

$$\begin{aligned} F(n;k) &= 2n\sum_{r=1}^{n}\binom{n-1}{r-1}\binom{r+k-2}{r-1} - (n-1)2^n - 1 - (k-1)2^{n-1}\binom{n}{2} \\ &= 2n\sum_{p=1}^{k}\binom{k-1}{p-1}\binom{n-1}{p-1}2^{n-p} - (n-1)2^n - 1 - (k-1)2^{n-1}\binom{n}{2}, \end{aligned}$$

for $k \geq 1$.

We can now deduce the order of $\mathcal{POR}_n$. The first expression is [4, Proposition 1.10].

Corollary 27. *For all natural number n we have*

$$\begin{aligned} \mid \mathcal{POR}_n \mid &= 1 + n(2^n - 1) = 2^{n-1}\binom{n}{2}^2 + \sum_{p=3}^{n} p2^{n-p+1}\binom{n}{p}^2 \\ &= 1 + 2n\sum_{r=1}^{n}\binom{n-1}{r-1}\binom{n+r-1}{n-1} - 2^{n-3}n(n-1)(n^2-n+8) - n. \end{aligned}$$

Corollary 28. *Let $S = \mathcal{OR}_n$. Then*

$$F(n;p,k) = \begin{cases} 2\binom{k-1}{p-1}\binom{n}{p}p, & k \geq p > 2; \\ \binom{k-1}{p-1}\binom{n}{p}p, & k \geq p = 2; \\ 1, & k = 1 \text{ or } p = 1. \end{cases}$$

Corollary 29. *Let $S = \mathcal{OR}_n$. Then*

$$F(n;p) = \begin{cases} 2p\binom{n}{p}^2, & p > 2; \\ 2\binom{n}{2}^2, & p = 2; \\ n, & p = 1. \end{cases}$$

Corollary 30. *Let $S = \mathcal{OR}_n$. Then*

$$F(n;k) = 2n\binom{n+k-2}{k-1} - 2(k-1)\binom{n}{2} - 2n + 1,$$

for $k \geq 1$.

Corollary 31. *[16, Theorem 5.2] & [1, Theorem 5.5] For any natural number n we have*

$$\mid \mathcal{OR}_n \mid = n\binom{2n}{n} - n^2(n^2 - 2n + 5)/2 + n.$$

Remark 32. The triangles of numbers $F_{POR}(n;r)$, $F_{POR}(n;p)$, $F_{POR}(n;k)$, $F_{OR}(n;p)$ and $F_{OR}(n;k)$ are as at the time of submitting this paper not in Sloane [18]. However, note that $F_{POR}(n;r_1)$ is [18, A000290], $F_{POR}(n;p_1)$ is [18, A066524], $F_{POR}(n;k_1)$ is [18, A000225], $F_{OR}(n;p_1)$ is [18, A163102] and $F_{OR}(n;k_1)$ is [18, A002061].

$n\backslash r$	0	1	2	3	4	5	6	$\sum F(n;r) = \mid \mathcal{POR}_n \mid$
0	1							1
1	1	1						2
2	1	4	4					9
3	1	9	27	27				64
4	1	16	96	256	180			549
5	1	25	250	1250	2025	1015		4566
6	1	36	540	4320	11790	12996	5028	34711

Table 3.1

$n\backslash p$	0	1	2	3	4	5	6	$\sum F(n;p) = \mid \mathcal{POR}_n \mid$
0	1							1
1	1	1						2
2	1	6	2					9
3	1	21	36	6				64
4	1	60	288	192	8			549
5	1	155	1600	2400	400	10		4566
6	1	378	7200	19200	7200	720	12	34711

Table 3.2

$n\backslash k$	0	1	2	3	4	5	6	$\sum F(n;k) = \mid \mathcal{POR}_n \mid$
0	1							1
1	1	1						2
2	1	3	5					9
3	1	7	19	37				64
4	1	15	63	159	311			549
5	1	31	191	591	1311	2441		4566
6	1	63	543	1983	4863	9783	17475	34711

Table 3.3

$n\backslash p$	1	2	3	4	5	6	7	$\sum F(n;p) = \mid \mathcal{OR}_n \mid$
1	1							1
2	2	2						4
3	3	18	6					27
4	4	72	96	8				180
5	5	200	600	200	10			1015
6	6	450	2400	1800	360	12		5028
7	7	882	7350	9800	4410	588	14	23051

Table 3.4

$n\backslash k$	1	2	3	4	5	6	7	$\sum F(n;k) = \mid \mathcal{OR}_n \mid$
1	1							1
2	1	3						4
3	1	7	19					27
4	1	13	49	117				180
5	1	21	101	281	611			1015
6	1	31	181	571	1381	2863		5028
7	1	43	295	1037	2759	6245	12671	23051

Table 3.5

4 Orientation-Preserving or Orientation-Reversing Partial One-to-one Transformations

The semigroups of orientation-preserving and orientation-preserving or reversing partial one-to-one transformations of X_n will be denoted by $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$, respectively.

Proposition 33. *Let $S = \mathcal{POPI}_n$. Then $F(n;p,k) = \binom{n}{p}\binom{k-1}{p-1}p$, for $n \geq k \geq p > 0$.*

Proof. First observe that the p elements of $\mathrm{Dom}\,\alpha$ can be chosen from X_n in $\binom{n}{p}$ ways, and since k is the maximum element in $\mathrm{Im}\,\alpha$ then the remaining $p-1$ elements of $\mathrm{Im}\,\alpha$ can be chosen from $\{1, 2, \ldots, k-1\}$ in $\binom{k-1}{p-1}$ ways. Finally, observe that if $p > 0$, then the p elements of $\mathrm{Dom}\,\alpha$ can be tied to the p images in a one-to-one fashion (whilst preserving the orientation), in p ways. The result now follows. □

The following corollaries can be deduced in exactly the same manner as their corresponding results in Section 2:

Corollary 34. *Let $S = \mathcal{POPI}_n$. Then*

$$F(n;p) = \begin{cases} \binom{n}{p}^2 p, & n \geq p > 1; \\ n^2, & p = 1. \end{cases}$$

Corollary 35. *Let* $S = \mathcal{POPI}_n$*. Then* $F(n;k) = n\binom{n+k-2}{k-1}$*, for* $n \geq k \geq 1$*.*

Corollary 36. *Let* $S = \mathcal{POPI}_n$*. Then* $F(n;k_n) = n\binom{2n-2}{n-1}$*, for* $n \geq 1$*.*

Corollary 37. *[2, Corollary 2.8] For any natural number* n *we have*

$$| \mathcal{POPI}_n | = 1 + \frac{n}{2}\binom{2n}{n}.$$

Proposition 38. *Let* $S = \mathcal{PORI}_n$*. Then*

$$F(n;p,k) = \begin{cases} 2\binom{n}{p}\binom{k-1}{p-1}p, & n \geq k \geq p > 2; \\ n^2, & k = 2; \\ 2(k-1)\binom{n}{2}, & p = 2; \\ n, & k = 1 \, or \, p = 1. \end{cases}$$

Proof. The proof is similar to that of Proposition 33. □

Corollary 39. *Let* $S = \mathcal{PORI}_n$*. Then*

$$F(n;p) = \begin{cases} 2\binom{n}{p}^2 p, & n \geq p > 2; \\ 2\binom{n}{2}^2, & p = 2; \\ n^2, & p = 1. \end{cases}$$

Corollary 40. *Let* $S = \mathcal{PORI}_n$*. Then*

$$F(n;k) = 2n\binom{n+k-2}{n-1} - n - n(n-1)(k-1), \; for \; n \geq k > 0.$$

Corollary 41. *Let* $S = \mathcal{PORI}_n$*. Then*

$$F(n;k_n) = 2n\binom{2n-2}{n-1} - n - n(n-1)^2, \; for \; n \geq 1.$$

Corollary 42. *[3, Proposition 5.2] For any natural number* n *we have*

$$| \mathcal{PORI}_n | = 1 + n\binom{2n}{n} - n^2(n^2 - 2n + 3)/2.$$

Remark 43. The triangular arrays of numbers $F_{POPI}(n;p)$, $F_{POPI}(n;k)$, $F_{PORI}(n;p)$ and $F_{PORI}(n;k)$ are as at the time of submitting this paper not in Sloane [18]. However, note that $F_{PORI}(n;p_2)$ is [18, A163102].

$n \backslash p$	0	1	2	3	4	5	6	$\sum F(n;p) = \mid \mathcal{POPI}_n \mid$
0	1							1
1	1	1						2
2	1	4	2					7
3	1	9	18	3				31
4	1	16	72	48	4			141
5	1	25	200	300	100	5		631
6	1	36	450	1200	900	180	6	2773

Table 4.1

$n\backslash k$	0	1	2	3	4	5	6	$\sum F(n;k) = \mid \mathcal{POPI}_n \mid$
0	1							1
1	1	1						2
2	1	2	4					7
3	1	3	9	18				31
4	1	4	16	40	80			141
5	1	5	25	75	175	350		631
6	1	6	36	126	336	756	1512	2773

Table 4.2

$n\backslash p$	0	1	2	3	4	5	6	$\sum F(n;p) = \mid \mathcal{PORI}_n \mid$
0	1							1
1	1	1						2
2	1	4	2					7
3	1	9	18	6				34
4	1	16	72	96	8			193
5	1	25	200	600	200	10		1036
6	1	36	450	2400	1800	360	12	5059

Table 4.3

$n\backslash k$	0	1	2	3	4	5	6	$\sum F(n;k) = \mid \mathcal{PORI}_n \mid$
0	1							1
1	1	1						2
2	1	2	4					7
3	1	3	9	21				34
4	1	4	16	52	120			193
5	1	5	25	105	285	615		1036
6	1	6	36	166	576	1386	2868	5059

Table 4.4

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