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Iterative Procedure for Hypersums of Powers of Integers

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Abstract

Relying on a recurrence relation for the hypersums of powers of integers put forward recently, we develop an iterative procedure which allows us to express a hypersum of arbitrary order in terms of ordinary (zeroth order) power sums. Then, we derive the coefficients of the hypersum polynomial as a function of the Bernoulli numbers and the Stirling numbers of the first kind.

1 Introduction

For every integer m, $m \geq 1$, the hypersums of powers of integers are defined recursively as follows: $P_k^{(m)}(n) = \sum_{j=1}^{n} P_k^{(m-1)}(j)$, where $P_k^{(0)}(n)$ is the sum of the first n positive integers each raised to the integer power $k \geq 0$, $P_k^{(0)}(n) = 1^k + 2^k + \cdots + n^k$ [1, 2]. The latter is given by a polynomial in n of degree $k+1$ with zero constant term. Hence, inductively $P_k^{(m)}(n)$ is given by a polynomial in n of degree $k+m+1$ with zero constant term:

$$P_k^{(m)}(n) = \sum_{r=1}^{k+m+1} c_{k,m}^r n^r. \qquad (1)$$

An explicit formula for the coefficients $c_{k,m}^r$ involving the Stirling numbers of the first and second kinds has been given by the author [3]. In this paper (Section 2), by an iterative procedure, we obtain a new representation of the m-th order hypersum $P_k^{(m)}(n)$ in terms of ordinary (zeroth order) power sums. Specifically, we will show that $P_k^{(m)}(n)$ can be expressed as a linear combination of $P_k^{(0)}(n), P_{k+1}^{(0)}(n), \ldots, P_{k+m}^{(0)}(n)$, as follows:

$$P_k^{(m)}(n) = \sum_{i=0}^{m} \frac{(-1)^i q_{m,i}(n)}{m!} P_{k+i}^{(0)}(n), \tag{2}$$

where $q_{m,i}(n)$ is a polynomial in n of degree $m-i$. (Note that formula (2) holds for $m=0$ if we set $q_{0,0}(n)=1$.) In Section 3, we determine the explicit form of the coefficients of $q_{m,i}(n)$. Then, using (2), we obtain the coefficients $c_{k,m}^r$ in terms of the Bernoulli numbers B_k and the (unsigned) Stirling numbers of the first kind $\left[{n \atop k}\right]$ (see A008275 in [4]). In particular, we proved that

$$c_{k,m}^1 = \frac{1}{m!} \sum_{i=0}^{m} (-1)^i \left[{m+1 \atop i+1}\right] B_{k+i}, \tag{3}$$

in accordance with the result obtained by Inaba [2, Proposition 1]. (Please note that, throughout this paper, we use the convention that $B_1 = \frac{1}{2}$.)

For later reference, we recall that the recurrence relation defining the numbers $\left[{n \atop k}\right]$ is given by [5, p. 214]:

$$\left[{m+1 \atop i+1}\right] = m \left[{m \atop i+1}\right] + \left[{m \atop i}\right], \tag{4}$$

with $\left[{0 \atop 0}\right] = 1$, and $\left[{n \atop 0}\right] = \left[{0 \atop n}\right] = 0$ for $n \geq 1$.

Formula (2) is noteworthy since it neatly shows how the hypersum $P_k^{(m)}(n)$ is constructed out of the building blocks $P_{k+i}^{(0)}(n)$, $i = 0, 1, \ldots, m$. Moreover, we point out that the polynomials $q_{m,i}(n)$ are interesting in their own right. Indeed, for fixed m, the coefficients corresponding to the set of polynomials $\{q_{m,i}(n)\}_{i=0}^m$ may be arranged in a Pascal-like triangular array with a specific rule of formation.

2 Iterative procedure for the hypersum

The basic tool we use to obtain $P_k^{(m)}(n)$ in terms of ordinary power sums is the following recurrence relation, a proof of which has been given by the author [3, Theorem 8]:

Theorem 1. *The hypersums $P_k^{(j)}(n)$, $P_k^{(j-1)}(n)$, and $P_{k+1}^{(j-1)}(n)$ satisfy the recurrence relation*

$$P_k^{(j)}(n) = \frac{n+j}{j} P_k^{(j-1)}(n) - \frac{1}{j} P_{k+1}^{(j-1)}(n), \quad k \geq 0,\ j \geq 1. \tag{5}$$

To obtain $P_k^{(m)}(n)$, we repeatedly apply the recurrence (5) to successive values of $j =$

$1, 2, \ldots,$ up to $j = m$. Proceeding in this way, it is easy to see that, for example,

$$\begin{aligned} P_k^{(3)}(n) = \frac{1}{6}(n+1)(n+2)(n+3)P_k^{(0)}(n) \\ - \frac{1}{6}\big[(n+1)(n+2) + (n+1)(n+3) + (n+2)(n+3)\big]P_{k+1}^{(0)}(n) \\ + \frac{1}{6}\big[(n+1) + (n+2) + (n+3)\big]P_{k+2}^{(0)}(n) - \frac{1}{6}P_{k+3}^{(0)}(n), \end{aligned}$$

which is of the form (2) with $q_{3,0}(n) = (n+1)(n+2)(n+3)$, $q_{3,1}(n) = (n+1)(n+2) + (n+1)(n+3) + (n+2)(n+3)$, $q_{3,2}(n) = (n+1) + (n+2) + (n+3)$, and $q_{3,3}(n) = 1$.

From this procedure, the general form of the polynomial $q_{m,i}(n)$ is argued to be

$$q_{m,i}(n) = \sum_{1 \leq s_1 < s_2 < \cdots < s_{m-i} \leq m} \prod_{t=1}^{m-i} (n + s_t), \quad i = 0, 1, \ldots, m, \tag{6}$$

where s_t, $t = 1, 2, \ldots, m-i$, is an integer. Note the special cases $q_{m,m-1}(n) = \sum_{i=1}^{m}(n+i)$ and $q_{m,0}(n) = \prod_{i=1}^{m}(n+i)$. Furthermore,

$$q_{m,i}(0) = \sum_{1 \leq s_1 < s_2 < \cdots < s_{m-i} \leq m} \prod_{t=1}^{m-i} s_t = \sigma_{m-i}(1, 2, \ldots, m), \tag{7}$$

where $\sigma_{m-i}(1, 2, \ldots, m)$ is the $(m-i)$-th elementary symmetric polynomial evaluated on the first m integers $\{1, 2, \ldots, m\}$ [6, Chapter 6].

Lemma 2. *For $m \geq 1$, the polynomials $q_{m,i}(n)$ satisfy the recurrence relation*

$$(n+m)q_{m-1,i}(n) = q_{m,i}(n) - q_{m-1,i-1}(n), \quad i = 0, 1, \ldots, m-1, \tag{8}$$

where it is understood that $q_{m-1,i-1}(n) = 0$ for $i = 0$.

Proof. Relation (8) follows directly from the definition of $q_{m,i}(n)$. Hence, from (6), we obtain

$$(n+m)q_{m-1,i}(n) = \sum_{1 \leq s_1 < \cdots < s_{m-i-1} \leq m-1} \prod_{t=1}^{m-i-1} (n + s_t)(n+m). \tag{9}$$

On the other hand, we have

$$q_{m-1,i-1}(n) = \sum_{1 \leq s_1 < \cdots < s_{m-i} \leq m-1} \prod_{t=1}^{m-i} (n + s_t). \tag{10}$$

Clearly, the sum of the right-hand side of (9) and (10) is identical to (6). $\square$

Now we show by induction on m that $P_k^{(m)}(n)$ have the form (2) with $q_{m,i}(n)$ given by (6). This statement is readily verified for the base cases $m = 0, 1, 2$, and 3. Assuming the inductive hypothesis holds for $P_k^{(m-1)}(n)$ (with $m \geq 1$), Equation (5) yields

$$P_k^{(m)}(n) = \frac{1}{m!}\left[(n+m)\sum_{i=0}^{m-1}(-1)^i q_{m-1,i}(n)P_{k+i}^{(0)}(n) - \sum_{i=0}^{m-1}(-1)^i q_{m-1,i}(n)P_{k+i+1}^{(0)}(n)\right].$$

Using (8), it follows that

$$\begin{aligned}
P_k^{(m)}(n) &= \frac{1}{m!}\left[\sum_{i=0}^{m-1}(-1)^i q_{m,i}(n)P_{k+i}^{(0)}(n)\right. \\
&\quad \left. - \sum_{i=1}^{m-1}(-1)^i q_{m-1,i-1}(n)P_{k+i}^{(0)}(n) + \sum_{i=1}^{m}(-1)^i q_{m-1,i-1}(n)P_{k+i}^{(0)}(n)\right] \\
&= \frac{1}{m!}\left[\sum_{i=0}^{m-1}(-1)^i q_{m,i}(n)P_{k+i}^{(0)}(n) + (-1)^m q_{m-1,m-1}(n)P_{k+m}^{(0)}(n)\right] \\
&= \frac{1}{m!}\sum_{i=0}^{m}(-1)^i q_{m,i}(n)P_{k+i}^{(0)}(n),
\end{aligned}$$

where we used that $q_{m-1,m-1}(n) = q_{m,m}(n) = 1$ to justify the last equation. This completes the inductive step and the proof of the above statement. We formally state this result in the following theorem.

Theorem 3. *The hypersum $P_k^{(m)}(n)$ admits a representation of the form (2) with $q_{m,i}(n)$ given by (6).*

3 The coefficients of the hypersum polynomial

In this section, we provide an explicit expression for the coefficients $c_{k,m}^r$ in terms of the Bernoulli numbers and the Stirling numbers of the first kind. To this end, we first put $q_{m,i}(n)$ in polynomial form as $q_{m,i}(n) = \sum_{s=0}^{m-i} q_{m,i}^s n^s$. On the other hand, according to the well-known Bernoulli formula, $P_{k+i}^{(0)}(n)$ can be written as [7, Equation 9]

$$P_{k+i}^{(0)}(n) = \frac{1}{k+i+1}\sum_{t=1}^{k+i+1}\binom{k+i+1}{t}B_{k+i+1-t}n^t.$$

(Remember that we are taking $B_1 = \frac{1}{2}$ in the above formula.) Then, substituting the aforementioned expressions for $q_{m,i}(n)$ and $P_{k+i}^{(0)}(n)$ into (2) and comparing the resulting polynomial with (1), gives

$$c_{k,m}^r = \frac{1}{m!}\sum_{i=0}^{m}(-1)^i Q_{k,m,i}^r, \quad r = 1, 2, \ldots, k+m+1, \tag{11}$$

where

$$Q^r_{k,m,i} = \frac{1}{k+i+1} \sum_{h=0}^{r-1} q^h_{m,i} \binom{k+i+1}{r-h} B_{k+i+h+1-r}. \tag{12}$$

In particular, from (11) and (12), we quickly obtain

$$c^1_{k,m} = \frac{1}{m!} \sum_{i=0}^{m} (-1)^i q^0_{m,i} B_{k+i}, \quad k, m \geq 0. \tag{13}$$

Now let us address the question of the nature of the coefficients $q^s_{m,i}$, $s = 0, 1, \ldots, m-i$, of $q_{m,i}(n)$. Let us first look at the constant term $q^0_{m,i}$. This is the value of $q_{m,i}(n)$ at $n = 0$. Hence, from (7), we have $q^0_{m,i} = \sigma_{m-i}(1, 2, \ldots, m)$. On the other hand, the Stirling numbers of the first kind $\left[{n \atop k}\right]$ enumerate all the permutations of size n with k cycles. It turns out that $\sigma_k(1, 2, \ldots, n) = \left[{n+1 \atop n+1-k}\right]$ [5, pp. 213–214], and then

$$\sigma_{m-i}(1, 2, \ldots, m) = \begin{bmatrix} m+1 \\ i+1 \end{bmatrix}. \tag{14}$$

Thus, we have $q^0_{m,i} = \left[{m+1 \atop i+1}\right]$. Putting this in (13), we obtain formula (3).

In order to systematically derive the coefficients $q^s_{m,i}$, it is useful to note that

$$\prod_{t=1}^{m-i} (n + s_t) = \sum_{s=0}^{m-i} \sigma_{m-i-s}(s_1, s_2, \ldots, s_{m-i}) n^s,$$

where $\sigma_{m-i-s}(s_1, s_2, \ldots, s_{m-i})$ is the $(m-i-s)$-th elementary symmetric polynomial on the variables $s_1, s_2, \ldots, s_{m-i}$. Substituting this expression into (6), we deduce that

$$q^s_{m,i} = \sum_{1 \leq s_1 < s_2 < \cdots < s_{m-i} \leq m} \sigma_{m-i-s}(s_1, s_2, \ldots, s_{m-i}). \tag{15}$$

Clearly, the right-hand side of (15) is a symmetric function on $\{s_1, s_2, \ldots, s_{m-i}\}$. This function is a sum of products of $m-i-s$ distinct integers chosen from $\{1, 2, \ldots, m\}$, with a total of $\binom{m}{i}$ times $\binom{m-i}{s}$ terms. On the other hand, the elementary symmetric polynomial $\sigma_{m-i-s}(1, 2, \ldots, m)$ is a sum of $\binom{m}{i+s}$ terms, each of which is a product of $m-i-s$ distinct integers chosen from $\{1, 2, \ldots, m\}$. Therefore, since $\binom{m}{i}\binom{m-i}{s} = \binom{i+s}{s}\binom{m}{i+s}$, we conclude that the right-hand side of (15) is necessarily $\binom{i+s}{s}$ times $\sigma_{m-i-s}(1, 2, \ldots, m)$. Hence, using (14), we find that

$$q^s_{m,i} = \binom{i+s}{s} \begin{bmatrix} m+1 \\ i+s+1 \end{bmatrix}, \quad s = 0, 1, \ldots, m-i. \tag{16}$$

Note, in particular, that $q^{m-i}_{m,i} = \binom{m}{i}$. From (16), we also deduce the symmetry property $q^s_{m,i} = q^i_{m,s}$. As a concrete example, Table 1 displays the coefficients of the polynomials $q_{8,i}(n)$, $i = 0, 1, \ldots, 8$, where we use $[n^s]$ to denote the coefficient of n^s. Note that the

	$[n^0]$	$[n^1]$	$[n^2]$	$[n^3]$	$[n^4]$	$[n^5]$	$[n^6]$	$[n^7]$	$[n^8]$
$q_{8,8}(n)$	1	–	–	–	–	–	–	–	–
$q_{8,7}(n)$	36	8	–	–	–	–	–	–	–
$q_{8,6}(n)$	546	252	28	–	–	–	–	–	–
$q_{8,5}(n)$	4536	3276	756	56	–	–	–	–	–
$q_{8,4}(n)$	22449	22680	8190	1260	70	–	–	–	–
$q_{8,3}(n)$	67284	89796	45360	10920	1260	56	–	–	–
$q_{8,2}(n)$	118124	201852	134694	45360	8190	756	28	–	–
$q_{8,1}(n)$	109584	236248	201852	89796	22680	3276	252	8	–
$q_{8,0}(n)$	40320	109584	118124	67284	22449	4536	546	36	1

Table 1: The coefficients of the polynomials $q_{8,i}(n)$, $i = 0, 1, \ldots, 8$.

symmetry property implies that the table of coefficients is symmetric about a 45° diagonal. For example, we have $q^4_{8,2} = q^2_{8,4} = 8190$.

Finally, combining the Equations (11), (12), and (16), we obtain

$$c^r_{k,m} = \frac{1}{m!}\sum_{i=0}^{m}\frac{(-1)^i}{k+i+1}\sum_{h=0}^{r-1}\binom{i+h}{h}\binom{k+i+1}{r-h}\left[\begin{matrix} m+1 \\ i+h+1 \end{matrix}\right] B_{k+i+h+1-r},$$

which constitutes the generalization of Inaba's formula (3) to arbitrary $r = 1, 2, \ldots, k+m+1$, with $k, m \geq 0$, and $B_1 = \frac{1}{2}$.

On the other hand, from (8), we immediately derive the following recurrence relation for the coefficients $q^s_{m,i}$:

$$q^s_{m,i} = mq^s_{m-1,i} + q^s_{m-1,i-1} + q^{s-1}_{m-1,i}. \tag{17}$$

For $s = 0$, relation (17) becomes $q^0_{m,i} = mq^0_{m-1,i} + q^0_{m-1,i-1}$. Therefore, comparing this relation with (4) and noting that $q^0_{0,0} = 1 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, we retrieve the result $q^0_{m,i} = \left[\begin{smallmatrix} m+1 \\ i+1 \end{smallmatrix}\right]$. For $s = 1$ we have $q^1_{m,i} = mq^1_{m-1,i} + q^1_{m-1,i-1} + q^0_{m-1,i}$, which is satisfied when we set $q^1_{m,i} = (i+1)q^0_{m,i+1} = (i+1)\left[\begin{smallmatrix} m+1 \\ i+2 \end{smallmatrix}\right]$. In general, the solution of the recurrence (17) is given by

$$\begin{aligned} q^s_{m,i} &= \frac{1}{s}(i+1)q^{s-1}_{m,i+1} \\ &= \frac{1}{s}\frac{1}{s-1}(i+1)(i+2)q^{s-2}_{m,i+2} \\ &\ \ \vdots \\ &= \frac{1}{s!}(i+1)(i+2)\cdots(i+s)q^0_{m,i+s}, \end{aligned}$$

so that $q^s_{m,i} = \binom{i+s}{s} q^0_{m,i+s} = \binom{i+s}{s}\left[\begin{smallmatrix} m+1 \\ i+s+1 \end{smallmatrix}\right]$, in accordance with (16).

Thus, Table 1 is generated by the rule $q^s_{m,i} = \frac{1}{s}(i+1)q^{s-1}_{m,i+1}$, $s \geq 1$, which enables one to determine the element $q^s_{m,i}$ in row $m-i$ and column s from the preceding element $q^{s-1}_{m,i+1}$ in row $m-i-1$ and column $s-1$, the elements of the starting 0-th column being given by $q^0_{m,i} = \left[{m+1 \atop i+1}\right]$.

We conclude with three brief remarks.

Remark 4. For $k = 0$ the hypersum $P^{(m)}_k(n)$ is equal to $P^{(m)}_0(n) = \binom{n+m}{m+1}$. Then, letting $k = 0$ in (2), we will have $\sum_{i=0}^m \frac{(-1)^i q_{m,i}(n)}{m!} P^{(0)}_i(n) = \binom{n+m}{m+1}$. Solving for $P^{(0)}_m(n)$, we get

$$(-1)^m P^{(0)}_m(n) = m!\binom{n+m}{m+1} + \sum_{i=0}^{m-1}(-1)^{i+1} q_{m,i}(n) P^{(0)}_i(n), \quad m \geq 1,$$

which allows us to compute recursively $P^{(0)}_m(n)$ from the power sums $P^{(0)}_0(n)$, $P^{(0)}_1(n), \ldots,$ $P^{(0)}_{m-1}(n)$, and the polynomials $q_{m,i}(n)$, $i = 0, 1, \ldots, m-1$.

Remark 5. The leading coefficient of the hypersum polynomial (1) has been given by the author [3]: $c^{k+m+1}_{k,m} = \frac{k!}{(k+m+1)!}$. On the other hand, the leading coefficients of $q_{m,i}(n)$ and $P^{(0)}_{k+i}(n)$ are given by $q^{m-i}_{m,i} = \binom{m}{i}$ and $c^{k+i+1}_{k+i,0} = \frac{1}{k+i+1}$, respectively. Therefore, equating the terms of maximum degree on the two sides of (2) yields the combinatorial identity

$$\sum_{i=0}^m \frac{(-1)^i}{k+i+1}\binom{m}{i} = \frac{k!\,m!}{(k+m+1)!}, \quad k, m \geq 0.$$

Remark 6. From formula (3), we deduce an identity relating the harmonic number $H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$ to the Bernoulli numbers and the Stirling numbers of the first kind. Indeed, from $c^1_{0,m} = 1/(m+1)$ [2, 3], recalling that $\left[{m+1 \atop 2}\right] = m!H_m$, and from (3) we obtain

$$H_m = \frac{2m}{m+1} + \frac{2}{m!}\sum_{j=1}^{\lfloor m/2 \rfloor}\left[{m+1 \atop 2j+1}\right] B_{2j}.$$

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