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A Combinatorial Proof of the Log-Convexity of Catalan-Like Numbers

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Abstract

The Catalan-like numbers $c_{n,0}$, defined by

$$c_{n+1,k} = r_{k-1}c_{n,k-1} + s_k c_{n,k} + t_{k+1}c_{n,k+1} \text{ for } n, k \geq 0,$$
$$c_{0,0} = 1, c_{0,k} = 0 \text{ for } k \neq 0,$$

unify a substantial amount of well-known counting coefficients. Using an algebraic approach, Zhu showed that the sequence $(c_{n,0})_{n\geq 0}$ is log-convex if $r_k t_{k+1} \leq s_k s_{k+1}$ for all $k \geq 0$. Here we give a combinatorial proof of this result from the point of view of weighted Motzkin paths.

1 Introduction

A sequence $a_0, a_1, a_2, \ldots$ of nonnegative numbers is called *log-convex* if $a_n^2 \leq a_{n+1}a_{n-1}$ for all $n \geq 1$. Many well-known combinatorial sequences are log-convex (see Liu and Wang [9] for instance). One of the simplest classes of log-convex sequences, yet still large enough to include many important instance, is the class of the Catalan-like numbers. The Catalan-like numbers $c_{n,0}$, introduced by Aigner [1, 2], are defined by the recursive system

$$c_{n+1,k} = r_{k-1}c_{n,k-1} + s_k c_{n,k} + t_{k+1}c_{n,k+1} \text{ for } n, k \geq 0, \tag{1}$$
$$c_{0,0} = 1, c_{0,k} = 0 \text{ for } k \neq 0,$$

where $(r_n)_{n\geq 0}$, $(s_n)_{n\geq 0}$ and $(t_n)_{n\geq 1}$ are three sequences of nonnegative numbers. The Catalan-like numbers unify many well-known counting coefficients. For example, let $r_k \equiv 1, \sigma = (s_0, s_1, \ldots)$ and $\tau = (t_1, t_2, \ldots)$ in (1). Then $c_{n,0}$ are

(1) the Catalan numbers C_n if $\sigma = (1, 2, 2, \ldots)$ and $\tau = (1, 1, 1, \ldots)$;

(2) the Motzkin numbers M_n if $\sigma = \tau = (1, 1, 1, \ldots)$;

(3) the middle binomial coefficients $\binom{2n}{n}$ if $\sigma = (2, 2, 2, \ldots)$ and $\tau = (2, 1, 1, \ldots)$;

(4) the Schröder numbers S_n if $\sigma = (2, 3, 3, \ldots)$ and $\tau = (2, 2, 2, \ldots)$;

(5) the Bell numbers B_n if $\sigma = \tau = (1, 2, 3, 4, \ldots)$;

(6) the (restricted) hexagonal numbers H_n if $\sigma = (3, 3, 3, \ldots)$ and $\tau = (1, 1, 1, \ldots)$;

(7) the Fine numbers F_n if $\sigma = (0, 2, 2, \ldots)$ and $\tau = (1, 1, 1, \ldots)$;

(8) the Riordan numbers R_n if $\sigma = (0, 1, 1, \ldots)$ and $\tau = (1, 1, 1, \ldots)$.

Recently, Zhu [14] established the following criterion for the log-convexity of the Catalan-like numbers.

Theorem 1. ([14, Theorem 3.1]) *Assume that $r_k t_{k+1} \leq s_k s_{k+1}$ for all $k \geq 0$. Then the sequence $(c_{n,0})_{n\geq 0}$ defined recursively by (1) is log-convex.*

Theorem 1 leads to the log-convexity of many well-known combinatorial sequences, including the Catalan numbers, the Motzkin numbers, the Bell numbers, the middle binomial coefficients, the (restricted) hexagonal numbers, the Schröder numbers, and so on (see [14] for details). On the other hand, Callan [3] gave an injective proof for the log-convexity of the Motzkin numbers. Liu and Wang [9] did the same for the Catalan numbers, and asked for a combinatorial proof for the log-convexity of the Bell numbers. The object of the present paper is to give a combinatorial proof for Theorem 1 from the point of view of weighted Motzkin paths.

A *Motzkin path* of length n is a lattice path from $(0, 0)$ to $(n, 0)$ consisting of up steps $U = (1, 1)$, down steps $D = (1, -1)$ and horizontal steps $H = (1, 0)$ that never falls below the x-axis. The *height* of a step in a Motzkin path is the y coordinate of the starting point. We assign a weight r_k (s_k, t_k, resp.) to all up steps (all horizontal steps, all down steps, resp.) of height k. Define the *weight of a Motzkin path* to be the product of weights of its steps. Similarly, we may define a general Motzkin path from $(0, 0)$ to (n, k) for $n \geq k$ (see [7] for instance). Then $c_{n,k}$ defined by (1) is the sum of weights of all paths from $(0, 0)$ to (n, k). In particular, the Catalan-like number

$$c_{n,0} = \sum_{P\in\mathcal{M}_n} w(P), \tag{2}$$

where $\mathcal{M}_n$ denotes the set of the Motzkin paths of length n [2].

In the next section we give a combinatorial proof of Theorem 1 by means of (2). In Section 3 we point out that the same idea may be applied to show the q-version of Theorem 1. We also propose a couple of problems.

2 Combinatorial proof of Theorem 1

To show the log-convexity of the sequence $(c_{n,0})_{n\geq 0}$, it suffices to show that

$$\sum_{P\in\mathcal{M}_n} w(P) \sum_{P\in\mathcal{M}_n} w(P) \leq \sum_{P\in\mathcal{M}_{n+1}} w(P) \sum_{P\in\mathcal{M}_{n-1}} w(P),$$

or equivalently,

$$\sum_{(P_1,P_2)\in(\mathcal{M}_n,\mathcal{M}_n)} w(P_1,P_2) \leq \sum_{(P_1',P_2')\in(\mathcal{M}_{n+1},\mathcal{M}_{n-1})} w(P_1',P_2'), \tag{3}$$

where $w(Q_1,Q_2) = w(Q_1)w(Q_2)$ denotes the weight of a pair of paths (Q_1,Q_2).

Let $\mathcal{S}$ be a set of pairs of paths. Denote

$$w(\mathcal{S}) = \sum_{(Q_1,Q_2)\in\mathcal{S}} w(Q_1,Q_2).$$

Then (3) is equivalent to

$$w(\mathcal{M}_n,\mathcal{M}_n) \leq w(\mathcal{M}_{n+1},\mathcal{M}_{n-1}). \tag{4}$$

Recall that the Motzkin number M_n is precisely the Catalan-like number $c_{n,0}$ when the all weights r_k, s_k, t_k equal to 1. In other words, M_n counts the Motzkin paths of length n. Callan [3] proved the log-convexity of the Motzkin numbers by establishing an injection from $(\mathcal{M}_n,\mathcal{M}_n)$ to $(\mathcal{M}_{n+1},\mathcal{M}_{n-1})$. We rewrite his proof as follows for our purposes.

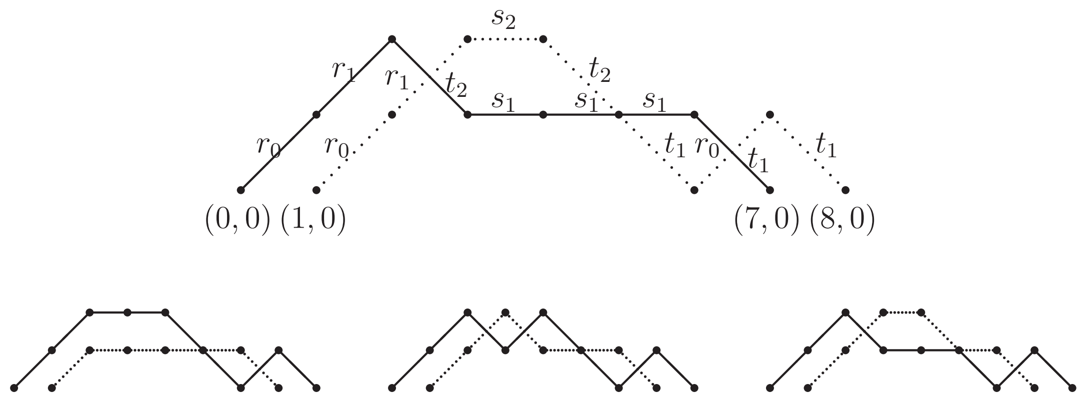
For each pair of Motzkin paths (Q_1,Q_2) in $(\mathcal{M}_n,\mathcal{M}_n)$ or $(\mathcal{M}_{n+1},\mathcal{M}_{n-1})$, place the first beginning from $(0,0)$ and the second beginning from $(1,0)$. Callan introduced the following operation on (Q_1,Q_2) at each possible "close encounter" to obtain a new pair of Motzkin paths.

1. Assume that Q_1 and Q_2 intersect at a lattice point. Then reassign the two initial segments to the other path.

2. Assume that Q_1 and Q_2 intersect at the center point of crossing diagonal steps. Then swing the crossing steps $45°$ so they become horizontal.

3. Assume that Q_1 and Q_2 possess a pair of flatsteps forming the top and bottom of a unit square. Then change the lower horizontal step to an upstep and the upper one to a downstep.

For example, for a pair of Motzkin paths (P_1, P_2) in $(\mathcal{M}_n, \mathcal{M}_n)$, we place P_1 from $(0,0)$ to $(n,0)$ and P_2 from $(1,0)$ to $(n+1,0)$. Clearly, P_1 and P_2 must intersect, so they have at least one close encounter. Callan's operation at each close encounter of (P_1, P_2) will lead to a pair of Motzkin paths in $(\mathcal{M}_{n+1}, \mathcal{M}_{n-1})$.



(1) (2) (3)

Figure 1. Callan's operation on a pair of Motzkin paths.

The above row in Figure 1 is a pair of Motzkin paths of length 7 with one lattice point of intersection, two pairs of crossing steps and one pair of flatsteps. The below low is obtained by carrying out Callan's operation at the first three close encounters of the pair of Motzkin paths.

Now for $(P_1, P_2) \in (\mathcal{M}_n, \mathcal{M}_n)$, let (P_1', P_2') be the pair of paths obtained by carrying out Callan's operation at the first close encounter of P_1 and P_2. Note that the location of the first close encounter will remain invariant. Hence the mapping

$$\tau : \quad (P_1, P_2) \mapsto (P_1', P_2')$$

is an injection from $(\mathcal{M}_n, \mathcal{M}_n)$ to $(\mathcal{M}_{n+1}, \mathcal{M}_{n-1})$. This gives a combinatorial interpretation of the log-convexity of the Motzkin numbers.

Note that the inequality $w(P_1, P_2) \le w(\tau(P_1, P_2))$ does not hold in general. For example, if the first close encounter of (P_1, P_2) is a pair of flatsteps with heights k and $k+1$ respectively, then $\tau(P_1, P_2)$ is the pair of paths obtained by changing this pair of flatsteps of (P_1, P_2) to a pair of crossing steps. Thus $w(P_1, P_2) = s_k s_{k+1} \overline{w} \ge r_k t_{k+1} \overline{w} = w(\tau(P_1, P_2))$, where $\overline{w}$ is the product of weights of steps in (P_1, P_2) except that pair of flatsteps. Hence Callan's injection τ can not be used to prove the inequality (4). However, we may still prove (4) by means of Callan's operation.

We divide $(\mathcal{M}_n, \mathcal{M}_n)$ into two parts: $\mathscr{L}$ consists of pairs of paths that there is at least one lattice point of intersection and $\mathscr{C}$ consists of the remain pairs. Let $(\mathcal{M}_{n+1}, \mathcal{M}_{n-1})^*$ consist of pairs of paths in $(\mathcal{M}_{n+1}, \mathcal{M}_{n-1})$ that there is at least one close encounter and divide $(\mathcal{M}_{n+1}, \mathcal{M}_{n-1})^*$ into two parts $\mathscr{L}^*$ and $\mathscr{C}^*$ similarly. We show the inequality (4) by showing that

$$w(\mathscr{L}) = w(\mathscr{L}^*) \tag{5}$$

and

$$w(\mathscr{C}) \leq w(\mathscr{C}^*). \tag{6}$$

Consider a pair of paths $(P_1, P_2) \in \mathscr{L}$. Let p be their first lattice point of intersection. Assume that p splits P_1 into parts P_{11} and P_{12}, and splits P_2 into parts P_{21} and P_{22}. Then the concatenation P_1^* of P_{11} and P_{22} is a path of length $n+1$, and the concatenation P_2^* of P_{12} and P_{21} is a path of length $n-1$. Clearly, $(P_1^*, P_2^*) \in \mathscr{L}^*$ and p is also their first lattice point of intersection. Furthermore,

$$w(P_1^*, P_2^*) = w(P_{11})w(P_{22})w(P_{12})w(P_{21}) = w(P_1, P_2).$$

So the mapping

$$\tau^*: \quad (P_1, P_2) \mapsto (P_1^*, P_2^*)$$

is a weight-preserving bijection from $\mathscr{L}$ to $\mathscr{L}^*$. Thus the equality (5) follows.

It remains to show that the inequality (6) holds. Let $(P_1, P_2) \in \mathscr{C}$. Then each close encounter of (P_1, P_2) is a pair of crossing steps or a pair of flatsteps. Define the *height* of such a close encounter as the smaller of heights of two corresponding steps. Callan's operation changes a pair of crossing steps to a pair of flatsetps of the same height, and vice versa. If (P_1', P_2') is obtained by carrying out Callan's operation on (P_1, P_2) at a pair crossing steps of height k, then $w(P_1, P_2) = s_k s_{k+1}\overline{w}$ and $w(P_1', P_2') = r_k t_{k+1}\overline{w}$, where $\overline{w}$ is the product of weights of steps in (P_1, P_2) except that pair of crossing steps, and vice versa.

Denote by $\mathcal{O}(P_1, P_2)$ and $\mathcal{E}(P_1, P_2)$ the sets of pairs of paths obtained by carrying out Callan's operation on (P_1, P_2) at odd and even close encounters respectively. Then $\mathcal{O}(P_1, P_2) \subseteq (\mathcal{M}_{n+1}, \mathcal{M}_{n-1})$ and $\mathcal{E}(P_1, P_2) \subseteq (\mathcal{M}_n, \mathcal{M}_n)$. Let $\{i_1, i_2, \ldots, i_m\}$ be the set of heights of close encounters of (P_1, P_2). Note that each pair of paths in $\mathcal{O}(P_1, P_2)$ and $\mathcal{E}(P_1, P_2)$ has even and odd pairs of crossing steps respectively. Hence

$$\begin{aligned} w(\mathcal{O}(P_1, P_2)) - w(\mathcal{E}(P_1, P_2)) &= \sum_{(Q_1,Q_2)\in\mathcal{O}(P_1,P_2)} w(Q_1, Q_2) - \sum_{(Q_1',Q_2')\in\mathcal{E}(P_1,P_2)} w(Q_1', Q_2') \\ &= (S_{i_1} - R_{i_1})(S_{i_2} - R_{i_2}) \cdots (S_{i_m} - R_{i_m})W, \end{aligned}$$

where $R_k = r_k t_{k+1}, S_k = s_k s_{k+1}$ and W is the product of weights of steps in (P_1, P_2) except those close encounters.

For example, let (P_1, P_2) be the first pair of paths in Figure 2. Then

$$\mathcal{E}(P_1, P_2) = \{(P_1, P_2), (Q_1, Q_2)\}, \quad O(P_1, P_2) = \{(P_1', P_2'), (Q_1', Q_2')\}.$$

So

$$\begin{aligned} w(\mathcal{O}(P_1, P_2)) - w(\mathcal{E}(P_1, P_2)) &= [w(P_1', P_2') + w(Q_1', Q_2')] - [w(P_1, P_2) + w(Q_1, Q_2)] \\ &= (R_0R_1 + S_0S_1 - S_0R_1 - R_0S_1)r_0^2 r_1 t_0^2 t_1 \\ &= (S_0 - R_0)(S_1 - R_1)r_0^2 r_1 t_0^2 t_1. \end{aligned}$$

(P_1, P_2) (Q_1, Q_2) (P_1', P_2') (Q_1', Q_2')

Figure 2.

It follows that $w(\mathcal{E}(P_1, P_2)) \leq w(\mathcal{O}(P_1, P_2))$ since $R_k \leq S_k$ for all $k \geq 0$ by the assumption in Theorem 1. Clearly, Callan's operation induces an equivalence relation on $\mathscr{C}$ and $\mathcal{O}(P_1, P_2)$ is an equivalence class. So we have

$$w(\mathscr{C}) = \sum w(\mathcal{E}(P_1, P_2)) \leq \sum w(\mathcal{O}(P_1, P_2)) \leq w(\mathscr{C}^*),$$

where the summation runs over such equivalence classes of (P_1, P_2). This completes the proof of the inequality (6), as required.

3 Concluding remarks and further work

Our proof adapts to all Catalan-like numbers satisfying the assumption in Theorem 1, and in particular, to the Catalan numbers and the Bell numbers. A more interesting problem is to construct an injection $\sigma : (\mathcal{M}_n, \mathcal{M}_n) \to (\mathcal{M}_{n+1}, \mathcal{M}_{n-1})$ such that $w(P_1, P_2) \leq w(\sigma(P_1, P_2))$. The Catalan-like numbers have various combinatorial interpretations [1, 2, 12, 13], so it is natural to find out other combinatorial proof of Theorem 1. We refer the reader to [6, 8, 10, 11] for some related techniques and topics.

It is well known that the sequence $(a_n)_{n\geq 0}$ is log-convex if and only if $a_m a_n \leq a_{n+1}a_{m-1}$ for $n \geq m \geq 1$. So the inequality (4) is equivalent to the inequality

$$w(\mathcal{M}_m, \mathcal{M}_n) \leq w(\mathcal{M}_{n+1}, \mathcal{M}_{m-1}). \tag{7}$$

Place each pair of Motzkin paths in $(\mathcal{M}_m, \mathcal{M}_n)$ or $(\mathcal{M}_{n+1}, \mathcal{M}_{m-1})$ such that the first begins from $(0, 0)$ and the second ends at $(n + 1, 0)$. Then the inequality (7) can be showed by the same technique used in the proof of the inequality (4).

Zhu [14] also gave the q-version of Theorem 1. For two polynomials $f(q)$ and $g(q)$, denote $f(q) \leq_q g(q)$ if the difference $g(q) - f(q)$ has only nonnegative coefficients. Let $(f_n(q))_{n\geq 0}$ be a polynomial sequence. It is called *q-log-convex* if $f_n^2(q) \leq_q f_{n+1}(q)f_{n-1}(q)$ for all $n \geq 1$. It is called *strongly q-log-convex* if $f_m(q)f_n(q) \leq_q f_{n+1}(q)f_{m-1}(q)$ for any $n \geq m \geq 1$. It is well known that the strong q-log-convexity implies the q-log-convexity but the converse is not true (see [5] for instance). The method of proof used in the log-convexity can be carried over verbatim to the q-log-convexity.

Theorem 2. ([14, Theorem 2.1]) *Let $(r_n(q))_{n\geq 0}$, $(s_n(q))_{n\geq 0}$ and $(t_n(q))_{n\geq 1}$ be three sequences of polynomials with nonnegative coefficients. Assume that the triangular array $(c_{n,k}(q))_{0\leq k\leq n}$*

satisfies the recurrence

$$c_{n+1,k}(q) = r_{k-1}(q)c_{n,k-1}(q) + s_k(q)c_{n,k}(q) + t_{k+1}(q)c_{n,k+1}(q) \text{ for } n, k \geq 0,$$
$$c_{0,0}(q) = 1, c_{0,k}(q) = 0 \text{ for } k \neq 0.$$

If $r_k(q)t_{k+1}(q) \leq_q s_k(q)s_{k+1}(q)$ *for all* $k \geq 0$*, then the sequence* $(c_{n,0}(q))_{n\geq 0}$ *is strongly* q*-log-convex.*

Liu and Wang [9] conjectured the q-log-convexity of sequences of Narayana polynomials of two types and Chen et al. [4, 5] proved them by means of the theory of symmetric functions. It immediately follows from Theorem 2 that such two sequences are actually strongly q-log-convex respectively. See Zhu [14] for details.

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