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# Pattern Popularity in Multiply Restricted Permutations

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**Abstract**

We derive explicit formulae or generating functions for the popularity of all the length-3 patterns in multiply restricted permutations, and provide combinatorial interpretations for some non-trivial equipopular patterns as well.

## 1 Introduction

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation in the symmetric group $S_n$. We say that $\sigma$ *contains a pattern* $q = q_1q_2\cdots q_k \in S_k$ if there exist $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that the entries $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ have the same relative order as the entries of $q$, i.e., $q_j < q_l$ if and only if $\sigma_{i_j} < \sigma_{i_l}$ whenever $1 \le j, l \le k$. We say that $\sigma$ avoids $q$ if $\sigma$ does not contain $q$ as a pattern. A permutation may contain multiple copies of a pattern. For example, permutation 43512 contains two copies of pattern 321, namely 431 and 432, but avoids pattern 123.

For a pattern $q$, let $S_n(q)$ denote the set of all permutations in $S_n$ that avoid the pattern $q$, and for $R \subseteq S_k$, let $S_n(R) = \bigcap_{q\in R} S_n(q)$ be the set of permutations in $S_n$ that avoid every pattern contained in $R$. For two permutations $\sigma$ and $q$, we set $f_q(\sigma)$ to be the number of copies of $q$ in $\sigma$ as a pattern. The *popularity* of pattern $q$ in $S_n(R)$ is defined as

$$f_q(S_n(R)) = \sum_{\sigma\in S_n(R)} f_q(\sigma).$$

| | | | |
|---|---|---|---|
| | $f_{213}(n) = (n-3)2^{n-2} + 1$ | | $\sum f_{231}(n)x^n = \sum f_{312}(n)x^n = \frac{x^3(1+2x)}{(1-x-x^2)^3}$ |
| $S_n(123, 132)$ | $f_{231}(n) = f_{312}(n) = (n^2 - 5n + 8)2^{n-3} - 1$ | $S_n(123, 132, 213)$ | $\sum f_{321}(n)x^n = \frac{x^3(1+6x+12x^2+8x^3)}{(1-x-x^2)^4}$ |
| | $f_{321}(n) = (n^3/3 - 2n^2 + 14n/3 - 5)2^{n-2} + 1$ | | $f_{213}(n) = f_{312}(n) = \binom{n}{3}$ |
| | $f_{123}(n) = (n-4)2^{n-1} + n + 2$ | $S_n(123, 132, 231)$ | $f_{321}(n) = (n-2)\binom{n}{3}$ |
| $S_n(132, 213)$ | $f_{231}(n) = f_{312}(n) = (\frac{n^2}{4} - \frac{7n}{4} + 4)2^n - n - 4$ | | $f_{123}(n) = f_{312}(n) = \binom{n+1}{4}$ |
| | $f_{321}(n) = (\frac{1}{12}n^3 - \frac{3}{4}n^2 + \frac{38}{12}n - 6)2^n + n + 6$ | $S_n(132, 213, 231)$ | $f_{321}(n) = \frac{1}{12}n(n-2)(n-1)^2$ |
| $S_n(132, 231)$ | $f_{123}(n) = f_{213}(n) = f_{312}(n) = f_{321}(n) = \frac{2^n}{8}\binom{n}{3}$ | | $f_{213}(n) = f_{231}(n) = \binom{n}{3}$ |
| $S_n(132, 312)$ | $f_{123}(n) = f_{213}(n) = f_{231}(n) = f_{321}(n) = \frac{2^n}{8}\binom{n}{3}$ | $S_n(123, 132, 312)$ | $f_{321}(n) = (n-2)\binom{n}{3}$ |
| | $f_{213}(n) = f_{231}(n) = f_{312}(n) = \binom{n+2}{5}$ | | $f_{132}(n) = f_{213}(n) = \binom{n+1}{4}$ |
| $S_n(132, 321)$ | $f_{123}(n) = \frac{7n^5}{120} - \frac{n^4}{3} + \frac{17n^3}{24} - \frac{2n^2}{3} + \frac{7}{30}$ | $S_n(123, 231, 312)$ | $f_{321}(n) = \frac{1}{12}n(n-2)(n-1)^2$ |

Table 1: Pattern popularity in doubly and triply restricted permutations.

We say that $p$ and $q$ are equipopular if $f_p(S_n(R)) = f_q(S_n(R))$ for all $n$.

The *complement* of $\sigma$ is given by $\sigma^c = (n+1-\sigma_1)(n+1-\sigma_2)\cdots(n+1-\sigma_n)$, its *reverse* is defined as $\sigma^r = \sigma_n \cdots \sigma_2\sigma_1$ and the *inverse* $\sigma^{-1}$ is the regular group-theoretic inverse permutation. For any set of permutations $R$, let $R^c$ be the set obtained by complementing each element of $R$, and the sets $R^r$ and $R^{-1}$ are defined analogously. It is well known that

**Lemma 1.** *Let $R \subseteq S_k$ be any set of permutations in $S_k$, and $\sigma \in S_n$, we have*

$$\sigma \in S_n(R) \Leftrightarrow \sigma^c \in S_n(R^c) \Leftrightarrow \sigma^r \in S_n(R^r) \Leftrightarrow \sigma^{-1} \in S_n(R^{-1}).$$

Cooper [6] first raised the problem of determining the total number $f_q(S_n(r))$, and Bóna [2] derived the generating function of the sequence $(f_q(S_n(132)))_{n\geq 1}$ for monotone pattern, i.e., $q = 12\cdots k$ or $q = k(k-1)\cdots 21$. Further, Bóna [3] studied the generating functions for other length-3 patterns in $S_n(132)$, and showed both algebraically and bijectively that

$$f_{231}(S_n(132)) = f_{312}(S_n(132)) = f_{213}(S_n(132)).$$

According to the correspondence between 132-avoiding permutations and binary plane trees, Rudolph [13] showed that patterns of equal length are equipopular if their associated binary plane trees have identical spine structure. For the converse direction, Chua and Sankar [4] gave a complete classification of 132-avoiding permutations into equipopularity classes. Moreover, Homberger [9] presented exact formulae for the occurrences of each length-3 pattern in $S_n(123)$. From Lemma 1 and the existing results on $S_n(123)$ and $S_n(132)$, we can obtain the popularity of each length-3 pattern for the singly restricted permutations $S_n(r)$ with $r = 213, 231, 312, 321$. Therefore, it is well-studied for the popularity of length-3 patterns in singly restricted permutations, whereas it remains open for multiply restricted permutations.

In this paper, we focus on counting the number of occurrences of length-3 patterns in multiply restricted permutations $S_n(R)$ for $R \subset S_3$, especially for double and triple restrictions.

We obtain exact formulae or generating functions for popularity of each length-3 pattern, and the detailed results are summarized in Table 1. Moreover, we present combinatorial proofs for non-trivial equalities between the number of occurrences of different patterns. It is routine to consider the restricted permutations of higher multiplicity since there are only finite permutations, as shown in [14, Proposition 17]. Therefore, this work gives a complete study on the popularity of length-3 patterns in the multiply restricted permutations. For the distributions of other statistics in multiply restricted permutations, see [7, 8, 10, 11, 12].

## 2 Doubly restricted permutations

This section deals with the enumeration of the popularity for length-3 patterns in the doubly restricted permutations, i.e., permutations avoiding two different patterns in $S_3$. For doubly restricted permutations, we have the following proposition from [14].

**Proposition 2.** ([14, Lemma 5]) *For every symmetric group $S_n$,*

1. $|S_n(123,132)| = |S_n(123,213)| = |S_n(231,321)| = |S_n(312,321)| = 2^{n-1}$;
2. $|S_n(132,213)| = |S_n(231,312)| = 2^{n-1}$;
3. $|S_n(132,231)| = |S_n(213,312)| = 2^{n-1}$;
4. $|S_n(132,312)| = |S_n(213,231)| = 2^{n-1}$;
5. $|S_n(132,321)| = |S_n(123,231)| = |S_n(123,312)| = |S_n(213,321)| = \binom{n}{2} + 1$;
6. $|S_n(123,321)| = 0$ *for* $n \geq 5$.

Thus it is sufficient to consider the pattern popularity for the first set from class 1 to class 5, and the pattern popularity for the other sets can be derived by taking complement, reverse or inverse.

A *composition* of $n$ is an expression of $n$ as an ordered sum of positive integers, and we say that $c$ has $k$ parts or $c$ is a $k$-composition if there are exactly $k$ summands appeared in composition $c$. Let $\mathcal{C}_n$ and $\mathcal{C}_{n,k}$ denote the set of all compositions of $n$ and the set of $k$-compositions of $n$, respectively. It is known that $|\mathcal{C}_0| = 1$, and for $n \geq 1$, $1 \leq k \leq n$, $|\mathcal{C}_n| = 2^{n-1}$ and $|\mathcal{C}_{n,k}| = \binom{n-1}{k-1}$. For more details on compositions, see [16]. It is helpful to introduce a lemma as follows:

**Lemma 3.** *For $n \geq 1$, we have*

$$\begin{aligned} a(n) &:= \sum_{c_1+c_2+\cdots+c_k=n} c_k = 2^n - 1, \\ b(n) &:= \sum_{c_1+c_2+\cdots+c_k=n} c_k(c_k-1) = 2^{n+1} - 2n - 2, \\ c(n) &:= \sum_{c_1+c_2+\cdots+c_k=n} k = (n+1)2^{n-2}, \\ d(n) &:= \sum_{c_1+c_2+\cdots+c_k=n} k(k-1) = (n^2+n-2)2^{n-3}, \end{aligned}$$

*where the sums are taken over all compositions of $n$.*

*Proof.* For $c_k = m$ , we can regard $c_1 + c_2 + \cdots + c_{k-1}$ as a composition of $n - m$. Since the number of compositions of $n - m$ is $2^{n-m-1}$ for $1 \leq m \leq n-1$ and the number of compositions of $n$ with $k$ parts is $\binom{n-1}{k-1}$, we have

$$a(n) = n + \sum_{m=1}^{n-1} m2^{n-m-1}, \quad b(n) = n(n-1) + \sum_{m=1}^{n-1} m(m-1)2^{n-m-1},$$

and

$$c(n) = \sum_{k=1}^{n} k\binom{n-1}{k-1}, \quad d(n) = \sum_{k=1}^{n} k(k-1)\binom{n-1}{k-1}.$$

Let $g(x) = \sum_{i=0}^{n-1} x^i = \frac{1-x^n}{1-x}$ and $h(x) = x\sum_{i=1}^{n}\binom{n-1}{i-1}x^{i-1} = x(1+x)^{n-1}$. We have

$$\begin{aligned} g'(x) &= \sum_{i=1}^{n-1} ix^{i-1} = \frac{(n-1)x^n - nx^{n-1} + 1}{(1-x)^2}, \\ g''(x) &= \sum_{i=1}^{n-1} i(i-1)x^{i-2} = \frac{(3n-n^2-2)x^n + (2n^2-4n)x^{n-1} + (n-n^2)x^{n-2} + 2}{(1-x)^3}, \\ h'(x) &= \sum_{i=1}^{n} i\binom{n-1}{i-1}x^{i-1} = (nx+1)(1+x)^{n-2}, \\ h''(x) &= \sum_{i=1}^{n} i(i-1)\binom{n-1}{i-1}x^{i-2} = \left[n^2x + n(2-x) - 2\right](1+x)^{n-3}. \end{aligned}$$

It follows that

$$a(n) = 2^{n-2}g'(1/2) + n, \quad b(n) = 2^{n-3}g''(1/2) + n(n-1),$$

and

$$c(n) = h'(1), \quad d(n) = h''(1).$$

Lemma 3 holds by simple computations. □

## 2.1 Pattern popularity in $(123, 132)$-avoiding permutations

In this subsection, we calculate the popularity of all length-3 patterns in $S_n(123, 132)$. For a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$, $\sigma_i$ is said to be a *left-to-right maximum* (resp., *right-to-left maximum*) if $\sigma_i > \sigma_j$ for all $j < i$ (resp., $j > i$). We first recall a correspondence between $S_n(123, 132)$ and $\mathcal{C}_n$ as implicitly shown in [10].

**Lemma 4.** ([10, Theorem 3]) *There is a bijection $\varphi_1$ between $S_n(123, 132)$ and $\mathcal{C}_n$.*

*Proof.* Given $\sigma \in S_n(123, 132)$, let $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}$ be the $k$ right-to-left maxima with $i_1 < i_2 < \cdots < i_k$. Then $c = i_1 + (i_2 - i_1) + \cdots + (i_{k-1} - i_{k-2}) + (i_k - i_{k-1})$ is a composition of $n$ since $i_k = n$. On the converse, let $m_i = n - (c_1 + \cdots + c_{i-1})$ for any given composition $n = c_1 + c_2 + \cdots + c_k \in \mathcal{C}_n$. Set $\tau_i = m_i - 1, m_i - 2, \ldots, m_i - c_i + 1, m_i$ for $1 \le i \le k$. It is easy to check that $\sigma = \tau_1\, \tau_2\, \cdots\, \tau_k \in S_n(123, 132)$. □

For example, $\sigma = 8\,9\,7\,5\,4\,3\,6\,1\,2$ corresponds to the composition $9 = 2 + 1 + 4 + 2$.

Given a pattern $q$, for simplicity, let $f_q(n) := \sum_{\sigma \in S_n(123,132)} f_q(\sigma)$ be the number of occurrences of pattern $q$ in $S_n(123, 132)$, and we will use this notation in subsequent sections when the set in question is unambiguous. A *factor* of $\sigma$ is a subsequence consisting of contiguous letters in $\sigma$. From Lemma 4, we have

**Proposition 5.** *For $n \ge 3$,*

$$f_{213}(n) = \sum_{c_1+c_2+\cdots+c_k=n} \sum_{i=1}^{k} \binom{c_i - 1}{2}, \tag{1}$$

$$f_{231}(n) = \sum_{c_1+c_2+\cdots+c_k=n} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j(c_i - 1). \tag{2}$$

*Proof.* For each permutation $\sigma \in S_n(123, 132)$ with $\varphi_1(\sigma) = c_1 + c_2 + \cdots + c_k$, we can rewrite $\sigma$ as $\sigma = \tau_1\, \tau_2\, \cdots\, \tau_k$ from Lemma 4. We say that $\tau_i > \tau_j$ if all the elements in $\tau_i$ are larger than that in $\tau_j$. We see that the pattern 213 can only occur in every factor $\tau_i$ since the elements except the last one are decreasing in $\tau_i$ and $\tau_i > \tau_j$ for $j > i$. Thus, there are $\binom{c_i-1}{2}$ choices to select two elements in $\tau_i$ to play the role of "21", and the last element of $\tau_i$ plays the role of "3". If $c_i \le 2$, then there is no copy of the pattern 213 in $\tau_i$, this coincides with the value $\binom{c_i-1}{2} = 0$ for $c_i = 1$ or 2. Summing up all the number of 213-patterns in factors $\tau_1, \tau_2, \ldots, \tau_k$ yields formula (1).

For pattern 231, we have $c_i - 1$ choices in factor $\tau_i$ to select one element to play the role of "2" and one choice (always the last element of $\tau_i$) for "3". After this, we have $c_{i+1} + \cdots + c_k$ choices to select one element in $\tau_{i+1}, \ldots, \tau_k$ for the role of "1" since all the elements after $\tau_i$ are smaller than those in $\tau_i$. Summing up all the number of 231-patterns according to the position of "3" gives formula (2). □

**Theorem 6.** *For $n \geq 3$, in the set $S_n(123, 132)$, we have*

$$\begin{aligned} f_{213}(n) &= (n-3)2^{n-2} + 1, & (3) \\ f_{231}(n) &= f_{312}(n) = (n^2 - 5n + 8)2^{n-3} - 1, & (4) \\ f_{321}(n) &= (n^3/3 - 2n^2 + 14n/3 - 5)2^{n-2} + 1. & (5) \end{aligned}$$

*Proof.* From $S_3(123, 132) = \{213, 231, 312, 321\}$, we have

$$f_{213}(3) = f_{231}(3) = 1.$$

To prove formula (3), Proposition 5 gives that, for $n \geq 3$,

$$f_{213}(n+1) = \sum_{\substack{c_k=1 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k} \binom{c_i - 1}{2} + \sum_{\substack{c_k \geq 2 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k} \binom{c_i - 1}{2}.$$

If $c_k = 1$, then $k \geq 2$, and we have

$$\sum_{\substack{c_k=1 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k} \binom{c_i - 1}{2} = \sum_{c_1+c_2+\cdots+c_{k-1}=n} \sum_{i=1}^{k-1} \binom{c_i - 1}{2} = f_{213}(n).$$

If $c_k \geq 2$, then we set $c_k = 1 + r_k$ with $r_k \geq 1$. From Lemma 3, we find that

$$\begin{aligned} \sum_{\substack{c_k \geq 2 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k} \binom{c_i - 1}{2} &= \sum_{c_1+\cdots+c_{k-1}+r_k=n} \left[ \sum_{i=1}^{k-1} \binom{c_i - 1}{2} + \binom{r_k - 1}{2} + (r_k - 1) \right] \\ &= f_{213}(n) + \sum_{c_1+\cdots+c_{k-1}+r_k=n} (r_k - 1) \\ &= f_{213}(n) + a(n) - 2^{n-1}. \end{aligned}$$

Combining the above two cases, we have

$$f_{213}(n+1) = 2f_{213}(n) + 2^{n-1} - 1,$$

which proves formula (3) by solving the recurrence with initial value $f_{213}(3) = 1$.

For formula (4), we first have $f_{231}(n) = f_{312}(n)$ from $231^{-1} = 312$ and $\sigma \in S_n(123, 132) \Leftrightarrow \sigma^{-1} \in S_n(123, 132)$. Using the same method as in the proof of formula (3), we can show

$$\sum_{\substack{c_k=1 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j(c_i - 1) = f_{231}(n) - c(n) + n2^{n-1},$$

and

$$\sum_{\substack{c_k \geq 2 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j(c_i - 1) = f_{231}(n) - a(n) - c(n) + (n+1)2^{n-1}.$$

It follows that, from Lemma 3,

$$f_{231}(n+1) = 2f_{231}(n) + (n-2)2^{n-1} + 1.$$

Formula (4) is proved by solving this recurrence using $f_{231}(3) = 1$.

Since the total number of all length-3 patterns in a permutation $\sigma \in S_n$ is $\binom{n}{3}$, we have

$$f_{213}(n) + 2f_{231}(n) + f_{321}(n) = \binom{n}{3} 2^{n-1},$$

and formula (5) holds. □

The first few values of $f_q(S_n(123, 132))$ for $q$ of length 3 are shown below. Moreover, we observe that they appear in the On-Line Encyclopedia of Integer Sequences [15] as follows: $(f_{213}(n))_{n\geq 3}$ form sequence A000337, $(f_{231}(n))_{n\geq 3}$ form sequence A055580.

| $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ | $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0 | 0 | 1 | 1 | 1 | 1 | 6 | 0 | 0 | 49 | 111 | 111 | 369 |
| 4 | 0 | 0 | 5 | 7 | 7 | 13 | 7 | 0 | 0 | 129 | 351 | 351 | 1409 |
| 5 | 0 | 0 | 17 | 31 | 31 | 81 | 8 | 0 | 0 | 321 | 1023 | 1023 | 4801 |

## 2.2 Pattern popularity in $(132, 213)$-avoiding permutations

We first recall a correspondence between $S_n(132, 213)$ and $\mathcal{C}_n$ as follows:

**Lemma 7.** ([10, Theorem 8]) *There is a bijection $\varphi_2$ between $S_n(132, 213)$ and $\mathcal{C}_n$.*

*Proof.* Given $\sigma \in S_n(132, 213)$, let $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}$ be the $k$ right-to-left maxima with $i_1 < i_2 < \cdots < i_k$. It follows that $c = i_1 + (i_2 - i_1) + \cdots + (i_{k-1} - i_{k-2}) + (i_k - i_{k-1})$ is a composition of $n$ since $i_k = n$. On the converse, given a composition $n = c_1 + c_2 + \cdots + c_k \in \mathcal{C}_n$, let $m_i = n - (c_1 + \cdots + c_{i-1})$ and $\tau_i = m_i - c_i + 1, m_i - c_i + 2, \ldots, m_i - 1, m_i$ for $1 \leq i \leq k$. Set $\sigma = \tau_1 \tau_2 \cdots \tau_k$, and it is easy to check that $\sigma \in S_n(132, 213)$. □

For example, for the composition $9 = 3 + 3 + 1 + 2$, we get $\sigma = 7\,8\,9\,4\,5\,6\,3\,1\,2$. From this lemma, we have

**Proposition 8.** *For $n \geq 3$,*

$$f_{123}(n) = \sum_{c_1+c_2+\cdots+c_k=n} \sum_{i=1}^{k} \binom{c_i}{3}, \tag{6}$$

$$f_{231}(n) = \sum_{c_1+c_2+\cdots+c_k=n} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j \binom{c_i}{2}. \tag{7}$$

*Proof.* For a permutation $\sigma \in S_n(132, 213)$ with $\varphi_2(\sigma) = c_1 + c_2 + \cdots + c_k$, we rewrite $\sigma$ as $\sigma = \tau_1 \tau_2 \cdots \tau_k$. The pattern 123 can only occur in every factor $\tau_i$ as $\tau_i > \tau_j$ for $j > i$ and the elements in $\tau_i$ are increasing. Thus, we have $\binom{c_i}{3}$ choices to select three elements in $\tau_i$ to play the role of "123", and formula (6) follows by summing up all 123-patterns in factors $\tau_1, \tau_2, \ldots, \tau_k$.

For the pattern 231, we have $\binom{c_i}{2}$ choices in factor $\tau_i$ to select two elements to play the role of "23". After this, we have $c_{i+1} + \cdots + c_k$ choices to select one element in $\tau_{i+1}, \ldots, \tau_k$ for the role of "1" since $\tau_j < \tau_i$ for all $j > i$. Summing up all the number of 231-patterns according to the position of "23" gives formula (7). □

**Theorem 9.** *For $n \geq 3$, in the set $S_n(132, 213)$, we have*

$$f_{123}(n) = (n-4)2^{n-1} + n + 2, \tag{8}$$

$$f_{231}(n) = f_{312}(n) = (n^2 - 7n + 16)2^{n-2} - n - 4, \tag{9}$$

$$f_{321}(n) = (n^3/3 - 3n^2 + 38n/3 - 24)2^{n-2} + n + 6. \tag{10}$$

*Proof.* From Proposition 8, it follows that

$$f_{123}(n+1) = \sum_{\substack{c_k=1 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k} \binom{c_i}{3} + \sum_{\substack{c_k\geq 2 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k} \binom{c_i}{3}.$$

An argument similar to the proof of Theorem 6 shows that

$$f_{123}(n+1) = 2f_{123}(n) + 2^n - n - 1.$$

Solving this recurrence with initial value $f_{123}(3) = 1$ leads to formula (8).

From Lemma 1, we see that $\sigma \in S_n(132, 213) \Leftrightarrow \sigma^{-1} \in S_n(132, 213)$, which implies $f_{231}(n) = f_{312}(n)$ as $231^{-1} = 312$.

To calculate $f_{231}(n)$, by Proposition 8, we arrive at

$$f_{231}(n+1) = \sum_{\substack{c_k=1 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j \binom{c_i}{2} + \sum_{\substack{c_k\geq 2 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j \binom{c_i}{2}.$$

If $c_k = 1$, then $k \geq 2$, and we have

$$\sum_{\substack{c_k=1 \\ c_1+c_2+\cdots+c_k=n+1}} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j \binom{c_i}{2} = f_{231}(n) + \alpha(n),$$

where

$$\begin{aligned}
\alpha(n) &= \sum_{c_1+\cdots+c_k=n} \sum_{i=1}^{k} \binom{c_i}{2} = \sum_{c_1+\cdots+c_k=n} \sum_{i=1}^{k} \left[ \binom{c_i-1}{2} + c_i - 1 \right] \\
&= f_{213}(S_n(123,132)) + \sum_{c_1+\cdots+c_k=n} (n-k) \\
&= f_{213}(S_n(123,132)) - c(n) + n2^{n-1}.
\end{aligned}$$

Here we have used the deduced expression (1).
If $c_k \geq 2$, then we can derive that

$$\sum_{\substack{c_k\geq 2 \\ c_1+\cdots+c_k=n+1}} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j \binom{c_i}{2} = f_{231}(n) + \beta(n),$$

where

$$\begin{aligned}
\beta(n) &= \sum_{c_1+\cdots+c_k=n} \sum_{i=1}^{k-1} \binom{c_i}{2} \\
&= \sum_{c_1+\cdots+c_k=n} \sum_{i=1}^{k} \binom{c_i}{2} - \sum_{c_1+\cdots+c_k=n} \frac{c_k(c_k-1)}{2} = \alpha(n) - b(n)/2.
\end{aligned}$$

From Lemma 3, we get

$$f_{231}(n+1) = 2f_{231}(n) + (2n-6)2^{n-1} + n + 3.$$

Formula (9) holds by solving this recurrence with initial condition $f_{213}(3) = 1$.
Finally, formula (10) follows from $f_{123}(n) + 2f_{231}(n) + f_{321}(n) = \binom{n}{3}2^{n-1}$. □

The first few values of $f_q(S_n(132,213))$ for $q$ of length 3 are shown below. They appear in [15] as follows: $(f_{123}(n))_{n\geq 3}$ form sequence A045618, $(f_{231}(n))_{n\geq 3}$ form sequence A055581 and $(f_{321}(n))_{n\geq 3}$ form sequence A055586.

| $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ | $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 0 | 0 | 1 | 1 | 1 | 6 | 72 | 0 | 0 | 150 | 150 | 268 |
| 4 | 6 | 0 | 0 | 8 | 8 | 10 | 7 | 201 | 0 | 0 | 501 | 501 | 1037 |
| 5 | 23 | 0 | 0 | 39 | 39 | 59 | 8 | 522 | 0 | 0 | 1524 | 1524 | 3598 |

### 2.3 Pattern popularity in $(132, 231)$-avoiding permutations

For each $\sigma \in S_n(132, 231)$, we observe that $n$ must lie in the beginning or the end of $\sigma$, and $n-1$ must lie in the beginning or the end of $\sigma\backslash\{n\}$,..., and so on. Here $\sigma\backslash\{n\}$ denotes the sequence obtained from $\sigma$ by deleting the element $n$. In view of such special structure, we can derive the pattern popularity in $(132, 231)$-avoiding permutations directly.

**Theorem 10.** *For $n \geq 3$, in the set $S_n(132, 231)$, we have*

$$f_{123}(n) = f_{213}(n) = f_{312}(n) = f_{321}(n) = \binom{n}{3} 2^{n-3}. \tag{11}$$

*Proof.* Suppose that $q$ is a length-3 pattern in $\{123, 213, 312, 321\}$, and $abc$ is a copy of the pattern $q$. Set

$$[n]\backslash\{a, b, c\} := \{r_1 > r_2 > \cdots > r_{n-4} > r_{n-3}\}.$$

We will construct a permutation in the set $S_n(132, 231)$ which contains $abc$ as a copy of the pattern $q$. Start with the subsequence $\sigma^0 := abc$, and for $i$ from 1 to $n-3$, $\sigma^i$ is obtained by inserting $r_i$ into $\sigma^{i-1}$ such that

- If there are at least two elements in $\sigma^{i-1}$ that are smaller than $r_i$, then choose the two elements $A$ and $B$ such that $A$ is the leftmost one and $B$ is the rightmost one. We put $r_i$ immediately to the left of $A$ or immediately to the right of $B$;

- If there is only one element $A$ in $\sigma^{i-1}$ such that $A < r_i$, then we put $r_i$ immediately to the left or to the right of $A$;

- If all the elements in $\sigma^{i-1}$ are larger than $r_i$, then choose $A$ the smallest one, and put $r_i$ immediately to the left or to the right of $A$.

Finally, we set $\sigma := \sigma^{n-3}$ and $\sigma \in S_n(132, 231)$ from the above construction. It can be seen that, the number of permutations having a copy $abc$ is $2^{n-3}$ since each $r_i$ has 2 choices in the inserting procedure. Moreover, there are $\binom{n}{3}$ choices to select three elements $a, b, c$ as an appearance of the pattern $q$ in $\{123, 213, 312, 321\}$. Hence we deduce $f_q(n) = \binom{n}{3} 2^{n-3}$. $\square$

Here we give an illustration for constructing a permutation in $S_8(132, 231)$ which contains $abc = 256$ as a copy of the pattern 123. Set $\sigma^0 := 2\,5\,6$, we may have $\sigma^1 = 8\,2\,5\,6$, $\sigma^2 = 8\,7\,2\,5\,6$, $\sigma^3 = 8\,7\,2\,4\,5\,6$, $\sigma^4 = 8\,7\,3\,2\,4\,5\,6$, $\sigma := \sigma^5 = 8\,7\,3\,2\,1\,4\,5\,6$.

We can also give a combinatorial proof for Theorem 10. Since $\sigma \in S_n(132, 231) \Leftrightarrow \sigma^r \in S_n(132, 231)$, it is easy to show $f_{123}(n) = f_{321}(n)$ and $f_{213}(n) = f_{312}(n)$ from $123^r = 321$ and $213^r = 312$. It remains to give a bijection for $f_{213}(n) = f_{123}(n)$, and our construction is motivated from Bóna [3].

We first introduce some notation about trees. A *binary plane tree* is a rooted unlabelled tree in which each vertex has at most two children, and each child is a left child or a right child of its parent. For each $\sigma \in S_n(132)$, we can construct a binary plane tree $T(\sigma)$ as follows:

the root of $T(\sigma)$ corresponds to the entry $n$ of $\sigma$, the left subtree of the root corresponds to the string of entries of $\sigma$ on the left of $n$, and the right subtree of the root corresponds to the string of entries of $\sigma$ on the right of $n$. Both subtrees are constructed recursively by the same rule. For more details, see [1, 3, 13].

A *left descendant* (resp., *right descendant*) of a vertex $x$ in a binary plane tree is a vertex in the left (resp., right) subtree of $x$. Similarly, an *ascendant* of a vertex $x$ in a binary plane tree is a vertex whose subtree contains $x$. Given a tree $T$ and a vertex $v \in T$, let $T_v$ be the subtree of $T$ rooted at $v$. Let $R$ be an occurrence of the pattern 123 in $\sigma \in S_n(132)$, and let $R_1, R_2, R_3$ be the three vertices of $T(\sigma)$ that correspond to $R$, going left to right. Then, $R_1$ is a left descendant of $R_2$, and $R_2$ is a left descendant of $R_3$.

According to the correspondence between 132-avoiding permutations and binary plane trees, we see that for $\sigma \in S_n(132, 231)$, $T(\sigma)$ is a binary plane tree on $n$ vertices such that each vertex has at most one child from the forbiddance of the pattern 231. For simplicity, let $\mathcal{T}_n$ be the set of such binary plane trees on $n$ vertices. Let $Q$ be an occurrence of the pattern 213 in $\sigma \in S_n(132, 231)$, and let $Q_2, Q_1, Q_3$ be the three vertices of $T(\sigma)$ that correspond to $Q$, going left to right. From the characterization of trees in $\mathcal{T}_n$, $Q_2$ is a left descendant of $Q_3$, and $Q_1$ is a right descendant of $Q_2$.

*Combinatorial proof for $f_{213}(n) = f_{123}(n)$.* Let $\mathcal{A}_n$ be the set of binary plane trees in $\mathcal{T}_n$ where three vertices forming a 213-pattern are colored black. Let $\mathcal{B}_n$ be the set of all binary plane trees in $\mathcal{T}_n$ where three vertices forming a 123-pattern are colored black. We define a map $\rho : \mathcal{A}_n \to \mathcal{B}_n$ as follows.

Given a tree $T \in \mathcal{A}_n$ with $Q_2, Q_1, Q_3$ being the three black vertices as a 213-pattern, we define $\rho(T)$ be the tree obtained from $T$ by changing the right subtree of $Q_2$ to be its left subtree. See Figure 1 for an illustration.

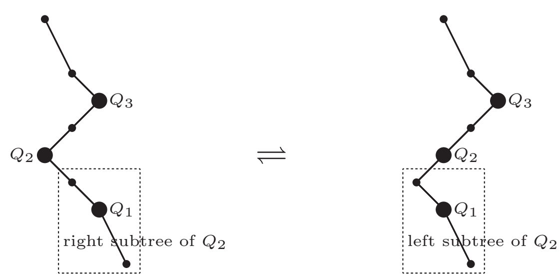


Figure 1: The bijection $\rho$.

In the tree $\rho(T)$, the relative positions of $Q_2$ and $Q_3$ keep the same, and $Q_1$ is a left descendant of $Q_2$. Therefore, points $Q_1Q_2Q_3$ form a 123-pattern in $\rho(T)$, and $\rho(T) \in \mathcal{B}_n$. On the converse, it is routine to verify that changing left subtree of $Q_2$ to be its right subtree is the desired reverse map. Therefore, $\rho$ is a bijection between $\mathcal{A}_n$ and $\mathcal{B}_n$.

The initial values for $f_q(S_n(132, 231))$ are

$$1, 8, 40, 160, 560, 1792, \ldots,$$

and this is essentially the sequence A001789 in [15].

## 2.4 Pattern popularity in $(132,312)$-avoiding permutations

We first present a lemma as follows:

**Lemma 11.** *There is a bijection $\varphi_4$ between $S_n(132,312)$ and $\mathcal{C}_n$.*

*Proof.* For $\sigma \in S_n(132,312)$, let $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}$ be the $k$ left-to-right maxima with $i_1 < i_2 < \cdots < i_k$. Then $c = (i_2-i_1)+(i_3-i_2)+\cdots+(i_k-i_{k-1})+(n+1-i_k)$ is a composition of $n$ since $i_1 = 1$. On the converse, let $n = c_k+c_{k-1}+\cdots+c_2+c_1 \in \mathcal{C}_n$. For $1 \le i \le k$, if $c_i = 1$ then set $\tau_i = n-i+1$; otherwise, set $m_i = c_1+\cdots+c_{i-1}-i+2$ and $\tau_i = n-i+1, m_i+c_i-2, \ldots, m_i+1, m_i$. It is easy to get $\sigma = \tau_k\,\tau_{k-1}\,\cdots\,\tau_2\,\tau_1 \in S_n(132,312)$. □

For example, if $9 = 3+1+2+3$, then $\sigma = 6\,5\,4\,7\,8\,3\,9\,2\,1$.

**Proposition 12.** *For $n \ge 3$,*

$$f_{123}(n) = \sum_{c_1+c_2+\cdots+c_k=n} \sum_{i=1}^{k-2} c_i \binom{k-i}{2}. \tag{12}$$

*Proof.* Let $\sigma = \tau_k \cdots \tau_2\,\tau_1$ be a permutation in $S_n(132,312)$ whose composition is given by $n = c_k + c_{k-1} + \cdots + c_2 + c_1$. It is evident that, for $i+1 \le j \le k$, the first element in $\tau_i$ is larger than all the elements in $\tau_j$, whereas the other elements in $\tau_i$ are smaller than that in $\tau_j$. Furthermore, the left-to-right maxima form an increasing subsequence and the other elements form a decreasing subsequence. Thus we have $c_i$ choices to select one element in $\tau_i$ to play the role of "1", and then $\binom{i-1}{2}$ choices to select two left-to-right maxima after $\tau_i$ to paly the role of "23". Summing up all the number of 123-patterns in factors $\tau_k, \ldots, \tau_2, \tau_1$ yields that

$$f_{123}(n) = \sum_{c_k+\cdots+c_2+c_1=n} \sum_{i=3}^{k} c_i \binom{i-1}{2}.$$

By setting $i := k-i+1$ and using the symmetry of the summands in compositions, it is equivalent to formula (12). □

**Theorem 13.** *For $n \ge 3$, in the set $S_n(132,312)$, we have*

$$f_{123}(n) = f_{321}(n) = \binom{n}{3} 2^{n-3}, \tag{13}$$

$$f_{213}(n) = f_{231}(n) = \binom{n}{3} 2^{n-3}. \tag{14}$$

*Proof.* From Lemma 1, we know that $\sigma \in S_n(132,312) \Leftrightarrow \sigma^c \in S_n(132,312)$. Hence it is obvious that $f_{123}(n) = f_{321}(n)$ and $f_{213}(n) = f_{231}(n)$ as $123^c = 321$ and $213^c = 231$.

To calculate $f_{123}(n)$, by using Proposition 12 and the similar argument in the proof of Theorem 6, we have

$$f_{123}(n+1) = 2f_{123}(n) + (n^2 - n)2^{n-3}.$$

Formula (13) holds by solving the recurrence with initial value $f_{123}(3) = 1$, and formula (14) is a direct computation of $2f_{123}(n) + 2f_{213}(n) = \binom{n}{3}2^{n-1}$. □

We will give a combinatorial interpretation for $f_{231}(n) = f_{123}(n)$. For each $\sigma \in S_n(132,312)$, we construct a binary plane tree $T(\sigma)$ on $n$ vertices such that each vertex with a right descendant of some vertex does not have a left descendant from the forbiddance of the pattern 312. Let $\mathscr{T}_n$ denote the set of such trees on $n$ vertices. Let $Q$ be an occurrence of the pattern 231 in $\sigma \in S_n(132,312)$, and let $Q_2, Q_3, Q_1$ be the three vertices of $T(\sigma)$ that correspond to $Q$, going left to right. Then, $Q_2$ is a left descendant of $Q_3$, and there exists a lowest ascendant $x$ of $Q_3$ or $x = Q_3$ so that $Q_1$ is a right descendant of $x$.

*Combinatorial proof for $f_{231}(n) = f_{123}(n)$.* Let $\mathscr{A}_n$ be the set of binary plane trees in $\mathscr{T}_n$ in which three vertices forming a 231-pattern are colored black. Let $\mathscr{B}_n$ be the set of all binary plane trees in $\mathscr{T}_n$ in which three vertices forming a 123-pattern are colored black. We define a map $\varrho : \mathscr{A}_n \to \mathscr{B}_n$ as follows.

Given a tree $T \in \mathscr{A}_n$ with $Q_2, Q_3, Q_1$ being the three black vertices forming a 231-pattern, let $y$ be the parent of $x$ if it exists. We can see that $x$ is the left child of $y$ from $T \in \mathscr{A}_n$. Let $T^u := T - T_x$ be the tree obtained from $T$ by deleting the subtree $T_x$, and $T^d := T_x - T_{Q_1}$ be the tree obtained from $T_x$ by deleting $T_{Q_1}$. Now we define $\varrho(T)$ to be the tree obtained from $T$ by first adjoining $T_{Q_1}$ to the vertex $y$ as its left subtree, then adjoining $T^d$ to $Q_1$ as its left subtree and keeping all three black vertices the same if $y$ exits; otherwise, we adjoin $T^d$ to $Q_1$ as its left subtree directly. An illustration is given in Figure 2.

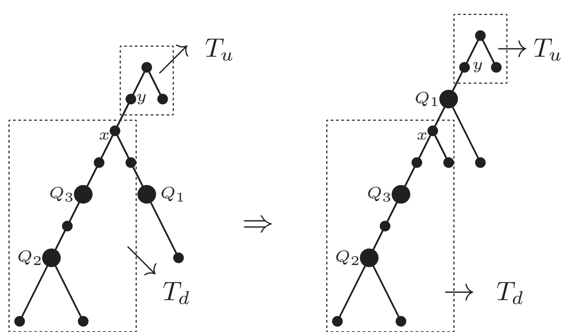


Figure 2: The bijection $\varrho$.

In the tree $\varrho(T)$, the relative positions of $Q_2$ and $Q_3$ are unchanged, and $Q_3$ is a left descendant of $Q_1$, thus the three black points $Q_2Q_3Q_1$ form a 123-pattern in $\varrho(T)$, and $\varrho(T) \in \mathscr{B}_n$. It is easy to describe the inverse map and we omit here.

## 2.5 Pattern popularity in $(132, 321)$-avoiding permutations

We first introduce a lemma as follows:

**Lemma 14.** [14, Proposition 13] *There is a bijection $\varphi_5$ between $S_n(132, 321)\backslash\{id\}$ and the set of 2-element subsets of $[n]$.*

*Proof.* For a permutation $\sigma \in S_n(132, 321)\backslash\{\text{id}\}$, suppose $\sigma_k = m$ $(k < m)$ and define $\varphi_5(\sigma) = \{k, m\}$. On the converse, given two elements $1 \le k < m \le n$, set $\tau_1 = m-k+1, m-k+2, \ldots, m-1, m$, $\tau_2 = 1, 2, \ldots, m-k$ and $\tau_3 = m+1, m+2, \ldots, n-1, n$. We have $\sigma = \varphi_5^{-1}(k, m) = \tau_1\,\tau_2\,\tau_3$. □

For example, if $k = 4, m = 6$, then $\sigma = 3\,4\,5\,6\,1\,2\,7\,8$.

**Proposition 15.** *For $n \ge 3$,*

$$f_{213}(n) = \sum_{1\le k<m\le n} k(m-k)(n-m), \tag{15}$$

$$f_{312}(n) = \sum_{1\le k<m\le n} k\binom{m-k}{2}. \tag{16}$$

*Proof.* Given a permutation $\sigma = \tau_1\,\tau_2\,\tau_3$ in $S_n(132, 321)$ with $\varphi_5(\sigma) = \{k, m\}$, we see that the elements in each $\tau_i$ $(1 \le i \le 3)$ are increasing, and $\tau_2 < \tau_1 < \tau_3$. Hence we have $k$ choices to select one element in $\tau_1$ to play the role of "2", $m-k$ choices to select one element in $\tau_2$ to play the role of "1", and $n-m$ choices to select one element in $\tau_3$ to play the role of "3". Summing up all possible $k$ and $m$ gives formula (15).

For the pattern 312, we have $k$ choices to select one element in factor $\tau_1$ to play the role of "3", and then have $\binom{m-k}{2}$ choices to select two elements in factor $\tau_2$ to play the role of "12". Summing up all $k$ and $m$ proves formula (16). □

We now derive the exact formulae for the popularity of patterns in $S_n(132, 321)$ as follows.

**Theorem 16.** *For $n \ge 3$, in the set $S_n(132, 321)$, we have*

$$f_{213}(n) = f_{231}(n) = f_{312}(n) = \binom{n+2}{5}, \tag{17}$$

$$f_{123}(n) = n(7n^4 - 40n^3 + 85n^2 - 80n + 28)/120. \tag{18}$$

*Proof.* It is simple to prove $f_{312}(n) = f_{231}(n)$ from Lemma 1 and $312^{-1} = 231$. By Proposition 15, we have

$$
\begin{aligned}
f_{312}(n) &= \sum_{1\le k<m\le n} k\binom{m-k}{2} \\
&= \sum_{k=1}^{n-1} k \sum_{m=k+1}^{n} \binom{m-k}{2} = \sum_{k=1}^{n-1} k\binom{n-k+1}{3} \\
&= \sum_{k=1}^{n-1} \left[(n^3-n)k + (1-3n^2)k^2 + 2nk^3 - k^4\right],
\end{aligned}
$$

and

$$
\begin{aligned}
f_{213}(n) &= \sum_{1\le k<m\le n} k(m-k)(n-m) = \sum_{k=1}^{n-1}\sum_{m=k+1}^{n} k(m-k)(n-m) \\
&= \sum_{k=1}^{n-1}\sum_{m'=1}^{n-k} km'(n-m'-k) = \sum_{k=1}^{n-1} k(n-k)\sum_{m'=1}^{n-k} m' - \sum_{k=1}^{n-1} k \sum_{m'=1}^{n-k} m'^2 \\
&= \sum_{k=1}^{n-1} \left[\left(\frac{n^3}{6}-\frac{n}{6}\right)k + \left(\frac{1}{6}-\frac{n^2}{2}\right)k^2 + \frac{n}{2}k^3 - \frac{1}{6}k^4\right].
\end{aligned}
$$

We get formula (17) by substituting the closed forms of $\sum_{k=1}^{n} k^p$ $(p=1,2,3,4)$ into the above expressions, and this theorem holds from $2f_{231}(n) + f_{213}(n) + f_{123}(n) = \binom{n}{3}\left[\binom{n}{2}+1\right]$. □

Notice that $f_{213}(n) = f_{231}(n)$ can be proved by Bóna's bijection [3] on the set of binary plane trees on $n$ vertices such that the vertex which is a right descendant of some node has no right descendant.

The first few values of $f_q(S_n(132, 321))$ for $q$ of length 3 are shown below, and $(f_{213}(n))_{n\ge 3}$ form sequence A000389 in [15].

| $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ | $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 0 | 1 | 1 | 1 | 0 | 6 | 152 | 0 | 56 | 56 | 56 | 0 |
| 4 | 10 | 0 | 6 | 6 | 6 | 0 | 7 | 392 | 0 | 126 | 126 | 126 | 0 |
| 5 | 47 | 0 | 21 | 21 | 21 | 0 | 8 | 868 | 0 | 252 | 252 | 252 | 0 |

# 3 Triply restricted permutations

This section studies the pattern popularity in the permutations which avoid simultaneously any three patterns of length 3. We begin with the following proposition from [14].

**Proposition 17.** ([14, Lemma 6]) *The numbers of triply restricted permutations in $S_n$ satisfy the following equalities:*

1. $|S_n(123, 132, 213)| = |S_n(231, 312, 321)| = F_{n+1}$;

2. $|S_n(123, 132, 231)| = |S_n(123, 213, 312)| = |S_n(132, 231, 321)| = |S_n(213, 312, 321)| = n$;

3. $|S_n(132, 213, 231)| = |S_n(132, 213, 312)| = |S_n(132, 231, 312)| = |S_n(213, 231, 312)| = n$;

4. $|S_n(123, 132, 312)| = |S_n(123, 213, 231)| = |S_n(132, 312, 321)| = |S_n(213, 231, 321)| = n$;

5. $|S_n(123, 231, 312)| = |S_n(132, 213, 321)| = n$;

6. $|S_n(R)| = 0$ *for all* $R \supset \{123, 321\}$ *if* $n \geq 5$, *where* $F_n$ *is the Fibonacci number given by* $F_0 = 0, F_1 = 1$ *and* $F_n = F_{n-1} + F_{n-2}$ *for* $n \geq 2$.

An argument similar to the one used for doubly restricted permutations shows that we only need to consider the pattern popularity for the first set of class 1 to class 5.

## 3.1 Pattern popularity in $(123, 132, 213)$-avoiding permutations

It is well-known that Fibonacci number $F_{n+1}$ counts the number of 0-1 sequences of length $n-1$ in which there are no consecutive ones, see [5]. We call such a sequence a *Fibonacci binary word* for convenience. Let $B_n$ denote the set of all Fibonacci binary words of length $n$. Simion and Schmidt [14] showed that

**Lemma 18.** ([14, Proposition 15*]) *There is a bijection* $\psi_1$ *between* $S_n(123, 132, 213)$ *and* $B_{n-1}$.

*Proof.* For $w = w_1 w_2 \cdots w_{n-1} \in B_{n-1}$, we construct the permutation $\sigma$ as follows. For $1 \leq i \leq n-1$, let $X_i = [n] - \{\sigma_1, \ldots, \sigma_{i-1}\}$, and set

$$\sigma_i = \begin{cases} \text{largest element in } X_i, & \text{if } w_i = 0, \\ \text{second largest element in } X_i, & \text{if } w_i = 1. \end{cases}$$

Finally, $\sigma_n$ is the unique element in $X_n$. □

For example, if $w = 01001010$, then $\psi_1(w) = 9\,7\,8\,6\,4\,5\,2\,3\,1$.

Given a word $w = w_1 w_2 \cdots w_n \in B_n$, the index $i$ $(1 \leq i < n)$ is an *ascent* of $w$ if $w_i < w_{i+1}$. Let $\mathrm{asc}(w) = \{i \,|w_i < w_{i+1}\}$ be the set of ascents of $w$, and let $\mathrm{maj}(w) = \sum_{i \in \mathrm{asc}(w)} i$.

**Proposition 19.** *For* $n \geq 3$,

$$f_{312}(n) = \sum_{w \in B_{n-1}} \mathrm{maj}(w). \tag{20}$$

*Proof.* Suppose $\sigma \in S_n(123, 132, 213)$ and $\psi_1(\sigma) = w_1 w_2 \cdots w_{n-1}$. If $k$ is an ascent of $w$, then $w_k w_{k+1} = 01$ and $\sigma_k > \sigma_{k+1}$. From bijection $\psi_1$, we see that for all $i \in [n-1]$, there is at most one $j > i$ such that $\sigma_j > \sigma_i$. This implies that $\sigma_i > \sigma_{k+1}$ for all $i < k$. Since $\sigma_k$ is the largest element in $X_k$, we have $\sigma_i > \sigma_j$ for all $i < k+1$ and $j > k+1$. On the other hand, since $\sigma_{k+1}$ is the second largest element in $X_{k+1}$, there exists a unique $l > k+1$ such that $\sigma_l > \sigma_{k+1}$. Thus, we find that $\sigma_i \sigma_{k+1} \sigma_l$ forms a 312-pattern for all $i \leq k$, that is the ascent $k$ will produce $k$'s copies of 312-pattern in which $\sigma_{k+1}$ plays the role of "1". Summing up all the ascents, we derive that the number of copies of 312-pattern in $\sigma$ is $\mathrm{maj}(\psi_1(\sigma))$. □

Recall that the generating function of the Fibonacci number $F_n$ is given by

$$\sum_{n\geq 0} F_n x^n = \frac{x}{1-x-x^2}.$$

Hence we can deduce that

$$\sum_{n\geq 3} F_{n+1}x^n = x\sum_{n\geq 2} F_{n+2}x^n = \frac{1}{x}\left(\frac{x}{1-x-x^2} - x - x^2 - 2x^3\right) = \frac{x^3(3+2x)}{1-x-x^2}, \tag{21}$$

$$\sum_{n\geq 2} nF_{n+2}x^n = x\left(\frac{x^2(3+2x)}{1-x-x^2}\right)' = \frac{x^2(6+3x-4x^2-2x^3)}{(1-x-x^2)^2}, \tag{22}$$

$$\sum_{n\geq 3}\binom{n}{3}F_{n+1}x^n = \frac{x^3}{6}\left(\sum_{n\geq 3} F_{n+1}x^n\right)''' = \frac{x^3(3+8x+6x^2+4x^3)}{(1-x-x^2)^4}. \tag{23}$$

**Theorem 20.** *For $n \geq 3$, in the set $S_n(123,132,213)$, we have*

$$\sum_{n\geq 3} f_{231}(n)x^n = \sum_{n\geq 3} f_{312}(n)x^n = \frac{x^3(1+2x)}{(1-x-x^2)^3}, \tag{24}$$

$$\sum_{n\geq 3} f_{321}(n)x^n = \frac{x^3(1+6x+12x^2+8x^3)}{(1-x-x^2)^4}. \tag{25}$$

*Proof.* From Lemma 1, we have $f_{231}(n) = f_{312}(n)$ as $\sigma \in S_n(123,132,213) \Leftrightarrow \sigma^{-1} \in S_n(123,132,213)$ and $231^{-1} = 312$. By Proposition 19, we can write

$$\sum_{n\geq 3} f_{312}(n)x^n = \sum_{n\geq 3} x^n \sum_{w\in B_{n-1}} \mathrm{maj}(w) = x\sum_{n\geq 3}\sum_{w\in B_{n-1}} \mathrm{maj}(w)x^{n-1} = xu(x),$$

where $u(x) = \sum_{n\geq 2}\sum_{w\in B_n} \mathrm{maj}(w)x^n$. To calculate $u(x)$, we set

$$M_n(q) = \sum_{w\in B_n} q^{\mathrm{maj}(w)} \text{ and } M(x,q) = \sum_{n\geq 2} M_n(q)x^n.$$

It is easy to get

$$u(x) = \frac{\partial M(x,q)}{\partial q}\,\Big|_{q=1}.$$

Given a word $w = w_1w_2\cdots w_n \in B_n$, if $w_n = 0$, then $\mathrm{maj}(w) = \mathrm{maj}(w_1w_2\cdots w_{n-1})$; otherwise, $w_{n-1}w_n = 01$ and $\mathrm{maj}(w) = \mathrm{maj}(w_1w_2\cdots w_{n-2}) + n - 1$. Hence, we have

$$M_n(q) = M_{n-1}(q) + q^{n-1}M_{n-2}(q) \text{ for } n \geq 4,$$

with $M_2(q) = 2 + q$ and $M_3(q) = 2 + q + 2q^2$. Multiplying the recursion by $x^n$ and summing over $n \geq 4$ yields that

$$M(x,q) - (2+q)x^2 - (2+q+2q^2)x^3 = x\left[M(x,q) - (2+q)x^2\right] + qx^2M(xq,q).$$

Therefore

$$(1-x)M(x,q) = qx^2M(xq,q) + (2+q)x^2 + 2q^2x^3.$$

Differentiate both sides with respect to $q$, we get

$$(1-x)\frac{\partial M(x,q)}{\partial q} = x^2\left[M(xq,q) + q\frac{\partial M(xq,q)}{\partial q}\right] + x^2 + 4qx^3.$$

Setting $q = 1$ gives

$$(1-x)u(x) = x^2\left[M(x,1) + \frac{\partial M(xq,q)}{\partial q}\,|_{q=1}\right] + x^2 + 4x^3.$$

Notice that

$$M(x,1) = \sum_{n\geq 2}|B_n|x^n = \sum_{n\geq 2}F_{n+2}x^n,$$

and

$$\begin{aligned}\frac{\partial M(xq,q)}{\partial q}\,|_{q=1} &= \left(\sum_{n\geq 2}\sum_{w\in B_n}(n+\mathrm{maj}(w))q^{n+\mathrm{maj}(w)-1}x^n\right)|_{q=1}\\ &= \sum_{n\geq 2}x^n\sum_{w\in B_n}(n+\mathrm{maj}(w))\\ &= \sum_{n\geq 2}nF_{n+2}x^n + u(x).\end{aligned}$$

Invoking formulae (21) and (22), this implies that

$$(1-x)u(x) = x^2\left[\frac{x^2(3+2x)}{1-x-x^2} + \frac{x^2(6+3x-4x^2-2x^3)}{(1-x-x^2)^2} + u(x)\right] + x^2 + 4x^3.$$

Therefore, $u(x) = x^2(1+2x)/(1-x-x^2)^3$. Multiplying $u(x)$ by $x$, we arrive at formula (24).

As for formula (25), we notice that

$$\sum_{n\geq 3}f_{321}(n)x^n = \sum_{n\geq 3}\binom{n}{3}F_{n+1}x^n - 2\sum_{n\geq 3}f_{312}(n)x^n \tag{26}$$

from the observation $2f_{312}(n) + f_{321}(n) = \binom{n}{3}F_{n+1}$. Thus formula (25) is obtained by substituting equation (23) and the generating function of $f_{312}(n)$ into formula (26). □

The first few values of $f_q(S_n(123,132,213))$ for $q$ of length 3 are shown below, and $(f_{231}(n))_{n\geq 3}$ form sequence A152881 in [15].

| $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ | $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0 | 0 | 0 | 1 | 1 | 1 | 6 | 0 | 0 | 0 | 40 | 40 | 180 |
| 4 | 0 | 0 | 0 | 5 | 5 | 10 | 7 | 0 | 0 | 0 | 95 | 95 | 545 |
| 5 | 0 | 0 | 0 | 15 | 15 | 50 | 8 | 0 | 0 | 0 | 213 | 213 | 1478 |

## 3.2 Pattern popularity in other triply restricted permutations

This subsection deals with the popularity of length-3 patterns in the other four classes of triply restricted permutations. We begin with a helpful lemma from [14] as follows:

**Lemma 21.** ([14, Proposition 16*]) *We have*

$$\sigma \in S_n(123,132,231) \Leftrightarrow \sigma = n, n-1, \ldots, k+1, k-1, k-2, \ldots, 2, 1, k \text{ for some } k. \tag{27}$$
$$\sigma \in S_n(132,213,231) \Leftrightarrow \sigma = n, n-1, \ldots, k+1, 1, 2, 3, \ldots, k-1, k \text{ for some } k. \tag{28}$$
$$\sigma \in S_n(123,132,312) \Leftrightarrow \sigma = n-1, n-2, \ldots, k+1, n, k, k-1, \ldots, 1 \text{ for some } k. \tag{29}$$
$$\sigma \in S_n(123,231,312) \Leftrightarrow \sigma = k-1, k-2, \ldots, 3, 2, 1, n, n-1 \ldots, k \text{ for some } k. \tag{30}$$

Appealing to the above structural characterizations, we can derive the pattern popularity in those classes as follows:

**Theorem 22.** *For $n \geq 3$, in the set $S_n(123,132,231)$, we have*

$$f_{213}(n) = f_{312}(n) = \binom{n}{3}, \tag{31}$$
$$f_{321}(n) = (n-2)\binom{n}{3}. \tag{32}$$

*Proof.* According to the structural formula (27), the identity $f_{213}(n) = f_{312}(n)$ can be proved by a direct bijection.

Let $q = abc$ $(b < a < c)$ be a copy of 213-pattern in $\sigma \in S_n(123,132,231)$. We have $\sigma(n) = c$ since $b < c$ and $\sigma \in S_n(123,132,231)$ has only one ascent at position $n-1$. Therefore, $q$ is a 213-pattern in the sole permutation

$$\sigma = n, n-1, \ldots, c+1, c-1, \ldots, \underline{a}, \ldots, \underline{b}, \ldots, 2, 1, \underline{c}.$$

For the sake of clarity, we underline the occurrence of the assumed pattern.

For $q' = cba$ (312-pattern), we find similarly that $q'$ is a 312-pattern in

$$\sigma' = n, n-1, \ldots, \underline{c}, \ldots, a+1, a-1, \ldots, \underline{b}, \ldots, 2, 1, \underline{a}.$$

For example, if $n = 7$ and $q = 326$, then $\sigma = 7\,5\,4\,\underline{3}\,\underline{2}\,1\,\underline{6}$, $q' = 623$ and $\sigma' = 7\,\underline{6}\,5\,4\,\underline{2}\,1\,\underline{3}$.

Hence, for every copy of 213-pattern $(q, \sigma)$, there is a unique copy of 312-pattern $(q', \sigma')$, and the converse is also true. This implies that $f_{213}(n) = f_{312}(n)$.

To calculate $f_{312}(n)$, we suppose $\sigma = n, n-1, \ldots, k+1, k-1, k-2, \ldots, 2, 1, k$ for some $k$. We construct a 312-pattern as follows: Choose one element from the first $n-k$ elements to play the role of "3", then choose one element from the next $k-1$ elements to play the role of "1", and the last element plays the role of "2". Thus, summing up $k$ gives

$$f_{312}(n) = \sum_{k=1}^{n}(n-k)(k-1) = -n^2 + (n+1)\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2 = \frac{n(n-1)(n-2)}{6} = \binom{n}{3}.$$

The proof is completed by the relation $f_{213}(n) + f_{312}(n) + f_{321}(n) = n\binom{n}{3}$. □

The first few values of $f_q(S_n(123, 132, 231))$ for $q$ of length 3 are shown below, and $(f_{213}(n))_{n\geq 3}$ form sequence A000292, $(f_{321}(n))_{n\geq 3}$ form sequence A002417 in [15].

| $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ | $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0 | 0 | 1 | 0 | 1 | 1 | 6 | 0 | 0 | 20 | 0 | 20 | 80 |
| 4 | 0 | 0 | 4 | 0 | 4 | 8 | 7 | 0 | 0 | 35 | 0 | 35 | 175 |
| 5 | 0 | 0 | 10 | 0 | 10 | 30 | 8 | 0 | 0 | 56 | 0 | 56 | 336 |

**Theorem 23.** *For $n \geq 3$, in the set $S_n(132, 213, 231)$, we have*

$$f_{123}(n) = f_{312}(n) = \binom{n+1}{4}, \tag{33}$$

$$f_{321}(n) = \frac{n(n-2)(n-1)^2}{12}. \tag{34}$$

*Proof.* Based on structural formula (28), we could also prove $f_{123}(n) = f_{312}(n)$ directly. Let $abc$ be a 123-pattern in

$$\sigma = n, n-1, \ldots, k+1, 1, \ldots, \underline{a}, a+1, \ldots, \underline{b}, b+1, \ldots, c-1, \underline{c}, c+1, \ldots, k-1, k.$$

Set

$$\sigma' = n, n-1, \ldots, \underline{n-k+c}, \ldots, c, 1, 2, \ldots, \underline{a}, a+1, \ldots, \underline{b}, b+1, \ldots, c-1.$$

It is easy to check that $(n-k+c)\,a\,b$ is a 312-pattern of $\sigma'$. For example, if $\sigma = 9\,8\,7\,1\,\underline{2}\,\underline{3}\,4\,\underline{5}\,6$, then $\sigma' = 9\,\underline{8}\,7\,6\,5\,1\,\underline{2}\,\underline{3}\,4$.

To calculate $f_{123}(n)$, we suppose $\sigma = n, n-1, \ldots, k+1, 1, 2, \ldots, k-1, k$ for some $k$. A 123-pattern can be obtained by picking three elements from the last $k$ elements to play the role of "123". Thus, summing up all possible $k$ gives

$$f_{123}(n) = \sum_{k=1}^{n}\binom{k}{3} = \binom{n+1}{4}.$$

We complete the proof from $f_{123}(n) + f_{312}(n) + f_{321}(n) = n\binom{n}{3}$. □

The first few values of $f_q(S_n(132,213,231))$ for $q$ of length 3 are shown below, and $(f_{123}(n))_{n\geq 3}$ form sequence A000332, $(f_{321}(n))_{n\geq 3}$ form sequence A002415 in [15].

| $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ | $n$ | $f_{123}$ | $f_{132}$ | $f_{213}$ | $f_{231}$ | $f_{312}$ | $f_{321}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 0 | 0 | 0 | 1 | 1 | 6 | 35 | 0 | 0 | 0 | 35 | 50 |
| 4 | 5 | 0 | 0 | 0 | 5 | 6 | 7 | 70 | 0 | 0 | 0 | 70 | 105 |
| 5 | 15 | 0 | 0 | 0 | 15 | 20 | 8 | 126 | 0 | 0 | 0 | 126 | 196 |

**Theorem 24.** *For $n \geq 3$, in the set $S_n(123,132,312)$, we have*

$$f_{213}(n) = f_{231}(n) = \binom{n}{3}, \tag{35}$$

$$f_{321}(n) = (n-2)\binom{n}{3}. \tag{36}$$

*Proof.* In view of the structural formula (29), the equality $f_{213}(n) = f_{231}(n)$ can be proved by a direct correspondence. Let $abn$ be a copy of 213-pattern in

$$\sigma = n-1, \ldots, \underline{a}, a+1, \ldots, \underline{b}, b+1, \ldots, k+1, \underline{n}, k, k-1, \ldots, 2, 1.$$

Set

$$\sigma' = n-1, \ldots, \underline{n-a+b}, \ldots, n-a+k+1, \underline{n}, n-a+k, n-a+k-1, \ldots, \underline{n-a}, \ldots, 2, 1.$$

Then $n-a+b, n, n-a$ is a 231-pattern of $\sigma'$. For example, if $\sigma = 8\,\underline{7}\,6\,\underline{5}\,4\,\underline{9}\,3\,2\,1$, then $\sigma' = \sigma = 8\,\underline{7}\,6\,\underline{9}\,5\,4\,3\,\underline{2}\,1$.

To calculate $f_{213}(n)$, we suppose that $\sigma = n-1, n-2, \ldots, k+1, n, k, k-1, \ldots, 2, 1$ for some $k$. A 213-pattern can be obtained by choosing two elements from the first $n-k-1$ elements to play the role of "21", and let $n$ play the role of "3". Thus, summing up all possible $k$, we have

$$f_{213}(n) = \sum_{k=0}^{n-1} \binom{n-k-1}{2} = \binom{n}{3}.$$

The proof is completed by using the relation $f_{213}(n) + f_{231}(n) + f_{321}(n) = n\binom{n}{3}$. $\square$

**Theorem 25.** *For $n \geq 3$, in the set $S_n(123,231,312)$, we have*

$$f_{132}(n) = f_{213}(n) = \binom{n+1}{4}, \tag{37}$$

$$f_{321}(n) = \frac{n(n-2)(n-1)^2}{12}. \tag{38}$$

*Proof.* From Lemma 1, we see that

$$\sigma \in S_n(123, 231, 312) \Leftrightarrow \sigma^r \in S_n(321, 132, 213) \Leftrightarrow (\sigma^r)^c \in S_n(123, 231, 312).$$

As a consequence, we have $f_{213}(n) = f_{132}(n)$ from $(213^r)^c = 312^c = 132$.

For $f_{213}(n)$, we will employ the structure in formula (30). Suppose $\sigma = k-1, k-2, \ldots, 3, 2, 1, n, n-1 \ldots, k$ for some $k$. A 213-pattern can be obtained as follows: Choose two elements from the first $k-1$ elements to play the role of "21", and choose one element from the last $n-k+1$ elements to play the role of "3". Thus, summing up all possible $k$, we have

$$f_{213}(n) = \sum_{k=1}^{n} \binom{k-1}{2}(n-k+1) = \sum_{k=0}^{n-1} \binom{k}{2}(n-k) = \binom{n+1}{4}.$$

The formula for $f_{321}(n)$ is obtained by the relation $2f_{213}(n) + f_{321}(n) = n\binom{n}{3}$. □

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# References

[1] M. Bóna, *A Walk Through Combinatorics*, 3rd edition, World Scientific, 2011.

[2] M. Bóna, The absence of a pattern and the occurrences of another, *Discrete Math. Theor. Comput. Sci.* **12** (2010), 89–102.

[3] M. Bóna, Surprising symmetries in objects counted by Catalan numbers, *Electron. J. Combin.* **19** (2012), P62.

[4] L. Chua and K. R. Sankary, Equipopularity classes of 132-avoiding permutations, *Electron. J. Combin.* **21** (2014), P1.59.

[5] L. Comtet, *Advanced Combinatorics*, Reidel, 1974.

[6] J. Cooper, *Combinatorial Problems I like*, available at http://www.math.sc.edu/~cooper/combprob.html.

[7] T. Dokosa, T. Dwyer, B. P. Johnson, B. E. Sagan, and K. Selsor, Permutation patterns and statistics, *Discrete Math.* **312** (2012), 2760–2775.

[8] S. Elizalde, Multiple pattern avoidance with respect to fixed points and excedances, *Electron. J. Combin.* **11** (2004), R51.

[9] C. Homberger, Expected patterns in permutation classes, *Electron. J. Combin.* **19** (2012), P43.

[10] T. Mansour, Permutations avoiding a pattern from $S_k$ and at least two patterns from $S_3$, *Ars Combin.* **62** (2001), 227–239.

[11] T. Mansour, Permutations containing a pattern exactly once and avoiding at least two patterns of three letters, *Ars Combin.* **72** (2004), 213–222.

[12] T. Mansour and A. Robertson, Refined restricted permutations avoiding subsets of patterns of length three, *Ann. Combin.* **6** (2003), 407–418.

[13] K. Rudolph, Pattern popularity in 132-avoiding permutations, *Electron. J. Combin.* **20** (2013), P8.

[14] R. Simion and F. W. Schmidt, Restricted permutations, *European J. Combin.* **6** (1985), 383–406.

[15] N. J. A. Sloane, *On-Line Encyclopedia of Integer Sequences*, `http://oeis.org`.

[16] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1, Cambridge University Press, 1997.

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