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Power Sums of Binomial Coefficients

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Abstract

We establish an analog of Faulhaber's theorem for a power sum of binomial coefficients. We study reciprocal power sums of binomial coefficients and Faulhaber coefficients for a power sum of triangular numbers.

1 Introduction

While the sum of powers of consecutive nonnegative integers was studied by many mathematicians from ancient times, two names should be especially mentioned: Jacob Bernoulli (1654-1705) and Johann Faulhaber (1580-1635). It is well known that

$$\sum_{i=1}^{n} i = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2},$$
$$\sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n(n+1)(2n+1)}{6},$$
$$\sum_{i=1}^{n} i^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} = \left(\frac{n(n+1)}{2}\right)^2.$$

In general,

$$\sum_{i=1}^{N-1} i^m = \frac{1}{m+1} \sum_{i=0}^{m} \binom{m+1}{i} B_i N^{m+1-i},$$

where B_i are Bernoulli numbers.

Faulhaber [2, 5] noticed that odd power sums can be represented as a polynomial in $t = n(n+1)/2$. For example,

$$\sum_{i=1}^{n} i^3 = t^2,$$

$$\sum_{i=1}^{n} i^5 = \frac{4t^3 - t^2}{3},$$

$$\sum_{i=1}^{n} i^7 = \frac{12t^4 - 8t^3 + 2t^2}{6}.$$

He computed these sums up to degree 17. The first proof of Faulhaber's theorem was given by Jacobi [4]. The general formula for odd power sums can be written as

$$\sum_{i=1}^{n} i^{2p+1} = \frac{1}{2^{2p+2}(2p+2)} \sum_{i=0}^{p} \binom{2p+2}{2i} (2 - 2^{2i}) B_{2i}((8t+1)^{p+1-i} - 1).$$

Faulhaber knew that odd power sums are divisible by t^2 and even power sums can be expressed in terms of odd power sums. If

$$\sum_{i=1}^{n} i^{2p+1} = c_1 t^2 + c_2 t^3 + \cdots + c_p t^{p+1},$$

then

$$\sum_{i=1}^{n} i^{2p} = \frac{2n+1}{2(2p+1)} (2c_1 t + 3c_2 t^2 + \cdots + (p+1) c_p t^p).$$

In our paper we consider the power sum of binomial coefficients

$$f_{k,m}(N) = \sum_{i=0}^{N-1} \binom{i+k-1}{k}^m,$$

and for $k = 1$ we obtain the usual power sum

$$f_{1,m}(N) = \sum_{i=1}^{N-1} i^m.$$

We establish an analog of Faulhaber's theorem for any positive integer k. Namely, we show that:

- $f_{k,m}(N)$ is a polynomial in N and it can therefore be considered as a polynomial $f_{k,m}(x)$ in any variable x;
- $f_{k,m}(x)$ can be expressed as:

  $\binom{x+k-1}{k+1}^2 Q_{k,m}((2x+k-2)^2)$, if m, k are odd and $m > 1$;

  $\binom{x+k-1}{k+1} (2x+k-2) Q_{k,m}((2x+k-2)^2)$, if k is odd and m is even;

  $\binom{x+k-1}{k+1} Q_{k,m}((2x+k-2)^2)$, otherwise;

  for some polynomials $Q_{k,m}(x)$ with rational coefficients.

For example,

$$f_{3,3}(N) = \sum_{i=0}^{N-1} \binom{i+2}{3}^3 = \binom{N+2}{4}^2 \frac{(2N+1)^2-10}{15},$$

$$f_{3,2}(N) = \sum_{i=0}^{N-1} \binom{i+2}{3}^2 = \binom{N+2}{4}(2N+1)\frac{5(2N+1)^2-41}{420},$$

$$f_{2,2}(N) = \sum_{i=0}^{N-1} \binom{i+1}{2}^2 = \binom{N+1}{3}\frac{3N^2-2}{10}.$$

By Faulhaber's theorem, any odd power sum can be expressed as a combination of powers of triangular numbers

$$\sum_{i=0}^{N} i^{2m-1} = \frac{1}{2m}\sum_{i=0}^{m-1} F_i(m)(N(N+1))^{m-i},$$

and any even power sum can be expressed as

$$\sum_{i=0}^{N} i^{2m} = (N-\frac{1}{2})\sum_{i=0}^{m} \widetilde{F}_i(m)(N(N-1))^{m-i},$$

where $\widetilde{F}_i(m) = \frac{m+1-i}{(2m+1)(m+1)}F_i(m+1)$. [1]

Knuth [5] showed that the coefficients $F_i(m)$ have many interesting properties. Our generalization of Faulhaber's theorem tends to consider the inverse problem: expressing the power sum of triangular numbers $f_{2,m}(N)$ in terms of powers of N. We show that this expression can be presented as a combination of odd powers of N,

$$f_{2,m}(N) = \sum_{i=1}^{N-1} \left(\frac{i(i+1)}{2}\right)^m = \frac{1}{2^m}\sum_{i=0}^{m} \overline{F}_i(m)N^{2m-2i+1}.$$

We find the following duality relations between coefficients $\overline{F}_i^{(m)}$ and $F_i(m), \widetilde{F}_i(m)$:

$$\overline{F}_i(m+i) = (-1)^{i-1}\frac{m+i}{m(2m+1)}F_i(-m),$$

$$\overline{F}_i(m+i-1) = (-1)^i \widetilde{F}_i(-m).$$

Properties of $\overline{F}_i(m)$ which similar to the properties of $F_i(m)$ are also established.

We study integer divisibility properties of $f_{k,m}(N)$ for integer N. We consider an analog of the Riemann zeta function

$$\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{i^s}$$

[1] $F_i(m)$ is referred to the sequences A093556, A093557 and $\widetilde{F}_i(m)$ is referred to the sequences A093558, A093559 in [7].

for binomial coefficients. Let

$$\zeta_k(m) = \sum_{i=1}^{\infty} \binom{i+k-1}{k}^{-m}.$$

We prove that (for positive integers k, m and $l = \lceil m/2 \rceil$)

$$\zeta_k(m) \in \begin{cases} \mathbb{Q} + \mathbb{Q}\zeta(2) + \cdots + \mathbb{Q}\zeta(2l), & \text{if } km \text{ is even;} \\ \mathbb{Q} + \mathbb{Q}\zeta(3) + \cdots + \mathbb{Q}\zeta(2l-1), & \text{if } km \text{ is odd and } km > 1. \end{cases}$$

In case of $k = 2$ we obtain the formula for $\zeta_2(m)$,

$$\frac{\zeta_2(m)}{2^m} = (-1)^{m-1}\binom{2m-1}{m} + (-1)^m 2 \sum_{i=1}^{\lfloor m/2 \rfloor} \binom{2m-2i-1}{m-1}\zeta(2i).$$

The structure of our paper is as follows. In section 2, we consider the sum of products of binomial coefficients, which has combinatorial origin and is used in establishing polynomial properties of $f_{k,m}(x)$. In section 3, an analog of Faulhaber's theorem for the powers of binomial coefficients is proved. In section 4, we study integer properties for $f_{k,m}(x)$ and for $f_{k,-1}$. In section 5, the properties of infinite sum $\zeta_k(m)$ are derived. In section 6, we focus on the partial case $k = 2$ and express the power sum of triangular numbers $f_{2,m}(N)$ as a sum of powers of N.

2 Sum of products of binomial coefficients

2.1 A generalized Worpitzky identity

Let

$$f_{k_1,\ldots,k_m}(N) = \sum_{i=0}^{N-1} \binom{i+k_1-1}{k_1} \cdots \binom{i+k_m-1}{k_m}, \tag{1}$$

where $k_1 \le \cdots \le k_m$.

To study $f_{k_1,\ldots,k_m}$, we need a generalization of Worpitzky identity for multisets. To formulate this identity let us introduce Eulerian numbers for multisets.

Let $\mathbf{m} = 1^{k_1} \cdots m^{k_m}$ be a multiset where l repeats k_l times, $l = 1, \ldots, m$. Let $S_{\mathbf{m}}$ be the set of permutations of $\mathbf{m}$ and set $K = k_1 + \cdots + k_m$.

For a permutation $\sigma = \sigma(1)\ldots\sigma(K) \in S_{\mathbf{m}}$, let i be a *descent* index if $i = K$ or $\sigma(i) > \sigma(i+1), i < K$. A descent number $\operatorname{des}(\sigma)$ is defined as a number of descent indices of σ and Eulerian number $a_{\mathbf{m},p}$ is defined as a number of permutations with p descents,

$$a_{\mathbf{m},p} = |\{\sigma \in S_{\mathbf{m}} | des(\sigma) = p\}|.$$

For $\mathbf{m} = 1^1 2^1 \cdots m^1$, we obtain the usual Eulerian numbers $a_{m,p}$ (A008292 in [7]) and the well-known Worpitzky identity

$$x^m = \sum_{p>0} \binom{x+m-p}{m} a_{m,p}.$$

Example 1. Let $\mathbf{m} = 1^2 2^2$. Then

$$S_{1^2 2^2} = \{1122, 1212, 2112, 2121, 2211, 1221\},$$

and

$$\operatorname{des}(1122) = 1, \operatorname{des}(1212) = 2, \operatorname{des}(2112) = 2, \operatorname{des}(2121) = 3,$$
$$\operatorname{des}(2211) = 2, \operatorname{des}(1221) = 2.$$

Therefore,

$$a_{1^2 2^2,1} = 1,\ a_{1^2 2^2,2} = 4,\ a_{1^2 2^2,3} = 1,\ \text{and}\ a_{1^2 2^2,i} = 0,\ \text{if}\ i \neq 1, 2, 3.$$

Theorem 2 ([1])**.** *For any nonnegative integers* $k_1, \ldots, k_m$*,*

$$\prod_{i=1}^{m} \binom{x + k_i - 1}{k_i} = \sum_{p>0} \binom{x + K - p}{K} a_{\mathbf{m},p},$$

where $a_{\mathbf{m},p}$ *are Eulerian numbers of permutations of the multiset* $\mathbf{m} = 1^{k_1} \cdots m^{k_m}$*.*

Example 3. If $\mathbf{m} = 1^2 2^2$, then

$$\binom{x+1}{2}^2 = \binom{x+3}{4} + 4\binom{x+2}{4} + \binom{x+1}{4}.$$

For a more detailed overview and other properties of Eulerian numbers on multisets and generalized Worpitzky identity, see [1].

2.2 Sum of products of binomial coefficients as a polynomial

Theorem 4. *Let* $0 \leq k_1 \leq \cdots \leq k_m$*. Then the sum* (1) *induces a polynomial* $f_{k_1,\ldots,k_m}(x)$ *of degree* $K + 1$ *with rational coefficients. As a polynomial with rational coefficients, the polynomial* $f_{k_1,\ldots,k_m}(x)$ *is divisible by* $\binom{x+k_m-1}{k_m+1}$*.*

Proof. By Theorem 2,

$$f_{k_1,\ldots,k_m}(N) = \sum_{p>0} \sum_{i=0}^{N-1} \binom{i + K - p}{K} a_{\mathbf{m},p} = \sum_{p>0} \binom{N + K - p}{K + 1} a_{\mathbf{m},p},$$

for any positive integer N. Therefore, we can substitute any variable x in N and see that

$$f_{k_1,\ldots,k_m}(x) \in \mathbb{Q}[x], \qquad \deg\ f_{k_1,\ldots,k_m}(x) = K + 1.$$

If $k_m = 0$, then $k_1 = \cdots = k_m = 0$, and

$$f_{0,\ldots,0}(N) = N - 1.$$

Therefore, $f_{0,\ldots,0}(x) = x - 1$. So, in this case divisibility of $f_{k_1,\ldots,k_m}(x)$ by $\binom{x+k_m-1}{k_m+1}$ is evident.

Suppose now that $k_m > 0$. Let us consider a difference polynomial

$$\Delta f(x) = f_{k_1,\dots,k_m}(x+1) - f_{k_1,\dots,k_m}(x).$$

By (1),

$$\Delta f(x) = \binom{x+k_1-1}{k_1} \cdots \binom{x+k_m-1}{k_m}.$$

Therefore, $\Delta f(x)$ has k_m zeros: $0, -1, \dots, -(k_m-1)$. Hence,

$$\begin{gathered} f_{k_1,\dots,k_m}(1) - f_{k_1,\dots,k_m}(0) = \Delta f(0) = 0, \\ f_{k_1,\dots,k_m}(0) - f_{k_1,\dots,k_m}(-1) = \Delta f(-1) = 0, \\ \dots \\ f_{k_1,\dots,k_m}(-(k_m-2)) - f_{k_1,\dots,k_m}(-(k_m-1)) = \Delta f(-(k_m-1)) = 0. \end{gathered}$$

If $k_m > 0$, then $\binom{i+k_m-1}{k_m} = 0$ for $i = -k_m+1$. Therefore,

$$f_{k_1,\dots,k_m}(-k_m+1) = 0.$$

We thus obtain a polynomial $f_{k_1,\dots,k_m}(x)$ with k_m+1 zeros

$$1, 0, -1, \dots, -(k_m-1).$$

This means that $f_{k_1,\dots,k_m}(x)$ is divisible by $\binom{x+k_m-1}{k_m+1}$. □

Set $f_{k,m}(x) = f_{k,k,\dots,k}(x)$. In other words, $f_{k,m}(x)$ is a polynomial defined by the following relations

$$f_{k,m}(N) = \sum_{i=0}^{N-1} \binom{i+k-1}{k}^m. \tag{2}$$

Corollary 5. *The polynomial $f_{k,m}(x)$ has the following properties*

- $f_{k,m}(x) \in \mathbb{Q}[x]$,
- deg $f_{k,m}(x) = km+1$,
- $f_{k,m}(x)$ *is divisible by* $\binom{x+k-1}{k+1}$.

A more detailed version of this result is given in the next section.

3 Faulhaber's theorem for powers of binomial coefficients

3.1 Formulation of the main result

We know that $f_{1,m}(N) = \sum_{i=1}^{N-1} i^m$ is a polynomial in N of degree $m+1$. By Faulhaber's theorem [2, 5], the polynomial $f_{1,m}(x)$ is divisible by the polynomial $f_{1,1}(x) = x(x-1)/2$.

For odd m, the polynomial $f_{1,m}(x)$ is divisible by $f_{1,1}(x)^2$ and the quotient is a polynomial in $f_{1,1}(x)$. For even m, the polynomial $f_{1,m}(x)$ can be presented as product of $f_{1,1}(x)(2x-1)$ and a polynomial in $f_{1,1}(x)$.

The following theorem is an analog of Faulhaber's theorem for the sum of powers of binomial coefficients.

Theorem 6. *There exist polynomials $Q_{k,m}(x) \in \mathbb{Q}[x]$, such that*

$$f_{k,m}(x) = \begin{cases} \binom{x+k-1}{k+1}^2 Q_{k,m}((2x+k-2)^2), & \text{if } m, k \text{ are odd, } m > 1; \\ \\ \binom{x+k-1}{k+1}(2x+k-2)Q_{k,m}((2x+k-2)^2), & \text{if } k \text{ is odd, } m \text{ is even;} \\ \\ \binom{x+k-1}{k+1}Q_{k,m}((2x+k-2)^2), & \text{otherwise.} \end{cases}$$

Note that $f_{k,1}(x) = \binom{x+k-1}{k+1}$ and our theorem states that the polynomial $f_{k,m}(x)$ is divisible by $f_{k,1}(x)$. Moreover, if m and k are odd $(m > 1)$, then $f_{k,m}(x)$ is divisible by $f_{k,1}(x)^2$.

Let us show Theorem 6 for some small values of k.

If $k = 1$, then there are polynomials $Q_{1,m}(x) \in \mathbb{Q}[x]$, such that the polynomial $f_{1,m}(N) = \sum_{i=1}^{N-1} i^m$ can be expressed as:

$$f_{1,m}(x) = \begin{cases} \binom{x}{2}^2 Q_{1,m}((2x-1)^2), & \text{if } m \text{ is odd and } m > 1; \\ \\ \binom{x}{2}(2x-1)Q_{1,m}((2x-1)^2), & \text{if } m \text{ is even.} \end{cases}$$

Note that the polynomial $Q_{1,m}((2x-1)^2) = Q_{1,m}(8\frac{x(x-1)}{2}+1)$ can be written as a polynomial in $\binom{x}{2} = \frac{x(x-1)}{2}$. Hence, Faulhaber's theorem is a particular case of Theorem 6.

If $k = 2$, then for any $m > 0$, there exists a polynomial $Q_{2,m}(x) \in \mathbb{Q}[x]$, such that

$$f_{2,m}(x) = \binom{x+1}{3} Q_{2,m}(x^2).$$

Since

$$\binom{x+1}{3} = \frac{x(x^2-1)}{6},$$

this implies that $f_{2,m}(x)$ is an odd polynomial.

If $k = 3$, then $f_{3,3}(x) = \binom{x+2}{4}^2 Q_{3,3}(x)$, where

$$Q_{3,3}(x) = \frac{1}{15}(4x^2+4x-9).$$

Then for each positive integer m, $(m > 1)$, there exists a polynomial $Q_{3,m}(x) \in \mathbb{Q}[x]$, such that

$$f_{3,m}(x) = \binom{x+2}{4}^2 Q_{3,m}(Q_{3,3}(x)), \text{ if } m \text{ is odd, and}$$

$$f_{3,m}(x) = \binom{x+2}{4}(2x+1)Q_{3,m}(Q_{3,3}(x)), \text{ if } m \text{ is even.}$$

3.2 Reflective functions

The proof of Theorem 6 is based on the notion of reflective functions introduced by Knuth [5]. The function $f(x)$ is called *r-reflective* if for all x, we have

$$f(x) = f(-x-r);$$

and $f(x)$ is called *anti-r-reflective* if for all x, we have

$$f(x) = -f(-x-r).$$

In other words, reflective functions are even or odd functions shifted by $r/2$.

Note that the

- sum of two (anti)-r-reflective functions is (anti)-r-reflective;
- product of two r-reflective functions is r-reflective;
- product of anti-r-reflective and r-reflective is anti-r-reflective function and
- product of two anti-r-reflective functions is r-reflective.

Lemma 7. *Let $\nabla f(x) = f(x) - f(x-1)$. Suppose that $f(0) = f(-r) = 0$ and the function f is defined on the set of integers. Then the following is true:*

- *if the function ∇f is $(r-1)$-reflective, then f is anti-r-reflective and*
- *if the function ∇f is anti-$(r-1)$-reflective, then f is r-reflective.*

Proof. Suppose that ∇f is $(r-1)$-reflective. Then we have

$$f(N) - f(0) = \sum_{i=1}^{N} \nabla f(i) = \sum_{i=1}^{N} \nabla f(-i-r+1) = f(-r) - f(-N-r),$$

which gives $f(N) = -f(-N-r)$ and this implies that f is anti-r-reflective.

Now if ∇f is anti-$(r-1)$-reflective, then

$$f(N) - f(0) = \sum_{i=1}^{N} \nabla f(i) = -\sum_{i=1}^{N} \nabla f(-i-r+1) = -f(-r) + f(-N-r).$$

So, $f(N) = f(-N-r)$ and f is r-reflective. □

Lemma 8 ([5], Lemma 4)**.** *A polynomial $f(x)$ is r-reflective if and only if it can be presented as a polynomial in $x(x+r)$ (or $(2x+r)^2$); it is anti-r-reflective if and only if it can be presented as $2x+r$ times a polynomial in $x(x+r)$ (or $(2x+r)^2$).*

Lemma 9. *The polynomial $\binom{x+k-1}{k}$ is $(k-1)$-reflective if k is even and anti-$(k-1)$-reflective if k is odd.*

Proof. The proof follows from the identity $\binom{x+k-1}{k} = (-1)^k\binom{-x}{k}$. □

By Theorem 4, there exist polynomials $g_{k,m}(x) \in \mathbb{Q}[x]$, such that

$$f_{k,m}(x) = \binom{x+k-1}{k+1} g_{k,m}(x).$$

Lemma 10. *Reflective properties of functions $f_{k,m}$ and $g_{k,m}$ are*

- *$f_{k,m}(x)$ is $(k-2)$-reflective if $km+1$ is even and anti-$(k-2)$-reflective if $km+1$ is odd.*
- *$g_{k,m}(x)$ is $(k-2)$-reflective if $(m-1)k$ is even and anti-$(k-2)$-reflective if $(m-1)k$ is odd.*

Proof. Let $\nabla f = f(x) - f(x-1)$. Since $\nabla f_{k,m}(x) = \binom{x+k-2}{k}^m$, we see that $\nabla f_{k,m}(x)$ is $(k-3)$-reflective if km is even and anti-$(k-3)$-reflective, otherwise. By Theorem 4, $f(0) = f(-(k-2)) = 0$. Thus, by Lemma 7, $f_{k,m}(x)$ is anti-$(k-2)$-reflective if km is even and $(k-2)$-reflective, otherwise.

Note that $g_{k,m} = f_{k,m}/\binom{x+k-1}{k+1}$. Therefore by Lemma 9, $g_{k,m}(x)$ is $(k-2)$-reflective if $km-k$ is even and anti-$(k-2)$-reflective, otherwise. □

Lemma 11. *Let k be an odd number. Then $f_{k,m}(x)$ is divisible by $\binom{x+k-1}{k+1}(2x+k-2)$ if m is even and is divisible by $\binom{x+k-1}{k+1}^2$ if m is odd and $m > 1$.*

Proof. Suppose that m is even. By Lemma 10, the function $g_{k,m}$ is anti-$(k-2)$-reflective. The anti-$(k-2)$-reflectivity condition for $x = \frac{2-k}{2}$ gives us

$$g_{k,m}(-\frac{2-k}{2} - k + 2) = g_{k,m}(\frac{2-k}{2}) = -g_{k,m}(\frac{2-k}{2}).$$

Hence, $g_{k,m}(\frac{2-k}{2}) = 0$ and $g(x)$ is divisible by $(2x+k-2)$.

Now consider the case m is odd ($m > 1$). By Lemma 10, the function $g_{k,m}$ is $(k-2)$-reflective. We have

$$f_{k,m}(x+1) - f_{k,m}(x) = \binom{x+k}{k+1} g_{k,m}(x+1) - \binom{x+k-1}{k+1} g_{k,m}(x) = \binom{x+k-1}{k}^m.$$

Hence,

$$(x+k)g_{k,m}(x+1) - (x-1)g_{k,m}(x) = (k+1)\binom{x+k-1}{k}^{m-1}.$$

Therefore, for $i = 0, -1, \ldots, -(k-1)$, we obtain

$$(k+i)g_{k,m}(i+1) - (i-1)g_{k,m}(i) = 0.$$

In other words,

$$\begin{aligned} kg_{k,m}(1) &= -g_{k,m}(0), \\ (k-1)g_{k,m}(0) &= -2g_{k,m}(-1), \\ (k-2)g_{k,m}(-1) &= -3g_{k,m}(-2), \\ &\vdots \\ g_{k,m}(-(k-2)) &= -kg_{k,m}(-(k-1)). \end{aligned}$$

Hence,
$$g_{k,m}(1) = (-1)^k g_{k,m}(-(k-1)) = -g_{k,m}(-(k-1)).$$
Since $g_{k,m}(x)$ is $(k-2)$-reflective, the reflectivity condition for $x = 1$ gives us
$$g_{k,m}(1) = g_{k,m}(-(k-1)).$$
So, $g_{k,m}(1) = 0$ and
$$g_{k,m}(-(k-1)) = \cdots = g_{k,m}(-1) = g_{k,m}(0) = g_{k,m}(1) = 0.$$
Thus, $g_{k,m}(x)$ is divisible by $\binom{x+k-1}{k+1}$ and $f_{k,m}(x)$ is divisible by $\binom{x+k-1}{k+1}^2$. □

3.3 Proof of Theorem 6.

Let m and k be two odd positive integers and $m > 1$. By Lemma 11, there exist polynomials $R_{k,m}(x)$ such that $g_{k,m}(x) = \binom{x+k-1}{k+1} R_{k,m}(x)$. The function $g_{k,m}(x)$ is $(k-2)$-reflective and therefore, $R_{k,m}(x)$ (according to parity of k) is also $(k-2)$-reflective. So, by Lemma 8, there exist polynomials $Q_{k,m}(x) \in \mathbb{Q}[x]$ such that $R_{k,m}(x) = Q_{k,m}((2x+k-2)^2)$. In this case
$$f_{k,m}(x) = \binom{x+k-1}{k+1}^2 Q_{k,m}((2x+k-2)^2).$$

Now assume that k is odd and m is even. Then the function $g_{k,m}$ is anti-$(k-2)$-reflective. By Lemma 8, there exist polynomials $Q_{k,m}(x) \in \mathbb{Q}[x]$, such that $g_{k,m}(x) = (2x+k-2)Q_{k,m}((2x+k-2)^2)$. Therefore,
$$f_{k,m}(x) = \binom{x+k-1}{k+1}(2x+k-2)Q_{k,m}((2x+k-2)^2).$$
In all other cases $g_{k,m}$ is $(k-2)$-reflective. By Lemma 8, there exist polynomials $Q_{k,m}(x) \in \mathbb{Q}[x]$ such that $g_{k,m}(x) = Q_{k,m}((2x+k-2)^2)$. We have
$$f_{k,m}(x) = \binom{x+k-1}{k+1} Q_{k,m}((2x+k-2)^2).$$

□

4 Integer divisibility for $f_{k,m}(x)$

4.1 Formulation of the main result

By Theorem 6, the polynomial $f_{k,m}(x)$ is divisible by $\binom{x+k-1}{k+1}^2$ if m and k are odd $(m > 1)$. In particular, $f_{k,m}(x)$ is divisible by x^2. By Theorem 6, $f_{k,m}(x)$ is divisible by x for any k, m. Here divisibility refers to the divisibility of polynomials with rational coefficients. Now in this section, we study divisibility properties of $f_{k,m}(x)$ for integer x.

The divisibility properties of Theorem 6 do not hold for integers. Let us give counter-examples for all three cases:

$$\frac{f_{3,5}(7)}{7^2} = \frac{1}{7^2}\sum_{i=1}^{6}\binom{i+2}{3}^5 = \frac{86650668}{7} \notin \mathbb{Z},$$
$$\frac{f_{3,2}(5)}{5} = \frac{1}{5}\sum_{i=1}^{4}\binom{i+2}{3}^2 = \frac{517}{5} \notin \mathbb{Z},$$
$$\frac{f_{4,4}(11)}{11} = \frac{1}{11}\sum_{i=1}^{10}\binom{i+3}{4}^4 = \frac{335469880502}{11} \notin \mathbb{Z}.$$

Since $f_{k,m}(x) \in \mathbb{Q}[x]$, for any given k and m, there is a sufficiently large prime p, such that $f_{k,m}(p)$ is divisible by p and by p^2 if m, k are odd and $m > 1$. The following theorem is a more detailed version of these statements.

Theorem 12. *Let N, k, m be positive integers. Let M be an odd positive integer such that* $\gcd(M, k!) = 1$ *and $M - k + 1 \le N \le M + 1$. Then*

$$f_{k,m}(N) = \sum_{i=0}^{N-1}\binom{i+k-1}{k}^m \equiv 0 \pmod{M}$$

in the following cases

- *m, k are odd numbers, or*
- $\gcd(M, (km+1)!) = 1$ *(k, m may be even or odd).*

If m, k are odd numbers and $m > 1$, then

$$f_{k,m}(N) = \sum_{i=0}^{N-1}\binom{i+k-1}{k}^m \equiv 0 \pmod{M^2}$$

in the following cases

- *m is divisible by M;*
- $\gcd(M, (km+1)!) = 1$*.*

Remark 13. Theorem 12 do not present all conditions for divisibility of $f_{k,m}(N)$ by M and M^2. For example, if $M = 13, k = 4, m = 14, N = 13$, then

$$f_{4,14}(13) = \sum_{i=0}^{12}\binom{i+3}{4}^{14} = 13 \times 607543306976263500399956734407945023727885 6,$$

but $4, 14$ are even and $\gcd(13, 57!) \ne 1$. Similarly, if $M = 7, k = 3, m = 9, N = 7$, then

$$f_{3,9}(7) = \sum_{i=0}^{6}\binom{i+2}{3}^9 = 7^2 \times 112153022185284,$$

but 9 is not divisible by 7 and $\gcd(7, 28!) \ne 1$.

4.2 Proof

To prove Theorem 12 we need some preliminary facts.

Lemma 14. *Let M be an odd positive integer number such that* $\gcd(M,k!)=1$. *Then for odd numbers k,m and for all integers i such that $1\le i\le M-k$, the following relation holds*

$$\binom{M-i+k-1}{k}^m \equiv -\binom{i+k-1}{k}^m + Mm\binom{i+k-1}{k}^m\sum_{j=0}^{k-1}\frac{1}{i+j} \pmod{M^2}.$$

Proof. Let us consider an expression $\binom{M-i+k-1}{k}$ as a polynomial in M,

$$\begin{aligned}\binom{M-i+k-1}{k} &= \frac{(M-i+k-1)\cdots(M-i)}{k!}\\ &= a_kM^k+\cdots+a_1M+a_0\\ &= \frac{M^k}{k!}+\cdots+M(-1)^{k-1}\binom{i+k-1}{k}\sum_{j=0}^{k-1}\frac{1}{i+j}+(-1)^k\binom{i+k-1}{k}.\end{aligned}$$

Note that for $0\le j\le k-1$, all numbers $i+j$ are relatively prime with M. Therefore,

$$\binom{M-i+k-1}{k}\equiv(-1)^k\binom{i+k-1}{k}+(-1)^{k-1}M\binom{i+k-1}{k}\sum_{j=0}^{k-1}\frac{1}{i+j} \pmod{M^2}.$$

Hence,

$$\binom{M-i+k-1}{k}^m\equiv\left(-\binom{i+k-1}{k}+M\binom{i+k-1}{k}\sum_{j=0}^{k-1}\frac{1}{i+j}\right)^m \pmod{M^2}$$

On expanding the right side of this congruence, we obtain the result. □

Lemma 15. $(km+1)!f_{k,m}(x)\in\mathbb{Z}[x]$.

Proof. By Theorem 2, there are some integers a_j for which we can present $f_{k,m}(x)$ in the form $\sum_{j>0}a_j\binom{x+km-j}{km+1}$. Note that $(km+1)!\binom{x+km-j}{km+1}\in\mathbb{Z}[x]$. □

Proof of Theorem 12. For $0\le i\le M-k$ with odd k, by Lemma 14,

$$\binom{M-i+k-1}{k}^m+\binom{i+k-1}{k}^m\equiv 0 \pmod{M}.$$

Therefore,

$$\begin{aligned}\sum_{i=0}^{N-1}\binom{i+k-1}{k}^m &\equiv \sum_{i=1}^{M-k}\binom{i+k-1}{k}^m\\ &= \frac{1}{2}\sum_{i=1}^{M-k}\left(\binom{M-i+k-1}{k}^m+\binom{i+k-1}{k}^m\right)\equiv 0 \pmod{M}.\end{aligned}$$

Now suppose that m is divisible by N. Hence, by Lemma 14, we have

$$\begin{aligned}\binom{M-i+k-1}{k}^m &\equiv -\binom{i+k-1}{k}^m + Mm\binom{i+k-1}{k}^m \sum_{j=0}^{k-1}\frac{1}{i+j}\\ &\equiv -\binom{i+k-1}{k}^m \pmod{M^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{i=0}^{N-1}\binom{i+k-1}{k}^m &\equiv \sum_{i=1}^{M-k}\binom{i+k-1}{k}^m\\ &= \frac{1}{2}\sum_{i=1}^{M-k}\left(\binom{M-i+k-1}{k}^m + \binom{i+k-1}{k}^m\right) \equiv 0 \pmod{M^2}.\end{aligned}$$

Since $\gcd(M,(km+1)!) = 1$, by Lemma 15 and Theorem 6, $(km+1)!f_{k,m}(x) \in \mathbb{Z}[x]$ is divisible by $x(x+1)\cdots(x+k-1)$ and by $(x(x+1)\cdots(x+k-1))^2$ if m,k are odd numbers and $m>1$. Here divisibility refers to the divisibility of polynomials with integer coefficients. □

Remark 16. In Theorem 12 we can change M to a power of some prime number p. The property $\gcd(M,(km+1)!) = 1$ can be changed by an inequality $p > km+1$.

4.3 Integer divisibility in case $k = 2$

Theorem 17. *Assume that p is an odd prime number and $1 \le m \le p-1$. Then*

$$f_{2,m}(p) = \sum_{i=1}^{p-1}\binom{i+1}{2}^m \equiv \begin{cases} -2 \pmod p, & \text{if } m = p-1;\\ -\frac{1}{2^m}\binom{m}{p-1-m} \pmod p, & \text{otherwise.}\end{cases}$$

Proof. The following fact is known:

$$\sum_{i=1}^{p-1} i^t \equiv \begin{cases} 0 \pmod p, & \text{if } t \text{ is not divisible by } p-1;\\ -1 \pmod p, & \text{otherwise.}\end{cases}$$

Therefore,

$$f_{2,m}(p) = \sum_{i=1}^{p-1}\binom{i+1}{2}^m = \frac{1}{2^m}\sum_{i=1}^{p-1}\sum_{j=0}^{m}\binom{m}{j} i^{m+j} = \frac{1}{2^m}\sum_{j=0}^{m}\binom{m}{j}\sum_{i=1}^{p-1} i^{m+j}$$

If $m = p-1$, then $m+j$ is divisible by $p-1$ only in two cases: $j = 0, m$. Hence,

$$f_{2,p-1}(p) \equiv -\frac{1}{2^{p-1}}\left(\binom{p-1}{0} + \binom{p-1}{p-1}\right) \equiv -2 \pmod p.$$

If $1 \leq m < p-1$, there is only one integer $j \in [0, m]$ such that $m+j$ is divisible by $p-1$. Namely, $j = p-1-m$. In this case,

$$f_{2,m}(p) \equiv -\frac{1}{2^m}\binom{m}{p-1-m} \pmod{p}.$$

□

Remark 18. By Theorem 17, if $p-1-m > m$, then $f_{2,m}(p) \equiv 0 \pmod{p}$. This fact is compatible with Theorem 12, because $p > 2m+1$.

4.4 The case $f_{k,-1}(N)$

Theorem 19. *Let N, k be positive integer numbers and M be a positive integer such that* $\gcd(M, k!) = 1$. *Then, the rational number q defined as*

$$q = f_{(k,-1)}(N) = \sum_{i=1}^{N-1} \frac{1}{\binom{i+k-1}{k}},$$

is divisible by M (its denominator is relatively prime with M) in the following cases:
(i) $N \equiv 1 \pmod{M}$*;*
(ii) $N \equiv 1-k \pmod{M}$ *and k is odd.*

To prove this result we need one

Lemma 20. *Suppose that $k > 1$. Then*

$$f_{k,-1}(N) = \sum_{i=1}^{N-1} \frac{1}{\binom{i+k-1}{k}} = \frac{k}{k-1}\left(1 - \frac{1}{\binom{N+k-2}{k-1}}\right).$$

Proof. Let $\phi(x) = \frac{1}{\binom{x+k-1}{k-1}}$. Then it is easy to verify that

$$\frac{1}{\binom{i+k-1}{k}} = \frac{k}{k-1}\left(\frac{1}{\binom{i+k-2}{k-1}} - \frac{1}{\binom{i+k-1}{k-1}}\right) = -\frac{k}{k-1}\nabla\phi(i).$$

Therefore,

$$\begin{aligned} f_{(k,-1)}(N) &= \sum_{i=1}^{N-1} \frac{1}{\binom{i+k-1}{k}} = -\frac{k}{k-1}\sum_{i=1}^{N-1} \nabla\phi(i) \\ &= \frac{k}{k-1}(\phi(0) - \phi(N-1)) = \frac{k}{k-1}\left(1 - \frac{1}{\binom{N+k-2}{k-1}}\right). \end{aligned}$$

□

Proof of Theorem 19. By Lemma 20,

$$f_{k,-1}(N) = \frac{k!}{k-1}\left(\frac{1}{(k-1)!} - \frac{1}{(N+k-2)\cdots N}\right).$$

Notice that for both cases (i) and (ii), the numbers $N, \ldots, N+k-2$ are relatively prime with M. So, if $N \equiv 1 \pmod{M}$, then

$$(N+k-2)\cdots N \equiv (k-1)! \pmod{M}.$$

If $N \equiv 1-k \pmod{M}$ and k is odd, then

$$(N+k-2)\cdots N \equiv (-1)^{k-1}(k-1)! \equiv (k-1)! \pmod{M}.$$

□

5 Power sum of reciprocals of binomial coefficients

5.1 Formulation of the main result

In this section, we consider the case of power sums of binomial coefficients with negative powers,

$$\zeta_k(m) = \sum_{i=1}^{\infty} \binom{i+k-1}{k}^{-m}.$$

For $k = 1$ we have

$$\zeta_1(m) = \zeta(m) = \sum_{i=1}^{\infty} \frac{1}{i^m},$$

where $\zeta(m)$ is a Riemann zeta function.

In particular, for $m = 1$, by Lemma 20, one can obtain the exact value

$$\zeta_k(1) = \sum_{i=1}^{\infty} \binom{i+k-1}{k}^{-1} = \frac{k}{k-1}.$$

It is known that for any positive integer m,

$$\zeta(2m) = (-1)^{m+1}\frac{(2\pi)^{2m}}{2(2m)!}B_{2m}, \tag{3}$$

where B_{2m} is a Bernoulli number.

We prove similar results for binomial coefficients. Some examples that follow from our results:

$$\zeta_2(2) = \sum_{i=1}^{\infty} \binom{i+1}{2}^{-2} = \frac{4}{3}\pi^2 - 12,$$

$$\zeta_2(3) = \sum_{i=1}^{\infty} \binom{i+1}{2}^{-3} = -8\pi^2 + 80,$$

$$\zeta_3(2) = \sum_{i=1}^{\infty} \binom{i+2}{3}^{-2} = 9\pi^2 - \frac{351}{4}.$$

Similarly,

$$\zeta_3(3) = \sum_{i=1}^{\infty} \binom{i+2}{3}^{-3} = \frac{783}{4} - 162\zeta(3),$$
$$\zeta_5(3) = \sum_{i=1}^{\infty} \binom{i+4}{5}^{-3} = -\frac{1298125}{96} + 11250\zeta(3),$$
$$\zeta_3(5) = \sum_{i=1}^{\infty} \binom{i+2}{3}^{-5} = \frac{576639}{16} - \frac{47385}{2}\zeta(3) - 7290\zeta(5).$$

Below we show that these relations are based on reflectivity properties of binomial coefficients, and that $\zeta_k(m)$ can be expressed as a linear combination of certain values of the Riemann zeta function.

Theorem 21. *For any positive integers k and m, there exist rationals $\lambda_0, \lambda_1, \dots, \lambda_{\lceil m/2 \rceil}$, such that*

$$\zeta_k(m) = \begin{cases} \lambda_0 + \sum_{i=1}^{\lceil m/2 \rceil} \lambda_i \zeta(2i), & \text{if } km \text{ is even;} \\ \lambda_0 + \sum_{i=1}^{\lceil m/2 \rceil} \lambda_i \zeta(2i-1), & \text{if } km \text{ is odd.} \end{cases}$$

5.2 Proof of Theorem 21

Let $F(x) = \binom{x+k-1}{k}^{-m}$. Then $\zeta_k(m) = \sum_{i=1}^{\infty} F(i)$. Since the polynomial $F(x)$ has k zeros $x = 0, 1, \dots, k-1$,

$$F(x) = \frac{1}{\binom{x+k-1}{k}^m} = \sum_{j=0}^{k-1} \sum_{i=1}^{m} \frac{a_{i,j}}{(x+j)^i} = \frac{1}{2} \sum_{j=0}^{k-1} \sum_{i=1}^{m} \left(\frac{a_{i,j}}{(x+j)^i} + \frac{a_{i,k-1-j}}{(x+k-1-j)^i} \right) \tag{4}$$

for some rational numbers $a_{i,j}$ $(1 \le i \le m, 0 \le j \le k-1)$.

By Lemma 9, $1/F(x)$ is $(k-1)$-reflective if km is even, and $1/F(x)$ is anti-$(k-1)$-reflective, if km is odd. In other words, $F(x) = eF(-x-k+1)$ for almost all x (except zeros of denominator), where $e = \pm 1$. Thus, by (4) and based on the reflectivity property mentioned above,

$$\sum_{j=0}^{k-1} \sum_{i=1}^{m} \left(\frac{a_{i,j}}{(x+j)^i} + \frac{a_{i,k-1-j}}{(x+k-1-j)^i} \right) = e \quad \sum_{j=0}^{k-1} \sum_{i=1}^{m} \left(\frac{a_{i,j}}{(-x-k+1+j)^i} + \frac{a_{i,k-1-j}}{(-x-j)^i} \right)$$

or

$$\sum_{j=0}^{k-1} \sum_{i=1}^{m} \left(\frac{a_{i,j} - e(-1)^i a_{i,k-1-j}}{(x+j)^i} + \frac{a_{i,k-1-j} - e(-1)^i a_{i,j}}{(x+k-1-j)^i} \right) = 0.$$

Therefore,

$$a_{i,j} - e(-1)^i a_{i,k-1-j} = 0$$

for all i, j $(1 \le i \le m, 0 \le j \le k-1)$. Hence, equation (4) can be rewritten as

$$F(x) = \frac{1}{\binom{x+k-1}{k}^m} = \frac{1}{2}\sum_{j=0}^{k-1}\sum_{i=1}^{m}\left(\frac{a_{i,j}}{(x+j)^i} + e(-1)^i \frac{a_{i,j}}{(x+k-1-j)^i}\right).$$

Finally,

$$\begin{aligned}\zeta_k(m) &= \sum_{x=1}^{\infty} F(x)\\ &= \sum_{x=1}^{\infty}\frac{1}{2}\sum_{j=0}^{k-1}\sum_{i=1}^{m}\left(\frac{a_{i,j}}{(x+j)^i} + e(-1)^i\frac{a_{i,j}}{(x+k-1-j)^i}\right)\\ &= \frac{1}{2}\sum_{j=0}^{k-1}\sum_{i=1}^{m} a_{i,j}\left(\zeta(i) - \sum_{l=1}^{j}\frac{1}{l^i} + e(-1)^i\zeta(i) - e(-1)^i\sum_{l=1}^{k-1-j}\frac{1}{l^i}\right)\\ &= c_0 + \sum_{i=1}^{m} c_i(\zeta(i) + e(-1)^i\zeta(i)),\end{aligned}$$

for some rational constants c_i $(0 \le i \le m)$. Note that value $(\zeta(i) + e(-1)^i\zeta(i))$ vanishes if $e = 1$ and i is odd, or if $e = -1$ and i is even. □

Presentations of infinite series as a linear combination of odd (even) values of zeta functions play an important role in studying the irrationality problems of zeta functions. See, for example, the proof of Rivoal [6] that the sequence $\zeta(3), \zeta(5), \ldots$ contains infinitely many irrational values, or result of Zudilin [12] that at least one of four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.

Different kinds of sums of inverses of binomial coefficients were studied in many works, e.g., the sums of other types by Sofo [8], Sprugnoli [9], Sury, Wang and Zhao [10], Yang and Zhao [11].

5.3 Sums of reciprocals of powers of triangular numbers

Recall that

$$\zeta_2(m) = \sum_{x=1}^{\infty}\frac{2^m}{(x(x+1))^m}.$$

In this subsection we give exact presentation of $\zeta_2(m)$ as a combination of binomial coefficients and Bernoulli numbers. Namely, we prove the following result.

Theorem 22.

$$\sum_{x=1}^{\infty}\frac{1}{(x(x+1))^m} = (-1)^{m-1}\binom{2m-1}{m} + (-1)^m\sum_{i=1}^{\lfloor m/2\rfloor}\binom{2m-2i-1}{m-1}(-1)^{i+1}\frac{(2\pi)^{2i}}{(2i)!}B_{2i}.$$

To prove this theorem we need the following

Lemma 23.

$$\frac{1}{\binom{x+1}{2}^m} = 2^m \sum_{i=0}^{m-1} \binom{m+i-1}{i} \left((-1)^i \frac{1}{x^{m-i}} + (-1)^m \frac{1}{(x+1)^{m-i}} \right).$$

Proof. We use induction on m. For $m = 1$ we have

$$\frac{1}{\binom{x+1}{2}} = 2\left(\frac{1}{x} - \frac{1}{x+1}\right).$$

Assume that our statement holds for m and let us prove it for $m+1$. In our proof we need the following formulas (j is a positive integer):

$$\begin{aligned} \frac{1}{x^j(x+1)} &= \frac{1}{x^j} - \frac{1}{x^{j-1}} + \cdots + (-1)^j \frac{1}{x+1} \text{ and} \\ \frac{1}{x(x+1)^j} &= \frac{1}{x} - \frac{1}{x+1} - \cdots - \frac{1}{(x+1)^j}, \end{aligned}$$

which follow from:

$$\begin{aligned} 1-(-x)^j &= (1+x)(1+(-x)+\cdots+(-x)^{j-1}) \text{ and} \\ 1-(x+1)^j &= -x(1+(x+1)+\cdots+(x+1)^{j-1}). \end{aligned}$$

We have

$$\begin{aligned} \frac{1}{(x(x+1))^{m+1}} &= \frac{1}{(x(x+1))^m}\left(\frac{1}{x} - \frac{1}{x+1}\right) \\ &= \sum_{i=0}^{m-1} \binom{m+i-1}{i} \left(\frac{(-1)^i}{x^{m-i}} + \frac{(-1)^m}{(x+1)^{m-i}}\right)\left(\frac{1}{x} - \frac{1}{x+1}\right) \\ &= \sum_{i=0}^{m} \binom{m+i}{i} \left(\frac{(-1)^i}{x^{m-i+1}} + \frac{(-1)^{m+1}}{(x+1)^{m-i+1}}\right) \end{aligned}$$

(since $\sum_{j=0}^{i} \binom{m+j-1}{j} = \binom{m+i}{i}$). □

Proof of Theorem 22. By Lemma 23,

$$\begin{aligned} \sum_{x=1}^{\infty} \frac{1}{(x(x+1))^m} &= \sum_{x=1}^{\infty} \sum_{i=0}^{m-1} \binom{m+i-1}{i} \left(\frac{(-1)^i}{x^{m-i}} + \frac{(-1)^m}{(x+1)^{m-i}}\right) \\ &= \sum_{i=0}^{m-1} \binom{m+i-1}{i} \left(\sum_{x=1}^{\infty} \frac{(-1)^i}{x^{m-i}} + \sum_{x=1}^{\infty} \frac{(-1)^m}{(x+1)^{m-i}}\right). \end{aligned}$$

Thus, the last expression can be written as

$$\begin{aligned}
&\sum_{i=0}^{m-1}\binom{m+i-1}{i}\big((-1)^i\zeta(m-i)+(-1)^m(\zeta(m-i)-1)\big)\\
&=(-1)^{m-1}\sum_{i=0}^{m-1}\binom{m+i-1}{i}+\sum_{i=0,(m-i)\text{ even}}^{m-1}\binom{m+i-1}{i}(-1)^i 2\zeta(m-i)\\
&=(-1)^{m-1}\binom{2m-1}{m}+(-1)^m 2\sum_{i=1}^{\lfloor m/2\rfloor}\binom{2m-2i-1}{m-1}\zeta(2i)\\
&=(-1)^{m-1}\binom{2m-1}{m}+(-1)^m\sum_{i=1}^{\lfloor m/2\rfloor}\binom{2m-2i-1}{m-1}(-1)^{i+1}\frac{(2\pi)^{2i}}{(2i)!}B_{2i}.
\end{aligned}$$

□

6 Faulhaber coefficients and coefficients of the polynomial $f_{2,m}(x)$

6.1 Duality between F- and $\overline{F}$-Faulhaber coefficients

Recall that any number in the form $\binom{n}{2}=\frac{n(n-1)}{2}$ is called triangular (A000217 in [7]). In this section we study analogs of Faulhaber coefficients for power sums of triangular numbers. To stress similarities between Faulhaber coefficients $F_i(m)$ and $\overline{F}_i(m)$ (see definitions below), we use special notation for formulas like (A1), (B1), (A2), (B2), etc.

We define Faulhaber coefficients [3, 5] as numbers $F_i(m)$, such that

$$\sum_{i=0}^{N} i^{2m-1}=\frac{1}{2m}\sum_{i=0}^{m-1}F_i(m)(N(N+1))^{m-i}. \tag{A1}$$

These coefficients can be extended for all real numbers x by $F_i(x)$. In fact, $F_i(x)$ is a polynomial in x. In Sloane's OEIS [7], $F_i(m)$ is referred to the sequences A093556, A093557 and

$$\widetilde{F}_i(m)=\frac{m+1-i}{(2m+1)(m+1)}F_i(m+1) \tag{5}$$

is referred to the sequences A093558, A093559.

Knuth [5] has established the following properties of these coefficients,

$$F_0(x)=1;\quad \sum_{j=0}^{k}\binom{x-j}{2k+1-2j}F_j(x)=0,\quad k>0. \tag{A2}$$

$$F_k(x)=x(x-1)\cdots(x-k+1)\times q_k(x), \tag{A3}$$

where $q_k(x)$ is

- a polynomial of degree k
- with leading coefficient equal to $(2-2^{2k})B_{2k}/(2k)!$;
- and $q_k(k+1)=0$ if $k>0$.

Below we show that all zeros of this polynomial are real and distinct.

A recurrence formula due to Jacobi yields:

$$(2x-2k)(2x-2k+1)F_k(x)+(x-k+1)(x-k)F_{k-1}(x)=2x(2x-1)F_k(x-1). \quad \text{(A4)}$$

Gessel and Viennot [3] obtained the following explicit formula:

$$F_k(m)=(-1)^{m-k}\sum_j \binom{2m}{m-k-j}\binom{m-k+j}{j}\frac{m-k-j}{m-k+j}B_{m+k+j}, \quad \text{(A5)}$$

for $0\le k<m$.

In terms of determinants, this formula can be written as

$$F_k(m)=(-1)^{(m-k)}\frac{1}{(m-1)\cdots(m-k)}\det\left|\binom{m-k+i}{2i-2j+3}\right|_{i,j=1,\dots,k}. \quad \text{(A6)}$$

The last determinant has a combinatorial interpretation.

Theorem 24 (Gessel, Viennot, Theorem 14, [3])**.** *The number of sequences of positive integer numbers* $a_1,\dots,a_{3k}$ *satisfying inequalities* $a_{3i-2}<a_{3i-1}<a_{3i}$*,* $a_{3i+1}\le a_{3i-1}$*,* $a_{3i+2}\le a_{3i}$ *and* $a_{3i}\le m-k+i$*, for all* i*, is equal to*

$$\det\left|\binom{m-k+i}{2i-2j+3}\right|_{i,j=1,\dots,k}.$$

According to Theorem 6, the polynomial $f_{2,m}(x)$ is odd. Therefore we can consider the coefficients $\overline{F}_i(m)$ defined by

$$f_{2,m}(N)2^m=\sum_{i=0}^{N-1}(i(i+1))^m=\sum_{i=0}^{m}\overline{F}_i(m)N^{2m-2i+1}. \quad \text{(B1)}$$

The following relations are $\overline{F}_i^{(m)}$ analogs of relations (A2)–(A6),

$$\overline{F}_0(m)=1;\quad \sum_{j=0}^{k}\binom{m+2k-j}{2k+1-2j}(-1)^j\overline{F}_j(m+j)=0. \quad \text{(B2)}$$

$$\overline{F}_i(m)=\frac{m(m-1)\cdots(m-i)}{2m-2i+1}\times h_i(m), \quad \text{(B3)}$$

where $h_i(m)$ is

- a polynomial of degree i

- with leading coefficient $(-1)^{i-1}(2-2^{2i})B_{2i}/(2i)!$;
- and $h_i(-1) = 0$ if $i > 0$.

Proof of these properties will be presented in Theorem 29 below. In fact, Theorem 25 tells us that $h_i(m) = q_i(i-m)$.

A recurrence formula for $\overline{F}_i(m)$ is given by

$$2(m-i)(2m-2i+1)\overline{F}_i(m) = 2m(2m-1)\overline{F}_i(m-1) + m(m-1)\overline{F}_{i-1}(m-2). \quad \text{(B4)}$$

Coefficients $\overline{F}_i(m)$ are closely related to $F_i(m)$.

Theorem 25.

$$\overline{F}_i(m+i) = (-1)^{i-1}\frac{m+i}{m(2m+1)}F_i(-m) \quad (6)$$

Another formulation of Theorem 25 is given by

$$\overline{F}_i(m) = (-1)^{i-1}\frac{m}{(m-i)(2m-2i+1)}F_i(i-m).$$

Proof. From relations (A4) and (B4), we can see that the sequences $\overline{F}_i(m+i)$ and $F_i(-m)$ satisfy the same recurrence relations and they have equal initial values $F_0(x) = \overline{F}_0(x) = 1$. □

Knuth [5] proved that

$$\sum_{i=0}^{N-1} i^{2m} = (N - \frac{1}{2})\sum_{i=0}^{m} \widetilde{F}_i(m)(N(N-1))^{m-i},$$

with relation (5) between the coefficients $\widetilde{F}_i(m)$ and $F_i(m)$.

Corollary 26.

$$\overline{F}_i(m+i-1) = (-1)^i \widetilde{F}_i(-m). \quad (7)$$

The following is a general formula for $\overline{F}_i(m)$

$$\overline{F}_i(m) = \frac{1}{(2m-2i+1)}\sum_{k=0}^{2i}\binom{m}{2i-k}\binom{2m-2i+k}{k}B_k. \quad \text{(B5)}$$

Let us rewrite this formula in terms of determinants

$$\overline{F}_k(m) = \frac{(-1)^{(k-m+1)}(k-1)!}{(2m-2k+1)(m-1)!}\det\left|\binom{m+i-2j+2}{2i-2j+3}\right|_{i,j=1,\dots,k}, \quad \text{(B6)}$$

which follows from (A6) and Theorem 25.

Using [3, Theorem 16], we obtain one more combinatorial interpretation for the last determinant.

Theorem 27. *The number of sequences of positive integers* $a_1, \ldots, a_{3k}$ *such that* $a_{3i-2} \geq a_{3i-1} \geq a_{3i}$, $a_{3i+1} \geq a_{3i-1}$, $a_{3i+2} \geq a_{3i}$ *and* $a_{3i-2} \leq m - i$, *for all* i, *is equal to*

$$\det \left| \binom{m+i-2j+2}{2i-2j+3} \right|_{i,j=1,\ldots,k}.$$

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_k)$ be two non-increasing sequences of non-negative integers. *Plane partition of shape* $\lambda - \mu$ is a filling of the corresponding diagram with integers which are weakly decreased in every row and column. Such a diagram can be obtained from Ferrer diagram for λ by removing the diagram for μ (for details, see [3]) .

By [3, Theorem 16], the number of plane partitions of shape $\lambda - \mu$ whose parts in row i are at most A_i and at least B_i, where $A_i \geq A_{i+1} - 1$ and $B_i \geq B_{i+1} - 1$, is equal to

$$\det \left| \binom{A_j - B_i + \lambda_i - \mu_j}{\lambda_i - \mu_j + j - i} \right|_{i,j=1,\ldots,k}.$$

Taking $\lambda_i = k + 3 - i$, $\mu_j = k - j$, $A_j = m - j$ and $B_i = 1$, we obtain the result. □

6.2 The polynomial $f_{2,m}(x)$

Set $\lambda_{m,i} = \overline{F}_i(m)/2^m$.

Theorem 28. *The polynomial* $f_{2,m}(x) = \sum_{i=0}^{m} \lambda_{m,i} x^{2m-2i+1}$ *has the following properties.*

(I) *It satisfies the equation*

$$f''_{2,m}(x) = m(2m-1) f_{2,m-1}(x) + \frac{m(m-1)}{4} f_{2,m-2}(x), \qquad m \geq 2, \tag{8}$$

$$f_{0,2}(x) = x, \qquad f_{2,1}(x) = \binom{x+1}{3}.$$

(II) *The following recurrence relation holds for coefficients* $\lambda_{m,i}$,

$$\lambda_{m,i} = \frac{m(2m-1)}{2(m-i)(2m-2i+1)} \lambda_{m-1,i} + \frac{m(m-1)}{8(m-i)(2m-2i+1)} \lambda_{m-2,i-1}, \tag{9}$$

where $0 < i < m$.

(III) *The following general formula holds*

$$\lambda_{m,i} = \frac{1}{2^m(2m-2i+1)} \sum_{k=0}^{2i} \binom{m}{2i-k} \binom{2m-2i+k}{k} B_k. \tag{10}$$

Proof. **(I)** Since $\Delta f_{2,m}(x) = \binom{x+1}{2}^m$, we have

$$\Delta f''_{2,m}(x) = m(2m-1) \binom{x+1}{2}^{m-1} + \frac{m(m-1)}{4} \binom{x+1}{2}^{m-2},$$

and

$$f''_{2,m}(0) = 0.$$

Therefore, (8) is true.

(II) (9) is a consequence of (8).

(III) We have

$$\begin{aligned}
2^m f_{2,m}(N) &= \sum_{i=0}^{N-1} (i(i+1))^m = \sum_{i=0}^{N-1} \sum_{j=0}^{m} \binom{m}{j} i^{m+j} = \sum_{j=0}^{m} \binom{m}{j} \sum_{i=0}^{N-1} i^{m+j} \\
&= \sum_{j=0}^{m} \frac{1}{m+j+1} \binom{m}{j} \sum_{k=0}^{m+j} \binom{m+j+1}{k} B_k N^{m+j+1-k} \\
&= \sum_{k=1}^{2m+1} N^k \left(\sum_{j=0}^{m} \frac{1}{m+j+1} \binom{m}{j} \binom{m+j+1}{k} B_{m+j+1-k} \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lambda_{m,i} &= \frac{1}{2^m} \sum_{j \geq m-2i}^{m} \frac{1}{m+j+1} \binom{m}{j} \binom{m+j+1}{2m-2i+1} B_{j+2i-m} \\
&= \frac{1}{2^m} \sum_{k=0}^{2i} \frac{1}{2m-2i+k+1} \binom{m}{m+k-2i} \binom{2m-2i+k+1}{k} B_k \\
&= \frac{1}{2^m(2m-2i+1)} \sum_{k=0}^{2i} \binom{m}{2i-k} \binom{2m-2i+k}{k} B_k.
\end{aligned}$$

□

6.3 The polynomial part of $\overline{F}_i(m)$

Let us consider the function $\widetilde{h}_i(m)$ defined as

$$\widetilde{h}_i(m) = (-1)^i (i+1)(2m-2i+1) \frac{\overline{F}_i(m)}{\binom{m}{i}} = (-1)^i (i+1) \sum_{k=0}^{2i} \frac{\binom{m}{2i-k}\binom{2m-2i+k}{k}}{\binom{m}{i}} B_k.$$

In fact, $\widetilde{h}_i(m) = h_i(m)/(i+1)!$, where $h_i(m)$ is defined in relation (B3).

For example,

$$\widetilde{h}_0(m) = 1,$$

$$\widetilde{h}_1(m) = \frac{m+1}{3},$$

$$\widetilde{h}_2(m) = \frac{1}{60}(7m-6)(m+1),$$

$$\widetilde{h}_3(m) = \frac{(m+1)(31m^2-97m+60)}{630},$$

$$\widetilde{h}_4(m) = \frac{(m+1)(127m^3 - 833m^2 + 1606m - 840)}{5040}.$$

Analogous to Knuth's relation (A3), polynomial parts of $\overline{F}_i(m)$ are interesting by themselves. Below we study properties of $\widetilde{h}_i(x)$.

Theorem 29. *Let $i \geq 0$.*

(I) *The function $\widetilde{h}_i(x)$ is a polynomial of degree i.*

(II) *The following formula holds for $\widetilde{h}_i(m)$*

$$\begin{aligned}
&\widetilde{h}_i(m) \qquad (11)\\
&= (-1)^i(i+1)\left(\sum_{k=0}^{i} \frac{\binom{m-i}{i-k}\binom{2m-2i+k}{k}}{\binom{2i-k}{i}} B_k + \sum_{k=i+1}^{2i} \frac{\binom{i}{k-i}\binom{2m-2i+k}{k}}{\binom{m-2i+k}{k-i}} B_k\right)\\
&= (-1)^i \frac{(i+1)!}{(2i)!}\left(\sum_{k=0}^{i} p_k(m)\binom{2i}{k} B_k + \sum_{k=i+1}^{2i} q_k(m)\binom{2i}{k} B_k\right).
\end{aligned}$$

Here, $p_k(m), q_k(m)$ are polynomials given by

$$p_k(m) = (m-i)\cdots(m-2i+k+1)(2m-2i+k)\cdots(2m-2i+1)$$

and

$$\begin{aligned}
q_k(m) &= \frac{(2m-2i+k)\cdots(2m-2i+1)}{(m-2i+k)\cdots(m-i+1)}\\
&= 2^{k-i}(2m-2i+1)(2m-2i+3)\cdots\\
&\quad\times(2m-4i+2k+1)\ldots(2m-2i+k).
\end{aligned}$$

(If $k = 2i$, the last term $(2m-2i+k)$ in the product of $q_k(m)$ is cancelled).

(III) *The following recurrence relation holds for $\widetilde{h}_i(x)$*

$$(2i+1-2x)\widetilde{h}_i(x) = \frac{1}{2}(i+1)\widetilde{h}_{i-1}(x-2) - (2x-1)\widetilde{h}_i(x-1). \qquad (12)$$

(IV) *The initial value for $\widetilde{h}_i(x)$ is given by:*

$$\widetilde{h}_i(0) = (-1)^i(i+1)B_{2i}. \qquad (13)$$

(V) *If x approaches ∞, then*

$$\widetilde{h}_i(x) \sim (-1)^i\frac{(i+1)!}{(2i)!}(2-2^{2i})B_{2i}x^i. \qquad (14)$$

(VI) *If $i > 0$, then all zeros of the polynomial $\widetilde{h}_i(x)$ are real. Moreover, it has one negative zero $x_0 = -1$ and if $i > 1$, the other $i-1$ zeros $x_1, \ldots, x_{i-1}$ are positive, distinct and satisfy the following inequalities*

$$0 < x_1 < 1,\;\; 2 < x_2 < 3,\;\; 3 < x_3 < 4,\;\; \ldots,\;\; i-1 < x_{i-1} < i.$$

Proof. **(I)** Note that the terms $\binom{m}{2i-k}$ and $\binom{2m-2i+k}{k}$ are polynomials in m of degree $2i-k$ and k respectively. Hence, $\binom{m}{2i-k}\binom{2m-2i+k}{k}$ is a polynomial of degree $2i$ and it vanishes at $m = 0, 1, \ldots, i-1$ (if $i > 0$). This means that it is divisible by $m(m-1)\cdots(m-i+1)$ or $\binom{m}{i}$. So, $\frac{\binom{m}{2i-k}\binom{2m-2i+k}{k}}{\binom{m}{i}}$ is a polynomial in m of degree i. Therefore, $\widetilde{h}_i(m)$ is a polynomial in m of degree i.

(II) To prove (11), we have

$$\begin{aligned}
\widetilde{h}_i(m) &= (-1)^i(i+1)\sum_{k=0}^{2i}\frac{\binom{m}{2i-k}\binom{2m-2i+k}{k}}{\binom{m}{i}}B_k \\
&= (-1)^i(i+1)\sum_{k=0}^{2i}\frac{i!(m-i)!}{(2i-k)!(m-2i+k)!}\binom{2m-2i+k}{k}B_k \\
&= (-1)^i(i+1)\left(\sum_{k=0}^{i}\frac{\binom{m-i}{i-k}\binom{2m-2i+k}{k}}{\binom{2i-k}{i}}B_k + \sum_{k=i+1}^{2i}\frac{\binom{i}{k-i}\binom{2m-2i+k}{k}}{\binom{m-2i+k}{k-i}}B_k\right) \\
&= (-1)^i(i+1)\sum_{k=0}^{i}\frac{(m-i)\cdots(m-2i+k+1)(2m-2i+k)\cdots(2m-2i+1)i!}{k!(2i-k)!}B_k \\
&+ (-1)^i(i+1)\sum_{k=i+1}^{2i}\frac{i!(2m-2i+k)\cdots(2m-2i+1)}{k!(2i-k)!(m-2i+k)\cdots(m-i+1)}B_k \\
&= (-1)^i\frac{(i+1)!}{(2i)!}\left(\sum_{k=0}^{i}p_k(m)\binom{2i}{k}B_k + \sum_{k=i+1}^{2i}q_k(m)\binom{2i}{k}B_k\right).
\end{aligned}$$

(III) We use (9) for $x = m > i$, where m is a positive integer. We obtain (12) for positive integers $x = m > i$. Since $\widetilde{h}_i(x)$ is a polynomial, recurrence relation (12) must hold for all real numbers x.

(IV) A substitution $m = 0$ in (11) gives us

$$\begin{aligned}
\widetilde{h}_i(0) &= (-1)^i(i+1)\left(\sum_{k=0}^{i}\frac{\binom{-i}{i-k}\binom{-2i+k}{k}}{\binom{2i-k}{i}}B_k + \sum_{k=i+1}^{2i}\frac{\binom{i}{k-i}\binom{-2i+k}{k}}{\binom{-2i+k}{k-i}}B_k\right) \\
&= (-1)^i(i+1)\sum_{k=0}^{2i}\frac{1}{2}\binom{2i}{k}B_k = (-1)^i(i+1)B_{2i}.
\end{aligned}$$

Thus, (13) is proved.

(V) Let us calculate the leading coefficient A of $\widetilde{h}_i(m)$. By (11), we have

$$A = (-1)^i\frac{(i+1)!}{(2i)!}\sum_{k=0}^{2i}2^k\binom{2i}{k}B_k.$$

Let $B_n(x)$ be the Bernoulli polynomial,

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}.$$

It is well known that $B_n(1/2) = (2^{1-n} - 1)B_n$. Therefore,

$$\begin{aligned} A &= (-1)^i \frac{(i+1)!}{(2i)!} 2^{2i} \sum_{k=0}^{2i} (1/2)^{2i-k} \binom{2i}{k} B_k \\ &= (-1)^i \frac{(i+1)!}{(2i)!} 2^{2i} B_{2i}(1/2) \\ &= (-1)^i \frac{(i+1)!}{(2i)!} 2^{2i} (2^{1-2i} - 1) B_{2i} = (-1)^i \frac{(i+1)!}{(2i)!} (2 - 2^{2i}) B_{2i}, \end{aligned}$$

which yields (14).

(VI) We have

$$\begin{aligned} \widetilde{h}_i(-1) &= (-1)^i \frac{(i+1)}{\binom{-1}{i}} \sum_{k=0}^{2i} \binom{-1}{2i-k} \binom{-2-2i+k}{k} B_k \\ &= (i+1) \sum_{k=0}^{2i} \binom{2i+1}{k} B_k = 0. \end{aligned}$$

The part of the statement **(VI)** that relates to positive roots is evident for $i = 2$. Suppose that $i \geq 3$.

Let $\operatorname{sgn}(x) = -1$, if $x < 0$; $\operatorname{sgn}(x) = 0$, if $x = 0$; and $\operatorname{sgn}(x) = 1$, if $x > 0$.

Putting $m = 1$ in (12), we have

$$(2i-1)\widetilde{h}_i(1) = -\widetilde{h}_i(0).$$

Equation (13) gives us

$$\widetilde{h}_i(0) = (-1)^i (i+1) B_{2i}, \tag{15}$$

$$\widetilde{h}_i(1) = (-1)^{i+1} \frac{i+1}{2i-1} B_{2i}. \tag{16}$$

Therefore,

$$\operatorname{sgn}(\widetilde{h}(0)) = -\operatorname{sgn}(\widetilde{h}(1)).$$

Hence, there exists a real zero $x_1 \in (0,1)$ of $\widetilde{h}_i(x)$.

Putting $m = 2$ in (12), by (15) and (16), we have

$$\widetilde{h}_i(2) = \frac{1}{2i-3} \left(\frac{1}{2} (-1)^{i-1} i B_{2i-2} + 3(-1)^i \frac{i+1}{2i-1} B_{2i} \right).$$

By (3), for any positive integer i,

$$\operatorname{sgn}(B_{2i}) = (-1)^{i+1}.$$

So,

$$\operatorname{sgn}(\widetilde{h}_i(2)) = \operatorname{sgn}(\widetilde{h}_i(1)).$$

Therefore, relation (12) for $m = 3$ gives

$$\widetilde{h}_i(3) = \frac{1}{2i-5}\left((-1)^i \frac{11}{2}\frac{i}{2i-3}B_{2i-2} + 3(-1)^{i+1}\frac{i+1}{2i-1}B_{2i}\right).$$

Hence,

$$\operatorname{sgn}(\widetilde{h}(3)) = -\operatorname{sgn}(\widetilde{h}(2)).$$

Induction on i and on k shows that for any $k \in [2, i]$

$$\operatorname{sgn}(\widetilde{h}_i(k)) = (-1)^{i+k}\operatorname{sgn}(B_{2i}).$$

As we discussed above, this relation is also true for $k = 2, 3$.

By inductive hypothesis,

$$\operatorname{sgn}(\widetilde{h}_i(k-1)) = -\operatorname{sgn}(\widetilde{h}_{i-1}(k-2)).$$

Recurrence relation (12) yields

$$\begin{aligned}\operatorname{sgn}(\widetilde{h}_i(k)) &= \operatorname{sgn}\left(\frac{1}{2i-5}\left(\frac{1}{2}\widetilde{h}_{i-1}(k-2) - (2k-1)\widetilde{h}_i(k-1)\right)\right)\\ &= \operatorname{sgn}(\widetilde{h}_{i-1}(k-2) - \widetilde{h}_i(k-1))\\ &= \operatorname{sgn}((-1)^{i+k-3}B_{2i-2} - (-1)^{i+k-1}B_{2i})\\ &= (-1)^{i+k}\operatorname{sgn}(B_{2i}).\end{aligned}$$

It remains to note that

$$\operatorname{sgn}(\widetilde{h}(2)) = -\operatorname{sgn}(\widetilde{h}(3)) = \cdots = (-1)^{i-1}\operatorname{sgn}(\widetilde{h}(i-1)) = (-1)^i\operatorname{sgn}(\widetilde{h}(i)).$$

□

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