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Extremely Abundant Numbers and the Riemann Hypothesis

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Abstract

Robin's theorem states that the Riemann hypothesis is equivalent to the inequality $\sigma(n) < e^\gamma n \log\log n$ for all $n > 5040$, where $\sigma(n)$ is the sum of divisors of n and γ is Euler's constant. It is natural to seek the first integer, if it exists, that violates this inequality. We introduce the sequence of extremely abundant numbers, a subsequence of superabundant numbers, where one might look for this first violating integer. The Riemann hypothesis is true if and only if there are infinitely many extremely abundant numbers. These numbers have some connection to the colossally abundant numbers. We show the fragility of the Riemann hypothesis with respect to the terms of some supersets of extremely abundant numbers.

1 Introduction

There are several statements equivalent to the famous Riemann hypothesis (RH) [4]. Some of them are related to the asymptotic behavior of arithmetic functions. In particular, the well-known Robin theorem (inequality, criterion, etc.) deals with the upper bound of the sum-of-divisors function σ, which is defined by $\sigma(n) := \sum_{d|n} d$. Robin [24, Theorem 1] established an elegant connection between RH and the sum of divisors of n by proving that the RH is true if and only if

$$\frac{\sigma(n)}{n\log\log n} < e^{\gamma}, \qquad \text{for all} \quad n > 5040, \tag{1}$$

where γ is Euler's constant.

Throughout the paper, as in Robin [24], we define the function f by setting

$$f(n) = \frac{\sigma(n)}{n\log\log n}. \tag{2}$$

Gronwall, in his study of the asymptotic maximal size for the sum of divisors of n [11], found that the order of $\sigma(n)$ is always "very nearly n" [12, Theorem 323]. More precisely, he proved the following theorem.

Theorem 1 (Gronwall)**.** *Let f be defined as in (2). Then*

$$\limsup_{n\to\infty} f(n) = e^{\gamma}. \tag{3}$$

Let us call a positive integer n [2, 23]

(i) *colossally abundant*, if for some $\varepsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\varepsilon}} \geq \frac{\sigma(m)}{m^{1+\varepsilon}}, \qquad (m < n) \qquad \text{and} \qquad \frac{\sigma(n)}{n^{1+\varepsilon}} > \frac{\sigma(m)}{m^{1+\varepsilon}}, \qquad (m > n); \tag{4}$$

(ii) *generalized superior highly composite*, if there is a positive number ε such that

$$\frac{\sigma_{-s}(n)}{n^{\varepsilon}} \geq \frac{\sigma_{-s}(m)}{m^{\varepsilon}}, \qquad (m < n) \qquad \text{and} \qquad \frac{\sigma_{-s}(n)}{n^{\varepsilon}} > \frac{\sigma_{-s}(m)}{m^{\varepsilon}}, \qquad (m > n),$$

where $\sigma_{-s}(n) = \sum_{d|n} d^{-s}$. The parameter s is assumed to be positive in [23]. In the case $s = 1$, (ii) becomes (i).

Ramanujan initiated the study of these classes of numbers in an unpublished part of his 1915 work on highly composite numbers ([21, 23] and [22, pp. 78–129, 338–339]). More precisely, he defined rather general classes of these numbers. For instance, he defined generalized highly composite numbers, containing as a subset superabundant numbers [21, Section 59], and he introduced the generalized superior highly composite numbers, including as a particular case colossally abundant numbers. For more details we refer the reader to [2, 10, 23].

We denote by CA the set of all colossally abundant numbers. We also use CA as an abbreviation for the term "colossally abundant". Ramanujan [23] proved that if n is a generalized superior highly composite number, i.e., a CA number, then under the RH we have

$$\liminf_{n\to\infty}\left(\frac{\sigma(n)}{n}-e^{\gamma}\log\log n\right)\sqrt{\log n}\geq -\,e^{\gamma}(2\sqrt{2}+\gamma-\log 4\pi)\approx -1.558,$$

$$\limsup_{n\to\infty}\left(\frac{\sigma(n)}{n}-e^{\gamma}\log\log n\right)\sqrt{\log n}\leq -\,e^{\gamma}(2\sqrt{2}-4-\gamma+\log 4\pi)\approx -1.393.$$

Robin [24] also established (independent of the RH) the following inequality

$$f(n)\leq e^{\gamma}+\frac{0.648214}{(\log\log n)^2}\ ,\qquad (n\geq 3), \tag{5}$$

where $0.648214\approx(\frac{7}{3}-e^{\gamma}\log\log 12)\log\log 12$ and the left-hand side of (5) attains its maximum at $n=12$. In the same spirit, Lagarias [15] proved the equivalence of the RH to the problem

$$\sigma(n)\leq e^{H_n}\log H_n+H_n,\qquad (n\geq 1),$$

where $H_n:=\sum_{j=1}^{n}1/j$ is the nth harmonic number.

Investigating upper and lower bounds for arithmetic functions, Landau [16, pp. 216–219] obtained the following limits:

$$\liminf_{n\to\infty}\frac{\varphi(n)\log\log n}{n}=e^{-\gamma}\ ,\qquad \limsup_{n\to\infty}\frac{\varphi(n)}{n}=1,$$

where $\varphi(n)$ is the Euler totient function, which is defined as the number of positive integers not exceeding n that are relatively prime to n. It can also be expressed as a product extended over the distinct prime divisors of n [3, Theorem 2.4] by

$$\varphi(n)=n\prod_{p|n}\left(1-\frac{1}{p}\right).$$

Nicolas [18, 19] proved that if the RH is true, then we have for all $k\geq 2$

$$\frac{N_k}{\varphi(N_k)\log\log N_k}>e^{\gamma}, \tag{6}$$

where $N_k=\prod_{j=1}^{k}p_j$ and p_j is the jth prime. On the other hand, if the RH is false, then for infinitely many k, inequality (6) is true and for infinitely many k, inequality (6) is false.

Compared to numbers N_k which are the smallest integers that maximize $n/\varphi(n)$, there are integers which play this role for $\sigma(n)/n$ and they are called superabundant numbers. A positive integer n is said to be *superabundant* [2, 23] if

$$\frac{\sigma(n)}{n}>\frac{\sigma(m)}{m}\qquad \text{for all}\ \ m<n. \tag{7}$$

We will use the symbol SA to denote the set of superabundant numbers and also as an abbreviation for the term "superabundant". Briggs [5] described a computational study of the successive maxima of the relative sum-of-divisors function $\sigma(n)/n$. He also studied the density of these numbers. Wójtowicz [26] showed that the values of the function f defined in (2) are close to 0 on a set of asymptotic density 1. Another study on Robin's inequality (1) is due to Choie et al. [9]. They have shown that the RH holds true if and only if every natural number divisible by a fifth power greater than 1 satisfies Robin's inequality (1).

Akbary and Friggstad [1] established the following interesting theorem which enables us to limit our attention to a narrow sequence of positive integers, in order to find a probable counterexample to inequality (1).

Theorem 2 ([1, Theorem 3]). *If there is any counterexample to Robin's inequality (1), then the least such counterexample is a superabundant number.*

Unfortunately, to our knowledge, there is no known algorithm (except the formula (7) in the definition) to produce SA numbers. Alaoglu and Erdős [2] proved that

$$Q(x) > c\,\frac{\log x \log\log x}{(\log\log\log x)^2},$$

where $Q(x)$ denotes the number of superabundant numbers not exceeding x. Later, Erdős and Nicolas [10] demonstrated a stronger result that for every $\delta < 5/48$ we have

$$Q(x) > (\log x)^{1+\delta}, \qquad (x > x_0).$$

As a natural question in this direction, it is interesting to introduce and study a set of positive integers to which the first probable violation of inequality (1) belongs. Following this aim, we introduce the sequence of *extremely abundant numbers*. We will establish another criterion equivalent to the RH by proving that the RH is true if and only if there are infinitely many extremely abundant numbers. Also, we give a connection between extremely abundant numbers and CA numbers. Moreover, we present approximate formula for the prime factorization of (sufficiently large) extremely abundant numbers. Finally, we establish the fragility of the Riemann hypothesis with respect to the terms of certain subsets of superabundant numbers which are quite close to the set of extremely abundant numbers.

Before stating the main definition and results of this paper, we mention recent work of Caveney et al. [6]. They defined a positive integer n as an *extraordinary* number if n is composite and $f(n) \geq f(kn)$ for all

$$k \in \mathbb{N} \cup \{1/p : \ p \text{ is a prime factor of } n\}.$$

Under these conditions they showed that the smallest extraordinary number is $n = 4$. Then they proved that the RH is true if and only if 4 is the only extraordinary number. For more properties of these numbers and comparisons with SA and CA numbers, we refer the reader to [7].

2 Extremely abundant numbers

We define a new sequence of positive integers related to the RH. Our primary contribution and motivation of this definition are Theorems 6 and 7. Let us now state the main definition of this paper.

Definition 3. A positive integer n is *extremely abundant* if either $n = 10080$, or $n > 10080$ and

$$\frac{\sigma(n)}{n \log\log n} > \frac{\sigma(m)}{m \log\log m}, \qquad \text{for all} \quad 10080 \le m < n. \tag{8}$$

Here 10080 has been chosen as the smallest SA number greater than 5040. In Table 1 we list the first 20 extremely abundant numbers. To find them, we used a list of SA numbers (see Proposition 5) provided by Kilminster [14] and Noe [20].

Remark 4. If we choose (instead of 10080) n_1 such that $2520 < n_1 \le 5040$, and define n to be extremely abundant if either $n = n_1$, or $n > n_1$ and

$$\frac{\sigma(n)}{n \log\log n} > \frac{\sigma(m)}{m \log\log m}, \qquad \text{for all} \quad n_1 \le m < n,$$

then we have a finite number of elements $n \le 5040$ that satisfy the above inequality. Using inequality (5), we have

$$f(n) < e^\gamma + \frac{0.648214}{(\log\log n)^2} < f(5040), \qquad \text{for some } s_{10308} < n \le s_{10309},$$

where s_k denotes the kth SA number listed in [20]. Checking by computer for values n between 5040 and s_{10309}, we derive a finite set with maximum 5040. Similarly, one can easily check for $n_1 < 2520$, and get only sets with finite number of elements.

Let XA denote the set of all extremely abundant numbers. (We also use XA as an abbreviation for the term "extremely abundant".) Clearly, $XA \neq CA$ (see Table 1). Indeed, we shall prove that infinitely many elements of CA are not in XA and that, if RH holds, then infinitely many elements of XA are in CA. As an elementary result from the definition of XA numbers we have the following proposition.

Proposition 5. *The inclusion $XA \subset SA$ holds.*

Proof. First, $10080 \in SA$. Next, if $n > 10080$ and $n \in XA$, then for $10080 \le m < n$ we have

$$\frac{\sigma(n)}{n} = f(n) \log\log n > f(m) \log\log m = \frac{\sigma(m)}{m}.$$

In particular, for $m = 10080$ we get

$$\frac{\sigma(n)}{n} > \frac{\sigma(10080)}{10080}.$$

So for $m < 10080$, we have

$$\frac{\sigma(n)}{n} > \frac{\sigma(10080)}{10080} > \frac{\sigma(m)}{m}$$

since $10080 \in SA$. Therefore, n belongs to SA. $\square$

Next, motivating our construction of XA numbers, we will establish the first interesting result of the paper.

Theorem 6. *If there is any counterexample to Robin's inequality (1), then the least one is an XA number.*

Proof. By doing some computer calculations we observe that there is no counterexample to Robin's inequality (1) for $5040 < n \leq 10080$. Now let $n > 10080$ be the least counterexample to inequality (1). For m satisfying $10080 \leq m < n$ we have

$$f(m) < e^{\gamma} \leq f(n).$$

Therefore n is an XA number. $\square$

As we mentioned in Section 1 we will prove an equivalent criterion to the RH for which the proof is based on Robin's inequality (1) and Gronwall's theorem. Let $\#A$ denote the cardinality of the set A. The second stimulus result is the following theorem. This result also has its own interest that will be discussed in Section 5.

Theorem 7. *The RH is true if and only if $\#XA = \infty$.*

Proof. Sufficiency. Assume that RH is not true. Then by Theorem 6 we have $f(m) \geq e^{\gamma}$ for some $m \geq 10080$. From Gronwall's theorem, we know that $M = \sup_{n \geq 10080} f(n)$ is finite and there exists n_0 such that $f(n_0) = M \geq e^{\gamma}$ (if $M = e^{\gamma}$ then set $n_0 = m$). An integer $n > n_0$ satisfies $f(n) \leq M = f(n_0)$ and n can not be in XA, so $\#XA \leq n_0$.

Necessity. On the other hand, if RH is true, then Robin's inequality (1) holds. If $\#XA$ is finite, then let m be its largest element. For every $n > m$ the inequality $f(n) \leq f(m)$ holds and therefore

$$\limsup_{n \to \infty} f(n) \leq f(m) < e^{\gamma},$$

which is a contradiction to Gronwall's theorem. $\square$

There are some primes which cannot be the largest prime factor of any XA number. For example, referring to Table 1, there is no XA number with the largest prime factor $p(n) = 149$. Do there exist infinitely many such primes?

3 Auxiliary lemmas and inequalities

Chebyshev's functions $\vartheta(x)$ and $\psi(x)$ are defined by

$$\vartheta(x) = \sum_{p \le x} \log p, \qquad \psi(x) = \sum_{p^m \le x} \log p = \sum_{p \le x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p,$$

where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. It is known that the prime number theorem (PNT) ([12, Theorem 434] and [13, Theorem 3, 12]) is equivalent to

$$\psi(x) \sim x. \tag{9}$$

In his mémoir, Chebyshev proved the following lemma that we call Chebyshev's result.

Lemma 8 ([8, p. 379])**.** *For all $x > 1$*

$$\vartheta(x) < \frac{6}{5}Ax - Ax^{\frac{1}{2}} + \frac{5}{4\log 6}\log^2 x + \frac{5}{2}\log x + 2$$
$$\vartheta(x) > Ax - \frac{12}{5}Ax^{\frac{1}{2}} - \frac{5}{8\log 6}\log^2 x - \frac{15}{4}\log x - 3,$$

where

$$A = \log \frac{2^{\frac{1}{2}}3^{\frac{1}{3}}5^{\frac{1}{5}}}{30^{\frac{1}{30}}} \approx 0.92129202.$$

We will use the following corollary in the proof of Theorem 26.

Corollary 9. *We have*

$$\vartheta(x) > \frac{x}{3}, \qquad (x \ge 3).$$

The next lemma here provides Littlewood's result for oscillation of Chebyshev's ϑ function.

Lemma 10 ([7, Lemma 4])**.** *There exists a constant $c > 0$ such that for infinitely many primes p we have*

$$\vartheta(p) < p - c\sqrt{p}\log\log\log p, \tag{10}$$

and for infinitely many other primes p we have

$$\vartheta(p) > p + c\sqrt{p}\log\log\log p.$$

In what follows, we shall frequently use the following elementary inequalities:

$$\frac{t}{1+t} < \log(1+t) < t, \qquad (t > 0), \tag{11}$$

and

$$\frac{2t}{2+t} < \log(1+t), \qquad (t > 0). \tag{12}$$

4 Some properties of SA, CA and XA numbers

We divide this section into three subsections, for which we shall exhibit several properties of SA, CA and XA numbers, respectively. We denote by $p(n)$ the largest prime factor of n or, when there is no ambiguity, simply by p.

4.1 SA Numbers

Proposition 11. *Let $n < n'$ be two consecutive SA numbers. Then*

$$\frac{n'}{n} \leq 2.$$

Proof. Let $n = 2^{k_2} \cdots p$. We compare n with $2n$. In fact,

$$\frac{\sigma(2n)/(2n)}{\sigma(n)/n} = \frac{2^{k_2+2}-1}{2^{k_2+2}-2} > 1.$$

Hence $n' \leq 2n$. □

Corollary 12. *For any positive real number $x \geq 1$ there exists at least one SA number n such that $x \leq n < 2x$.*

Alaoglu and Erdős [2] have shown that if $n = 2^{k_2} \cdot 3^{k_3} \cdots p^{k_p}$ is an SA number, then $k_2 \geq k_3 \geq \cdots \geq k_p$ and $k_p = 1$, except for $n = 4,\ 36$.

Proposition 13 ([2, Theorem 2])**.** *Let $n \in SA$, and let $q < r$ be prime factors of n with corresponding exponents k_q and k_r. Set*

$$\beta := \left\lfloor \frac{k_q \log q}{\log r} \right\rfloor.$$

Then k_r has one of the three values: $\beta - 1$, $\beta + 1$, β.

As we observe, the above proposition determines the exponent of each prime factor of an SA number with error of at most 1 in terms of a smaller prime factor of that number. In the next lemma we prove a relation between the lower bound of an exponent of a prime factor of n and its largest prime factor p.

Lemma 14. *Let $n \in SA$, and let q be a prime factor of n. Then*

$$\left\lfloor \frac{\log p}{\log q} \right\rfloor \leq k_q.$$

Proof. If $q = p$ $(= p(n))$, then the result is trivial. Let $q < p$ and $k_q = k$. Suppose that $k \leq \lfloor \log p / \log q \rfloor - 1$. Then

$$q^{k+1} < p. \tag{13}$$

Now we compare values of $\sigma(\nu)/\nu$, taking $\nu = n$ and $\nu = m = nq^{k+1}/p$. Since $\sigma(\nu)/\nu$ is multiplicative, we restrict our attention to different factors. But n is an SA number and $m < n$. Hence

$$1 < \frac{\sigma(n)/n}{\sigma(m)/m} = \frac{q^{2k+2} - q^{k+1}}{q^{2k+2} - 1}\left(1 + \frac{1}{p}\right) = \frac{1}{1 + 1/q^{k+1}}\left(1 + \frac{1}{p}\right).$$

Consequently, $p < q^{k+1}$, which contradicts inequality (13). □

Proposition 15 ([2, Theorem 5])**.** *Let $n \in SA$. If $k_q = k$ and $q < (\log p)^{\alpha}$, where α is a constant, then*

$$\log \frac{q^{k+1} - 1}{q^{k+1} - q} > \frac{\log q}{p \log p}\left(1 + O\left(\frac{(\log \log p)^2}{\log p \log q}\right)\right), \tag{14}$$

$$\log \frac{q^{k+2} - 1}{q^{k+2} - q} < \frac{\log q}{p \log p}\left(1 + O\left(\frac{(\log \log p)^2}{\log p \log q}\right)\right). \tag{15}$$

Lemma 16. *Let $n \in SA$, and let q be a fixed prime factor of n. Then there exist two positive constants c and c' (depending on q) such that*

$$c\, p \frac{\log p}{\log q} < q^{k_q} < c' p \frac{\log p}{\log q}.$$

Proof. By inequality (11)

$$\log \frac{q^{k+1} - 1}{q^{k+1} - q} = \log\left(1 + \frac{q - 1}{q^{k+1} - q}\right) < \frac{q - 1}{q^{k+1} - q} \leq \frac{1}{q^k}$$

and (14), there exists a $c' > 0$ such that

$$q^k < c' \frac{p \log p}{\log q}.$$

On the other hand, again by inequality (11)

$$\log \frac{q^{k+2} - 1}{q^{k+2} - q} = \log\left(1 + \frac{q - 1}{q^{k+2} - q}\right) > \frac{q - 1}{q^{k+2} - 1} > \frac{1}{2q^{k+1}}$$

and (15), there exists a $c > 0$ such that

$$q^k > c \frac{p \log p}{\log q}.$$

□

Corollary 17. *Let $n = 2^k \cdots p$ be an SA number. Then for n sufficiently large we have*

$$\left\lfloor \frac{k \log 2}{\log p} \right\rfloor = 1.$$

We showed [17] that for sufficiently large $n \in SA$

$$\log n < p\left(1 + \frac{1}{2\log p}\right). \tag{16}$$

Our computation on the list of SA numbers in [20] suggests that a weaker inequality

$$\log n < p\left(1 + \frac{2}{3\log p}\right) \tag{17}$$

holds for all $n \geq s_{365}$. The product of exponent of a prime factor and the logarithm of the corresponding prime factor of an SA number can be controlled, on average, by the logarithm of the largest prime factor of that number. More precisely,

Proposition 18 ([2, Theorem 7])**.** *If $n \in SA$, then*

$$p(n) \sim \log n.$$

The next proposition gives a lower bound of $n \in SA$ in terms of Chebyshev's ψ function compared to the above-mentioned asymptotic relation.

Proposition 19. *Let $n \in SA$. Then*

$$\psi(p(n)) \leq \log n.$$

Moreover,

$$\lim_{\substack{n\to\infty \\ n\in SA}} \frac{\psi(p(n))}{\log n} = 1. \tag{18}$$

Proof. In fact, by Lemma 14

$$\psi(p(n)) = \sum_{q \leq p(n)} \left\lfloor \frac{\log p(n)}{\log q} \right\rfloor \log q \leq \sum_{q \leq p(n)} k_q \log q = \log n.$$

To prove (18) we appeal to (the equivalent of) the PNT (9) and Proposition 18. □

4.2 CA Numbers

By the definition of CA numbers (4) it is easily seen that $CA \subset SA$. Here we give a concise description of the algorithm (essentially borrowed from [7, 10, 24]) to produce CA numbers. For more details on this introduction we refer the reader to [2, 7, 10, 24].

Let F be defined by

$$F(x,k) = \frac{\log(1 + 1/(x + \cdots + x^k))}{\log x}.$$

For $\varepsilon > 0$ we define $x_1 = x_1(\varepsilon)$ to be the only number such that

$$F(x_1, 1) = \varepsilon, \tag{19}$$

and $x_k = x_k(\varepsilon)$ (for $k > 1$) to be the only number such that

$$F(x_k, k) = \varepsilon.$$

Let

$$E_p = \{F(p, \alpha) : \ \alpha \geq 1\}, \quad p \text{ is a prime}$$

and

$$E = \bigcup_p E_p = \{\varepsilon_1, \, \varepsilon_2, \, \ldots\}.$$

If $\varepsilon \notin E$, then the function $\sigma(n)/n^{1+\varepsilon}$ attains its maximum at a single point N_ε whose prime decomposition is

$$N_\varepsilon = \prod p^{\alpha_p(\varepsilon)}, \qquad \alpha_p(\varepsilon) = \left\lfloor \frac{\log \frac{p^{1+\varepsilon}-1}{p^\varepsilon - 1}}{\log p} \right\rfloor - 1,$$

or if one prefers

$$\alpha_p(\varepsilon) = \begin{cases} k, & \text{if } x_{k+1} < p < x_k, \, k \geq 1; \\ 0, & \text{if } p > x = x_1. \end{cases}$$

If $\varepsilon \in E$, then at most two x_k's are prime. Hence, there are either two or four CA numbers of parameter ε, defined by

$$N_\varepsilon = \prod_{k=1}^{K} \prod_{\substack{p < x_k \\ \text{or} \\ p \leq x_k}} p, \tag{20}$$

where K is the largest integer such that $x_K \geq 2$. In particular, if N is the largest CA number of parameter ε, then

$$F(p, 1) = \varepsilon \Rightarrow p(N) = p, \tag{21}$$

where $p(N)$ is the largest prime factor of N. Therefore for any ε, formula (20) gives all possible values of a CA number N of parameter ε [7].

Robin [24, Proposition 1] proved that the maximum order of the function f defined in (2) is attained by CA numbers. More precisely, if $3 \leq N < n < N'$, where N and N' are two successive CA numbers, then

$$f(n) \leq \max\{f(N), f(N')\}.$$

In the next proposition we improve the above inequality to a strict one.

Proposition 20. *Let $3 \leq N < n < N'$, where N and N' are two successive CA numbers. Then*

$$f(n) < \max\{f(N), f(N')\}. \tag{22}$$

Proof. In fact, due to the strict convexity of the function $t \mapsto \varepsilon t - \log\log t$, Robin's proof extends to the strict inequality (22). □

Proposition 20 shows that if there is a counterexample to inequality (1), then there exists at least one CA number that violates it.

Lemma 21. *Let $N < N'$ be two consecutive CA numbers. If there exists an XA number $n > 10080$ satisfying $N < n < N'$, then N' is also an XA.*

Proof. Let us set

$$B = \{m \in XA : \ N < m < N'\}.$$

By assumption $n \in XA$, we have $B \neq \emptyset$. Let $n' = \max B$. Since $n' \in XA$ and $n' > N$, it follows that $f(n') > f(N)$. From inequality (22) we must have $f(n') < f(N')$. Hence $N' \in XA$. □

Remark 22. If $n = 10080$, then we have $N = 5040$ and $N' = 55440$, and

$$f(N) \approx 1.7909, \qquad f(n) \approx 1.7558, \qquad f(N') \approx 1.7512.$$

Hence, $f(N') < f(n) < f(N)$ and inequality (22) is satisfied with

$$f(n) < f(N) = \max\{f(N), \ f(N')\}.$$

Therefore $N' \notin XA$. The point here is that $n = 10080$ is the initial XA number, so that it misses the property (8) of the definition of XA numbers which is used in the proof of Lemma 21.

Theorem 23. *If RH holds, then there exist infinitely many CA numbers that are also XA.*

Proof. If RH holds, then $\#XA = \infty$ by Theorem 7. Let $n \in XA$. Since $\#CA = \infty$ [2, 10], there exist two successive CA numbers $N, \ N'$ such that $N < n \leq N'$. If $N' = n$ then it is already in XA, otherwise N' belongs to XA via Lemma 21. □

It can be seen that there exist infinitely many CA numbers N for which the largest prime factor p $(= p(N))$ is greater than $\log N$. For this purpose we use the following lemma.

Lemma 24 ([7, Lemma 3]). *Let N be a CA number of parameter ε with*

$$\varepsilon < F(2,1) = \log(3/2)/\log 2$$

and define $x = x(\varepsilon)$ by (19). Then

(i) *for some constant $c > 0$*

$$\log N \leq \vartheta(x) + c\sqrt{x}.$$

(ii) *Moreover, if N is the largest CA number of parameter ε, then*

$$\vartheta(x) \leq \log N \leq \vartheta(x) + c\sqrt{x}.$$

Theorem 25. *There are infinitely many CA numbers N_ε such that $\log N_\varepsilon < p(N_\varepsilon)$.*

Proof. Let p be sufficiently large satisfying the inequality (10), and let N_ε be the largest CA number of parameter

$$\varepsilon = F(p,1).$$

Then from (21) it follows that $p(N_\varepsilon) = p$. By part (ii) of Lemma 24 we have

$$\log N_\varepsilon - \vartheta(p) < c\sqrt{p}, \qquad \text{(for some } c > 0\text{)}.$$

On the other hand, by Lemma 10 there exists a constant $c' > 0$ such that

$$\vartheta(p) - p < -c'\sqrt{p}\log\log\log p, \qquad (c' > 0).$$

Hence

$$\log N_\varepsilon - p < \{c - c'\log\log\log p\}\sqrt{p} < 0,$$

which is the desired conclusion. □

4.3 XA Numbers

We return to XA numbers and present some of their properties. We begin by the first interesting property of the XA numbers whose proof is essentially an application of the definition of XA numbers.

Theorem 26. *Let $n \in XA$. Then*

$$p(n) < \log n.$$

Proof. For $n = 10080$ we have

$$p(10080) = 7 < 9.218 < \log(10080).$$

Let $n > 10080$ be an XA number and $m = n/p(n)$. Then $m > 10080$, since for all primes p we have $\vartheta(p) > p/3$ (Corollary 9). Thus for $n \in SA$ we have

$$\log n \geq \psi(p(n)) \geq \vartheta(p(n)) > \frac{1}{3}p(n)$$

and $m = n/p(n) > n/(3\log n) > 10080$ if $n \geq 400,000$. For $n < 400,000$ we can check by computation. Hence by Definition 3 we obtain

$$1 + \frac{1}{p(n)} = \frac{\sigma(n)/n}{\sigma(m)/m} > \frac{\log\log n}{\log\log m}.$$

Therefore,

$$\frac{1}{p(n)} > \frac{\log\log n}{\log\log m} - 1 = \frac{\log(1 + \log p(n)/\log m)}{\log\log m}.$$

Using inequality (11) we have

$$\frac{1}{p(n)} > \frac{\log p(n)}{\log n \log\log m} > \frac{\log p(n)}{\log n \log\log n} \Rightarrow p(n) < \log n.$$

□

We mention here a similar result proved by Choie et al. [9].

Proposition 27 ([9, Lemma 6.1])**.** *Let $t \geq 2$ be fixed. Suppose that there exists a t-free integer exceeding* 5040 *that does not satisfy Robin's inequality (1). Let n be the smallest such integer. Then $p(n) < \log n$.*

In Theorem 23 we showed that if RH holds, then there exist infinitely many CA numbers that are also XA. Next theorem is a conclusion of Theorems 25 and 26 which is independent of the RH.

Theorem 28. *There exist infinitely many CA numbers that are not XA.*

We conclude this subsection with a result describing the structure of (sufficiently large) XA numbers. More precisely, the next theorem will determine the exponents of the prime factors of a (sufficiently large) XA number with an error at most 1.

Theorem 29. *Let $n = 2^{k_2} \cdots q^{k_q} \cdots p \in XA$. Set*

$$\alpha_q(p) = \left\lfloor \log_q \left(1 + (q-1)\frac{p\log p}{q\log q}\right)\right\rfloor.$$

Then for sufficiently large $n \in XA$ we have $|k_q - \alpha_q(p)| \leq 1$.

Proof. Assume that $k_q = k$ and $k - \alpha_q(p) \geq 2$. Then we have

$$q^k \geq q^{\alpha_q(p)+2} > q\left(1 + (q-1)\frac{p \log p}{q \log q}\right). \tag{23}$$

Now let us compare $f(n)$ with $f(m)$, where $m = n/q$. Since $n \in XA$ we must have

$$\frac{\sigma(n)/n}{\sigma(m)/m} = \frac{q^{k+1}-1}{q^{k+1}-q} > \frac{\log\log n}{\log\log m},$$

or using inequality (11),

$$q^k < 1 + (q-1)\frac{\log n \log\log m}{q \log q}. \tag{24}$$

Comparison of (23) and (24) gives that

$$\log n \log\log m - qp \log p > q \log q.$$

This contradicts inequality (16). Now we assume that $k - \alpha_q(p) \leq -2$. Then

$$\frac{q^{k+2}-1}{q-1} \leq \frac{p \log p}{q \log q}. \tag{25}$$

Choose $m = nq/p$. Comparing $f(n)$ with $f(m)$ we have

$$\frac{\sigma(n)/n}{\sigma(m)/m} = \left(1 - \frac{q-1}{q^{k+2}-1}\right)\left(1 + \frac{1}{p}\right) > \frac{\log\log n}{\log\log m} > 1 + \frac{\log p/q}{\log n \log\log m}.$$

or simply by (25)

$$-\frac{q \log q}{\log p}\left(1 + \frac{1}{p}\right) > \frac{p \log p/q}{\log n \log\log m} - 1.$$

Hence by (16) we have

$$-\frac{q \log q}{\log p}\left(1 + \frac{1}{p}\right) > \frac{\log p/q}{(\log p)\left(1 + \frac{1}{2\log p}\right)\left(1 + \frac{1}{2\log^2 p}\right)} - 1,$$

which is false for all $2 \leq q \leq p$. $\square$

Remark 30. Note that if inequality (17) holds for XA numbers $n \geq s_{365}$, then by performing computations for two smaller values of XA numbers, i.e., s_{20} and s_{356} (cf. Table 1) we see that the above theorem holds true for all $n \in XA$.

5 Fragility of the RH and certain supersets of XA numbers

In Theorem 7 we proved that under the RH the cardinality of the set of XA numbers is infinite. Here we present some interesting theorems which demonstrate the fragility of the RH showing the infinitude of some supersets of XA numbers independent of the RH. These sets are defined by inequalities quite close to that in (8). The basic inequalities used here to define these sets are (11) and (12).

Lemma 31. *If $m \geq 3$, then there exists $n > m$ such that*

$$\frac{\sigma(n)/n}{\sigma(m)/m} > 1 + \frac{\log n/m}{\log n \log\log m}.$$

Proof. Let $m \geq 3$. Then by inequality (5)

$$\frac{\sigma(m)}{m} \leq \left(e^{\gamma} + \frac{0.648214}{(\log\log m)^2}\right) \log\log m. \tag{26}$$

Since for $m' > m$

$$\frac{\log\log m}{\log\log m'} \left(1 + \frac{\log m'/m}{\log m' \log\log m}\right) < 1$$

and the left-hand side is decreasing with respect to m' and tends to zero as $m' \to \infty$, then for some $m' > m$ we have

$$\frac{\log\log m}{\log\log m'} \left(1 + \frac{\log m'/m}{\log m' \log\log m}\right) \left(e^{\gamma} + \frac{0.648214}{(\log\log m)^2}\right) = e^{\gamma} - \varepsilon, \tag{27}$$

where $\varepsilon > 0$ is arbitrarily small and fixed. Hence by Gronwall's theorem there exists $n \geq m'$ such that

$$\begin{aligned}
\frac{\sigma(n)}{n} &> (e^{\gamma} - \varepsilon) \log\log n \\
&= \frac{\log\log m}{\log\log m'} \left(1 + \frac{\log m'/m}{\log m' \log\log m}\right) \left(e^{\gamma} + \frac{0.648214}{(\log\log m)^2}\right) \log\log n \\
&\geq \left(1 + \frac{\log n/m}{\log n \log\log m}\right) \frac{\sigma(m)}{m},
\end{aligned}$$

where the last inequality holds by (26) and (27). □

Definition 32. Let $n_1 = 10080$, and let n_{k+1} be the least integer greater than n_k such that

$$\frac{\sigma(n_{k+1})/n_{k+1}}{\sigma(n_k)/n_k} > 1 + \frac{\log n_{k+1}/n_k}{\log n_{k+1} \log\log n_k}, \qquad (k = 1, 2, \ldots).$$

We define X' to be the set of all $n_1, n_2, n_3, \ldots$.

One can easily show that

$$XA \subset X' \subset SA. \tag{28}$$

Our first result towards the fragility of the RH is the following theorem.

Theorem 33. *The set X' has an infinite number of elements.*

Proof. If the RH is true, then the cardinality of X' is infinite by (28). If RH is not true, then by Theorem 7 there exists $m_0 \geq 10080$ such that

$$\frac{\sigma(m_0)/m_0}{\sigma(m)/m} \geq \frac{\log\log m_0}{\log\log m}, \qquad \text{for all } m \geq 10080.$$

By Lemma 31 there exists $m' > m_0$ such that m' satisfies

$$\frac{\sigma(m')/m'}{\sigma(m_0)/m_0} > 1 + \frac{\log m'/m_0}{\log m' \log\log m_0}.$$

Let n be the least number greater than m_0 for which

$$\frac{\sigma(n)/n}{\sigma(m_0)/m_0} > 1 + \frac{\log n/m_0}{\log n \log\log m_0}.$$

Hence $n \in X'$. $\square$

The following lemma can be proved in the same manner as Lemma 31.

Lemma 34. *If $m \geq 3$, then there exists $n > m$ such that*

$$\frac{\sigma(n)/n}{\sigma(m)/m} > 1 + \frac{2\log n/m}{(\log m + \log n)\log\log m}.$$

We continue our approach towards the fragility of the RH via (the stronger) inequality (12) defining a smaller superset of XA numbers as follows.

Definition 35. Let $n_1 = 10080$, and let n_{k+1} be the least integer greater than n_k, such that

$$\frac{\sigma(n_{k+1})/n_{k+1}}{\sigma(n_k)/n_k} > 1 + \frac{2\log n_{k+1}/n_k}{(\log n_k + \log n_{k+1})\log\log n_k}, \qquad (k = 1,\, 2,\, \ldots).$$

We define X'' to be the set of all $n_1,\, n_2,\, n_3,\, \ldots$.

By elementary inequality (12) and

$$\frac{t}{1+t} < \frac{2t}{2+t}, \qquad (t > 0)$$

one can easily show the inclusion $XA \subset X'' \subset X'$. The following theorem is a refinement of Theorem 33 with a similar proof.

Theorem 36. *The set X'' has an infinite number of elements.*

We calculated the number of elements in XA, X' and X'' up to the $300,000$th element of SA in [20]. Note that

$$\#XA = 9240, \qquad \#X' = 9535, \qquad \#X'' = 9279$$

and

$$\#(X' - XA) = 295, \qquad \#(X'' - XA) = 39.$$

It might be interesting to look at the list of elements of $X'' - XA$ up to $s_{300,000}$:

$$\begin{aligned} X'' - XA = \{ & s_{55}, s_{62}, s_{91}, s_{106}, s_{116}, s_{127}, s_{128}, s_{137}, s_{138}, s_{149}, s_{181}, s_{196}, s_{212}, s_{219}, \\ & s_{224}, s_{231}, s_{232}, s_{246}, s_{247}, s_{259}, s_{260}, s_{263}, s_{272}, s_{273}, s_{276}, s_{288}, s_{294}, \\ & s_{299}, s_{305}, s_{311}, s_{317}, s_{330}, s_{340}, s_{341}, s_{343}, s_{354}, s_{65343}, s_{271143}, s_{271151} \} \end{aligned}$$

Note that the second XA number is s_{356} (see Table 1) and only three out of 39 elements in the set $X'' - XA$ up to $s_{300,000}$, namely s_{65343}, s_{271143} and s_{271151}, are greater than s_{356}.

6 Numerical experiments

We present here some numerical results, mainly for the set of XA numbers (sorted in increasing order) up to 13770th element, which is less than $C_1 := s_{500,000}$, based on the list provided by Noe [20].

Property 37. *Let $n = 2^{k_2} \cdots q^{k_q} \cdots r^{k_r} \cdots p \in XA$, where $2 \le q < r \le p$. Then for* $10080 < n \le C_1$

(i) $\log n < q^{k_q+1}$,

(ii) $r^{k_r} < q^{k_q+1} < r^{k_r+2}$,

(iii) $q^{k_q} < k_q p$,

(iv) $q^{k_q} \log q < \log n \log\log n < q^{k_q+2}$.

Property 38. *Let $n = 2^{k_2} \cdots x_k^k \cdots p \in XA$, where $2 < x_k < p$ is the greatest prime factor of exponent k. Then*

$$\sqrt{p} < x_2 < \sqrt{2p}, \qquad \textit{for} \;\; 10080 < n \le C_1.$$

Property 39. *Let $n = 2^{k_2} \cdots q^{k_q} \cdots p$ and $n' = 2^{k'_2} \cdots q^{k'_q} \cdots p'$ be two consecutive XA numbers. Then for* $10080 < n < n' \le C_1$

$$|k_q - k'_q| \le 1, \qquad \textit{for all} \;\; 2 \le q \le p'.$$

Property 40. *If m, n are XA numbers, then for* $10080 \leq m < n \leq C_1$

(i) $p(m) \leq p(n)$,

(ii) $d(m) \leq d(n)$.

Remark 41. We note that Property 40 is not true for SA numbers. For instance,

$$s_{47} = (19\sharp)(3\sharp)^2 2, \qquad s_{48} = (17\sharp)(5\sharp)(3\sharp)2^3,$$
$$p(s_{47}) = 19 > 17 = p(s_{48}).$$

and

$$s_{173} = (59\sharp)(7\sharp)(5\sharp)(3\sharp)^2 2^3, \qquad s_{174} = (61\sharp)(7\sharp)(3\sharp)^2 2^2,$$
$$d(s_{173}) = 5308416 > 5160960 = d(s_{174}).$$

Property 42. *If n, $n' \in XA$ are consecutive, then for* $10080 \leq n < n' < C_1$

$$\frac{n'}{n} > 1 + c\,\frac{(\log\log n)^2}{\log n}, \qquad (0 < c \leq 4),$$

$$\frac{n'}{n} > 1 + c\,\frac{(\log\log n)^2}{\sqrt{\log n}}, \qquad (0 < c \leq 0.195).$$

Property 43. *If n, $n' \in XA$ are consecutive, then for* $10080 \leq n < n' < C_1$

$$\frac{f(n')}{f(n)} < 1 + \frac{1}{p'},$$

where p' is the largest prime factor of n'.

We have checked the following properties up to $C_2 = s_{250,000}$ and up to 8150th element of XA numbers which is less than C_2.

Property 44. *If n, $n' \in SA$ are consecutive, then*

$$\frac{\sigma(n')/n'}{\sigma(n)/n} < 1 + \frac{1}{p'}, \qquad (n' < C_2),$$

where p' is the largest prime factor of n'.

The number of distinct prime factors of a number n is denoted by $\omega(n)$ (see [25]).

Property 45. *If n, $n' \in XA$ are consecutive, then for* $10080 \leq n < n' < C_2$*, then*

$$\frac{n}{\omega(n)} \leq \frac{n'}{\omega(n')}.$$

The comparison of the sets CA and XA is given. We calculated them up to $C = s_{1,000,000}$ from the list of SA in [20].

$$\begin{aligned}
&\#\{n \in XA : n < C\} = 24875,\\
&\#\{n \in CA : n < C\} = 21187,\\
&\#\{n \in CA \cap XA : n < C\} = 20468,\\
&\#\{n \in XA \setminus CA : n < C\} = 4407,\\
&\#\{n \in CA \setminus XA : n < C\} = 719,
\end{aligned}$$

We conclude this paper with another remark on choosing the first element of XA as 10080.

Remark 46. If we replace $n_1 = 10080$, the initial number in the definition of XA numbers, by $n_1 = 665280$, we do not need to pose the condition (i.e., $n > n_1$) in Lemma 21. Indeed, if we choose the initial number $n_1 = 665280$, then $N = 55440 < 665280 < 720720 = N'$, where in this case N' is also an XA number. Therefore we do not need Remark 22. Moreover, if we choose $n_1 = 665280$, then there are 37 more XA numbers.

	n	Type	$f(n)$	$p(n)$	$\log n$	k_2
1	$s_{20} = (7\sharp)(3\sharp)2^3 = 10080$		1.75581	7	9.21831	5
2	$s_{356} = (113\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^3$	c	1.75718	113	126.444	8
3	$s_{368} = (127\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^3$	c	1.75737	127	131.288	8
4	$s_{380} = (131\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^3$	c	1.75764	131	136.163	8
5	$s_{394} = (137\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^3$	c	1.75778	137	141.083	8
6	$s_{408} = (139\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^3$	c	1.75821	139	146.018	8
7	$s_{409} = (139\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^4$	c	1.75826	139	146.711	9
8	$s_{438} = (151\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^3$		1.75831	151	156.039	8
9	$s_{440} = (151\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^4$	c	1.75849	151	156.732	9
10	$s_{444} = (151\sharp)(13\sharp)(7\sharp)(3\sharp)^2 2^4$	c	1.75860	151	158.678	9
11	$s_{455} = (157\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^4$		1.75864	157	161.788	9
12	$s_{458} = (157\sharp)(13\sharp)(7\sharp)(3\sharp)^2 2^3$		1.75866	157	163.041	8
13	$s_{459} = (157\sharp)(13\sharp)(7\sharp)(3\sharp)^2 2^4$	c	1.75892	157	163.734	9
14	$s_{476} = (163\sharp)(13\sharp)(7\sharp)(3\sharp)^2 2^4$	c	1.75914	163	168.828	9
15	$s_{486} = (163\sharp)(17\sharp)(7\sharp)(3\sharp)^2 2^4$		1.75918	163	171.661	9
16	$s_{493} = (167\sharp)(13\sharp)(7\sharp)(3\sharp)^2 2^4$	c	1.75943	167	173.946	9
17	$s_{502} = (167\sharp)(17\sharp)(7\sharp)(3\sharp)^2 2^4$	c	1.75966	167	176.779	9
18	$s_{519} = (173\sharp)(17\sharp)(7\sharp)(3\sharp)^2 2^4$	c	1.76006	173	181.933	9
19	$s_{537} = (179\sharp)(17\sharp)(7\sharp)(3\sharp)^2 2^4$	c	1.76038	179	187.120	9
20	$s_{555} = (181\sharp)(17\sharp)(7\sharp)(3\sharp)^2 2^4$	c	1.76089	181	192.318	9

Table 1: First 20 extremely abundant numbers ($p_k\sharp := \prod_{j=1}^{k} p_j$ and c represents a colossally abundant number).

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