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A New Characterization of Catalan Numbers Related to Hankel Transforms and Fibonacci Numbers

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Abstract

Cvetković, Rajković, and Ivković proved that the Hankel transform of the sequence of sums of two successive Catalan numbers is the sequence of Fibonacci numbers with odd indices. Later, Benjamin, Cameron, Quinn, and Yerger extended this result and proved that if we remove one term from this sequence of sums, then the Hankel transform is the sequence of Fibonacci numbers with even indices. In this paper, we prove that the Catalan numbers are the unique nonnegative integer sequence satisfying this property.

1 Introduction

The Hankel transform of a sequence $\{a_n\}_{n=0}^{\infty}$ is the sequence $H(\{a_n\}_{n=0}^{\infty}) = \{h_n\}_{n=0}^{\infty}$ defined as follows:

$$h_n = \det[a_{i+j}]_{0\le i,\, j\le n-1}. \tag{1}$$

The Hankel transform was introduced and first studied by Layman [7], who showed that the Hankel transform of a sequence is equal to the Hankel transform of the binomial transform of this sequence and conjectured that the Hankel transform of the sequence of sums of two adjacent Catalan numbers is a subsequence of Fibonacci number sequence.

Recall that the Catalan number sequence $\{C_n\}_{n=0}^{\infty}$ is defined as follows:

$$C_n = \frac{\binom{2n}{n}}{n+1},\ n \geq 0.$$

This sequence is the unique nonnegative integer sequence $\{a_n\}_{n=0}^{\infty}$ fulfilling (see [8])

$$\det[a_{i+j}]_{0\leq i,\, j\leq n-1} = 1$$

and

$$\det[a_{i+j}]_{1\leq i,\, j\leq n} = 1.$$

Cvetković, Rajković, and Ivković [6] proved the following formula, known as Layman's conjecture,

$$\det[C_{i+j+1} + C_{i+j+2}]_{0\leq i,\, j\leq n-1} = F_{2n+2},\ n \geq 1. \tag{2}$$

where $F_n,\ n \geq 0$ is the n-th Fibonacci number defined by the following recurrence:

$$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n,\ n \geq 0. \tag{3}$$

The proof depends on special properties of the corresponding orthogonal polynomials. Benjamin, Cameron, Quinn, and Yerger [1] used a combinatorial approach and proved that

$$\det[C_{i+j} + C_{i+j+1}]_{0\leq i,\, j\leq n-1} = F_{2n+1},\ n \geq 1. \tag{4}$$

In view of (4) and (2), it should be mentioned here that (4) and (2) are special cases of the more general determinant evaluation given in [3].

A natural question arises: Is the Catalan number sequence the unique nonnegative integer sequence fulfilling (4) and (2)? More precisely, is there any other nonnegative integer sequence $\{a_n\}_{n=0}^{\infty}$ fulfilling

$$\det[a_{i+j} + a_{i+j+1}]_{0\leq i,\, j\leq n-1} = F_{2n+1},\ n \geq 1 \tag{5}$$

and

$$\det[a_{i+j+1} + a_{i+j+2}]_{0\leq i,\, j\leq n-1} = F_{2n+2},\ n \geq 1. \tag{6}$$

The aim of this paper is to answer this question. Namely, we prove the main theorem of this paper:

Theorem 1. *The unique nonnegative integer sequence fulfilling (5) and (6) is the Catalan number sequence.*

Our paper is organized as follows: In Section 2, we state some basic results concerning the theory of orthogonal polynomial. These results will be used in Section 3 to prove the main theorem.

2 Notation and preliminary results

Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients and let $\mathcal{P}'$ be its algebraic dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $u_n = \langle u, x^n\rangle$, $n \geq 0$ the moments of u. If $u \in \mathcal{P}'$ and $\Phi \in \mathcal{P}$, then the left multiplication of the functional u by the polynomial Φ, denoted by Φu, is the functional in $\mathcal{P}'$ defined as follows:

$$\langle \Phi u, f\rangle = \langle u, \Phi f\rangle, \ f \in \mathcal{P}. \tag{7}$$

Definition 2. (see [5]) A sequence of polynomials $\{P_n\}_{n\geq 0}$ is said to be a *monic orthogonal polynomial sequence* (MOPS) with respect to a linear functional u if
$i)$ deg $P_n = n$ and the leading coefficient of $P_n(x)$ is equal to 1.
$ii)$ $\langle u, P_n P_m\rangle = r_n \delta_{n,m}, n, m \geq 0, r_n \neq 0, n \geq 0.$
A linear functional u is called *regular* if there exists a polynomial sequence $\{P_n\}_{n\geq 0}$ orthogonal with respect to u. Throughout this paper, we will take all regular linear functionals u normalized, i.e., $u_0 = 1$.

According to Favard-Shohat theorem, a sequence of monic orthogonal polynomials satisfies a three-term recurrence relation (see [5]):

$$\begin{gathered} P_0(x) = 1, \ P_1(x) = x - \beta_0, \\ P_{n+2} = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), n \geq 0, \end{gathered} \tag{8}$$

with

$$(\beta_n, \gamma_n) \in \mathbb{C} \times (\mathbb{C} \setminus \{0\}), n \geq 0.$$

3 Proof of the main theorem

First of all, recall that a linear functional L is regular if and only if (see [5])

$$\Delta_n(L) \neq 0, \tag{9}$$

where $\Delta_n(L)$ is the Hankel determinant of order $n+1$ of L defined as follows:

$$\Delta_{-1}(L) = 1, \ \Delta_n(L) = \begin{vmatrix} L_0 & L_1 & \cdots & L_n \\ L_1 & L_2 & \cdots & L_{n+1} \\ \vdots & \vdots & & \vdots \\ L_n & L_{n+1} & \cdots & L_{2n} \end{vmatrix}, \ n \geq 0. \tag{10}$$

In this condition, the coefficient γ_n in (8) is given as follows (see [2, 5]):

$$\gamma_{n+1} = \frac{\Delta_{n+1}(L)\Delta_{n-1}(L)}{\Delta_n^2(L)}, \ n \geq 0. \tag{11}$$

Conversely, the Hankel determinant $\Delta_n(L)$ can be written in terms of γ_n as follows (see [2]):

$$\Delta_{-1}(L) = \Delta_0(L) = 1, \ \Delta_n(L) = \prod_{k=1}^{n} \tau_k, \ n \geq 0, \tag{12}$$

where

$$\tau_k = \prod_{i=1}^{k} \gamma_i,\ k \geq 1. \tag{13}$$

Let us recall a result established in [4] concerning the so-called *kernel polynomials*. Given a MOPS $\{P_n\}_{n\geq 0}$ orthogonal with respect to a linear functional L. The linear functional L^* defined as follows:

$$L^* = \lambda(x-c)L,\ c \in \mathbb{C},$$

where λ is a normalization factor, is regular if and only if

$$P_n(c) \neq 0,\ n \geq 0.$$

In this case the MOPS corresponding to L^*, denoted by $\{P_n^*(c,x)\}_{n\geq 0}$, satisfies

$$P_n^*(c,x) = \frac{1}{x-c}(P_{n+1}(x) - \frac{P_{n+1}(c)}{P_n(c)}P_{n+1}(x)).$$

The sequence $\{P_n^*(c,x)\}_{n\geq 0}$ is called the sequence of kernel polynomials of K-parameter c corresponding to $\{P_n\}_{n\geq 0}$. The recurrence coefficients of $\{P_n^*(c,x)\}_{n\geq 0}$ denoted by β_n^* and γ_{n+1}^* are expressed in terms of those of $\{P_n\}_{n\geq 0}$ as follows:

$$\beta_n^* = \beta_{n+1} + \frac{P_{n+2}(c)}{P_{n+1}(c)} - \frac{P_{n+1}(c)}{P_n(c)}, \tag{14}$$

$$\gamma_{n+1}^* = \gamma_{n+1}\frac{P_{n+2}(c)P_n(c)}{P_{n+1}^2(c)},\ n \geq 0. \tag{15}$$

Let $\{a_n\}_{n=0}^{\infty}$ be a non-negative integer sequence fulfilling (5) and (6). Then for $n = 1$, we get $a_0 + a_1 = F_3 = 2 \neq 0$ and $a_1 + a_2 = F_4 = 3 \neq 0$. Define the linear functionals $u,\ v$ and v^* by

$$u_n = \frac{C_n + C_{n+1}}{C_0 + C_1},\ n \geq 0, \tag{16}$$

$$v_n = \frac{a_n + a_{n+1}}{a_0 + a_1},\ n \geq 0, \tag{17}$$

$$v_n^* = \frac{a_{n+1} + a_{n+2}}{a_1 + a_2},\ n \geq 0. \tag{18}$$

From (10), (16) and (4), we have

$$\Delta_n(u) = \frac{F_{2n+3}}{(C_0 + C_1)^{n+1}},\ n \geq 0. \tag{19}$$

Using the recurrence (3), we can easily see that $F_n \neq 0,\ n \geq 0$. Therefore, the condition (9) is satisfied by u. Hence, u is regular. Similarly, from (10), (5), (6), (17) and the relation (18), we get

$$\Delta_n(v) = \frac{F_{2n+3}}{(a_0 + a_1)^{n+1}},\ n \geq 0 \tag{20}$$

and

$$\Delta_n(v^*) = \frac{F_{2n+4}}{(a_1+a_2)^{n+1}},\ n \geq 0. \tag{21}$$

For the same reason, v and v^* are regular. We will denote by $\{P_n\}_{n\geq 0}$ and $\{P_n^*(c,x)\}_{n\geq 0}$ the MOPS corresponding to v and v^* respectively. From (17) and (18), we have

$$v^* = \frac{a_0+a_1}{a_1+a_2} x v.$$

So, the sequence $\{P_n^*(c,x)\}_{n\geq 0}$ is the sequence of kernel polynomials of K-parameter 0 corresponding to $\{P_n\}_{n\geq 0}$. Hence, using (12), (13) and (15), with $c = 0$, we get

$$\Delta_n(v^*) = \frac{P_{n+1}(0)}{(P_1(0))^{n+1}} \Delta_n(v),\ n \geq 0.$$

Taking into account (20) and (21), we obtain

$$F_{2n+4} = \left(\frac{a_1+a_2}{a_0+a_1}\right)^{n+1} \frac{P_{n+1}(0)}{(P_1(0))^{n+1}} F_{2n+3},\ n \geq 0. \tag{22}$$

Suppose that $\{P_n\}_{n\geq 0}$ satisfies the recurrence (8). We have

$$P_1(0) = -\beta_0.$$

But, from Definition 2 and the relation (8), we have

$$0 =< v, P_1 P_0 >=< v, P_1 >=< v, x > -\beta_0 < v, 1 >= v_1 - \beta_0.$$

Thus

$$\beta_0 = v_1 = \frac{a_1+a_2}{a_0+a_1}.$$

Therefore

$$P_1(0) = -\frac{a_1+a_2}{a_0+a_1}. \tag{23}$$

Substitution of (23) in (22) gives

$$F_{2n+4} = (-1)^{n+1} P_{n+1}(0) F_{2n+3},\ n \geq 0$$

or equivalently

$$P_{n+1}(0) = (-1)^{n+1} \frac{F_{2n+4}}{F_{2n+3}},\ n \geq 0. \tag{24}$$

From (11) and (20), we get

$$\gamma_{n+1} = \frac{F_{2n+5} F_{2n+1}}{F_{2n+3}^2},\ n \geq 0. \tag{25}$$

Taking $x = 0$ in (8) and using (24) and (25), we get

$$\beta_{n+1} = \frac{F_{2n+3} F_{2n+6}}{F_{2n+4} F_{2n+5}} + \frac{F_{2n+2} F_{2n+5}}{F_{2n+3} F_{2n+4}},\ n \geq 0.$$

On account of the recurrence relation (3), we get

$$\beta_{n+1} = \frac{F_{2n+4}}{F_{2n+3}} + \frac{F_{2n+3}}{F_{2n+5}},\ n \geq 0. \tag{26}$$

We have proved that the recurrence coefficients of the MOPS corresponding to any sequence $\{a_n\}_{n=0}^{\infty}$ satisfying (5) and (6) are given by (25) and (26). Since the sequence $\{C_n\}_{n=0}^{\infty}$ itself satisfies (5) and (6) then the recurrence coefficients of its MOPS are also given by (25) and (26). Consequently, by the uniqueness of the linear functional corresponding to a MOPS, we have

$$u = v$$

or, equivalently,

$$u_n = v_n,\ n \geq 0.$$

Therefore, owing to (16) and (17), we obtain

$$a_n + a_{n+1} = C_n + C_{n+1},\ n \geq 0. \tag{27}$$

On the other hand, we have

$$a_0 + a_1 = F_3 = 2.$$

Taking into account the fact that $\{a_n\}_{n=0}^{\infty}$ is a nonnegative sequence, we get $a_0 = 1 = C_0$ and $a_1 = 1 = C_1$. By induction and using (27), we can easily prove that,

$$a_n = C_n,\ n \geq 0.$$

Which completes the proof.

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