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# On $q$-Boson Operators and $q$-Analogues of the $r$-Whitney and $r$-Dowling Numbers

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**Abstract**

We define the $(q, r)$-Whitney numbers of the first and second kinds in terms of the $q$-Boson operators, and obtain several fundamental properties such as recurrence formulas, orthogonality and inverse relations, and other interesting identities. As a special case, we obtain a $q$-analogue of the $r$-Stirling numbers of the first and second kinds. Finally, we define the $(q, r)$-Dowling polynomials in terms of sums of $(q, r)$-Whitney numbers of the second kind, and obtain some of their properties.

## 1 Introduction

The investigation of $q$-analogues of combinatorial identities has proven to be a rich source of insight as well as of useful generalizations. Some examples of $q$-analogues are the $q$-*real* *number*, the *q-factorial* and the *q-falling factorial of order* $r$, respectively, given by

$$[x]_q = \frac{q^x - 1}{q - 1}, \ [n]_q! = \prod_{i=1}^{n} [i]_q, \ [x]_{q,n} = \prod_{i=0}^{r-1} [x - i]_q,$$

for any real number $x$ and non-negative integers $n$ and $r$, and the *q-binomial coefficients (also known as Gaussian polynomials)*

$$\binom{n}{r}_q = \frac{[n]_q!}{[r]_q![n-r]_q!} = \frac{[n]_{q,r}}{[r]_q!}.$$

The formulation of $q$-analogues is not unique, but some choices appear to allow more productive generalizations than others. In the present paper we apply the properties of the $q$-boson operators as a framework for the generation of $q$-deformations of a family of combinatorial identities involving the Whitney numbers.

A lattice $L$ in which every element is the join of elements $x$ and $y$ (in $L$) such that $x$ and $y$ cover the zero element 0, and is semimodular, is called a *geometric lattice*. Originally, if $L$ is a finite lattice of rank $n$, then the Whitney numbers $w(n, k)$ and $W(n, k)$ of the first and second kinds of $L$ are defined as the coefficients of the characteristic polynomial and as the number of elements of $L$ of corank $k$, respectively. Now, Dowling [20] defined a class of these geometric lattices, called *Dowling lattice*, which is a generalization of the partition lattice. Let $Q_n(G)$ be the Dowling lattice of rank $n$ associated to a finite group $G$ of order $m > 0$. Benoumhani [3] defined the *Whitney numbers of the first and second kind of* $Q_n(G)$, denoted by $w_m(n, k)$ and $W_m(n, k)$, respectively, in terms of the relations

$$m^n(x)_n = \sum_{k=0}^{n} w_m(n, k)(mx + 1)^k \tag{1}$$

and

$$(mx + 1)^n = \sum_{k=0}^{n} m^k W_m(n, k)(x)_k, \tag{2}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$ is the falling factorial of $x$ of order $n$. Notice that if the group $G$ is the trivial group ($m = 1$), multiplication of both equations (1) and (2) by $(x+1)$ yields the horizontal generating functions for the well-known Stirling numbers of the first and second kind [29], denoted by $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$, respectively. Hence,

$$w_1(n, k) = \left[{n+1 \atop k+1}\right], \ W_1(n, k) = \left\{{n+1 \atop k+1}\right\}.$$

We note that Benoumhani [3, 4] already established the fundamental properties of the numbers $w_m(n, k)$ and $W_m(n, k)$ while Dowling [20] gave a detailed discussion of geometric lattices. Other generalizations of the Stirling numbers $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ were already considered by several authors. For instance, Broder [5] defined the $r$-Stirling numbers $\widehat{\left[{n+r \atop k+r}\right]}_r$ and $\widehat{\left\{{n+r \atop k+r}\right\}}_r$ of the first and second kind whose relation to the Whitney numbers is stated in equations (21) and (22) below. Belbachir and Bousbaa [2] recently introduced the translated Whitney numbers $\widetilde{w}_{(\alpha)}(n,k)$ and $\widetilde{W}_{(\alpha)}(n,k)$ of the first and second kind, which are related to the Stirling numbers via

$$\widetilde{w}_{(\alpha)}(n,k) = \alpha^{n-k}\begin{bmatrix} n \\ k \end{bmatrix},\ \widetilde{W}_{(\alpha)}(n,k) = \alpha^{n-k}\begin{Bmatrix} n \\ k \end{Bmatrix}.$$

Furthermore, Mező [27] defined the $r$-Whitney numbers $w_{m,r}(n,k)$ and $W_{m,r}(n,k)$ of the first and second kind as the coefficients in the expressions

$$m^n(x)_n = \sum_{k=0}^{n} w_{m,r}(n,k)(mx+r)^k \tag{3}$$

and

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k)(x)_k. \tag{4}$$

respectively. Further developement of the numbers $w_{m,r}(n,k)$ and $W_{m,r}(n,k)$ were due to Cheon and Jung [7], Merca [26], Corcino et al. [10], Corcino et al. [19], C. B. Corcino and R. B. Corcino [9], and R. B. Corcino and C. B Corcino [14, 15].

Corcino and Hererra [17] introduced the *limit of the differences of the generalized factorial*

$$F_{\alpha,\gamma}(n,k) = \lim_{\beta\to 0} \frac{\left[\Delta_t^k\left(\beta t+\gamma|\alpha\right)_n\right]_{t=0}}{k!\beta^k}, \tag{5}$$

where

$$\left(\beta t+\gamma|\alpha\right)_n = \prod_{j=0}^{n-1}\left(\beta t+\gamma-j\alpha\right),\ \left(\beta t+\gamma|\alpha\right)_0 = 1, \tag{6}$$

which is a generalization of the Stirling numbers of the first kind. The numbers $F_{\alpha,-\gamma}(n,k)$ are actually the $r$-Whitney numbers of the first kind in (3). That is,

$$F_{\alpha,-\gamma}(n,k) = w_{\alpha,\gamma}(n,k).$$

Similarly, Corcino [11] defined the $(r,\beta)$-*Stirling numbers* $\left\langle{n \atop k}\right\rangle_{r,\beta}$ as coefficients in

$$t^n = \sum_{k=0}^{n}\binom{\frac{t-r}{\beta}}{k}\beta^k k!\left\langle{n \atop k}\right\rangle_{r,\beta}. \tag{7}$$

The numbers $\left\langle{n \atop k}\right\rangle_{r,\beta}$ are found to be equivalent to the $r$-Whitney numbers of the second kind in (4). To be precise,

$$\left\langle{n \atop k}\right\rangle_{r,\beta} = W_{\beta,r}(n,k).$$

Corcino et al. [16], and Corcino and Aldema [12] further studied the numbers $\left\langle {n \atop k} \right\rangle_{r,\beta}$.

Recall that the classical *Boson operators* $a$ and $a^\dagger$ satisfy the commutation relation

$$[a, a^\dagger] \equiv aa^\dagger - a^\dagger a = 1. \tag{8}$$

If we define the Fock space by the basis $\{|s\rangle \;;\; s = 0, 1, 2, \ldots\}$, to be referred to as Fock states, the relations $a|s\rangle = \sqrt{s}|s-1\rangle$ and $a^\dagger|s\rangle = \sqrt{s+1}|s+1\rangle$ form a representation that satisfies the commutation relation (8). The operator $\hat{n} \equiv a^\dagger a$, when acting on $|s\rangle$, yields

$$a^\dagger a|s\rangle = s|s\rangle,$$

and the operator $(a^\dagger)^k a^k$, when acting on the same state, yields

$$(a^\dagger)^k a^k|s\rangle = (s)_k|s\rangle.$$

Let $\{\langle s| \equiv (|s\rangle)^\dagger \;;\; s = 0, 1, 2, \ldots\}$ denote the Fock basis of the dual space. Requiring the normalization of the scalar product $\langle 0|0\rangle = 1$ we note that

$$\langle s+1|s+1\rangle = \frac{1}{s+1}\langle s|aa^\dagger|s\rangle = \frac{1}{s+1}\Big(\langle s|a^\dagger a|s\rangle + \langle s|s\rangle\Big) = \langle s|s\rangle.$$

Hence, from the normalization of $|0>$ it follows that all the Fock states are normalized. Moreover, since $\langle s+1|a^\dagger|s\rangle = \sqrt{s+1}\langle s+1|s+1\rangle$ and $(a|s+1\rangle)^\dagger|s\rangle = \sqrt{s+1}\langle s|s\rangle$, it follows that $a^\dagger$ is the Hermitian conjugate of $a$. That is, $a^\dagger a$ is Hermitian. Orthogonality follows from the fact that the Fock states are eigenstates of $a^\dagger a$ with distinct eigenvalues.

Hence, the *horizontal generating functions* of the Stirling numbers $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$,

$$(x)_n = \sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix} x^k \tag{9}$$

and

$$x^n = \sum_{k=0}^{n}\begin{Bmatrix} n \\ k \end{Bmatrix}(x)_k,$$

can be expressed as

$$(a^\dagger)^n a^n = \sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}(a^\dagger a)^k$$

and

$$(a^\dagger a)^n = \sum_{k=0}^{n}\begin{Bmatrix} n \\ k \end{Bmatrix}(a^\dagger)^k a^k,$$

respectively [22].

Now, the defining relations for the $r$-Whitney numbers, (3) and (4), can be expressed as

$$m^n(a^\dagger)^n a^n = \sum_{k=0}^{n} w_{m,r}(n,k)(ma^\dagger a + r)^k \tag{10}$$

and

$$(ma^{\dagger}a+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k)(a^{\dagger})^k a^k. \tag{11}$$

Making use of the $q$-Boson operators [1] that satisfy

$$[a,a^{\dagger}]_q \equiv aa^{\dagger} - qa^{\dagger}a = 1, \tag{12}$$

we have

$$a|s\rangle = \sqrt{[s]_q}|s-1\rangle,\ a^{\dagger}|s\rangle = \sqrt{[s+1]_q}|s+1\rangle,$$

hence,

$$a^{\dagger}a|s\rangle = [s]_q|s\rangle,$$

and

$$(a^{\dagger})^k a^k|s\rangle = [s]_{q,k}|s\rangle.$$

*Remark* 1. Although we use the same notation for the boson and for the $q$-boson operators, no confusion should arise because the meaning of these symbols should be clear from the context.

In line with this, the defining relations for Carlitz's [6] $q$-Stirling numbers of the first and second kind, $\begin{bmatrix} n \\ k \end{bmatrix}_q$ and $\begin{Bmatrix} n \\ k \end{Bmatrix}_q$, can be written in the form [22]

$$(a^{\dagger})^n a^n = \sum_{k=1}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q (a^{\dagger}a)^k \tag{13}$$

and

$$(a^{\dagger}a)^n = \sum_{k=1}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_q (a^{\dagger})^k a^k, \tag{14}$$

respectively.

We define $q$-analogues for the Whitney numbers $w_{m,r}(n,k)$ and $W_{m,r}(n,k)$ via the same pattern as in (13) and (14).

## 2 $(q,r)$-Whitney numbers

**Definition 2.** For non-negative integers $n$ and $k$ and complex numbers $r$ and $m$, the $(q,r)$-Whitney numbers of the first and second kind, denoted by $w_{m,r,q}(n,k)$ and $W_{m,r,q}(n,k)$, respectively, are defined by

$$m^n(a^{\dagger})^n a^n = \sum_{k=0}^{n} w_{m,r,q}(n,k)(ma^{\dagger}a+r)^k \tag{15}$$

and

$$(ma^\dagger a+r)^n=\sum_{k=0}^{n} m^k W_{m,r,q}(n,k)(a^\dagger)^k a^k \tag{16}$$

with initial conditions $w_{m,r,q}(0,0)=W_{m,r,q}(0,0)=1$ and $w_{m,r,q}(n,k)=W_{m,r,q}(n,k)=0$ for $k>n$ and for $k<0$, where the operators $a^\dagger$ and $a$ satisfy the relation in (12).

Before proceeding we note that from (15) and (16),

$$w_{m,0,q}(n,k) = (-m)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}_q, \tag{17}$$

$$W_{m,0,q}(n,k) = m^{n-k}\begin{Bmatrix} n \\ k \end{Bmatrix}_q. \tag{18}$$

Similarly, the $r$-Stirling numbers $\widehat{\left[\begin{smallmatrix} n+r \\ k+r \end{smallmatrix}\right]}_r$ and $\widehat{\left\{\begin{smallmatrix} n+r \\ k+r \end{smallmatrix}\right\}}_r$ are specified by the horizontal generating functions

$$(x-r)_n=\sum_{k=0}^{n}(-1)^{n-k}\widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r x^k,$$

or, equivalently,

$$(x)_n=\sum_{k=0}^{n}(-1)^{n-k}\widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r (x+r)^k,$$

and

$$(x+r)^n=\sum_{k=0}^{n}\widehat{\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}}_r (x)_r.$$

Hence, $\widehat{\left[\begin{smallmatrix} n+r \\ k+r \end{smallmatrix}\right]}_{q,r}$ and $\widehat{\left\{\begin{smallmatrix} n+r \\ k+r \end{smallmatrix}\right\}}_{q,r}$, the $q$-analogues of $\widehat{\left[\begin{smallmatrix} n+r \\ k+r \end{smallmatrix}\right]}_r$ and $\widehat{\left\{\begin{smallmatrix} n+r \\ k+r \end{smallmatrix}\right\}}_r$, respectively, are specified by the horizontal generating functions

$$(a^\dagger)^n a^n = \sum_{k=0}^{n}(-1)^{n-k}\widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_{q,r}(a^\dagger a+r)^k, \tag{19}$$

$$(a^\dagger a+r)^n = \sum_{k=0}^{n}\widehat{\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}}_{q,r}(a^\dagger)^k a^k. \tag{20}$$

It follows that

$$w_{1,r,q}(n,k) = (-1)^{n-k}\widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_{q,r}, \tag{21}$$

$$W_{1,r,q}(n,k) = \widehat{\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}}_{q,r}. \tag{22}$$

We will refer to the $q$-analogues in (19) and (20) as the $(q,r)$-Stirling numbers of the first and second kind, respectively.

**Theorem 3.** *The $(q,r)$-Whitney numbers $w_{m,r,q}(n,k)$ and $W_{m,r,q}(n,k)$ satisfy the following identities:*

$$w_{m,r,q}(n,k) = (-1)^{n-k} \sum_{i=k}^{n} \binom{i}{k} r^{i-k} m^{n-i} \begin{bmatrix} n \\ i \end{bmatrix}_q, \tag{23}$$

$$W_{m,r,q}(n,k) = \sum_{i=k}^{n} \binom{n}{i} r^{n-i} m^{i-k} \begin{Bmatrix} i \\ k \end{Bmatrix}_q. \tag{24}$$

*Proof.* From Eq. (13), we get

$$\begin{aligned} m^n (a^\dagger)^n a^n &= m^n \sum_{i=0}^{n} (-1)^{n-i} \begin{bmatrix} n \\ i \end{bmatrix}_q (a^\dagger a)^i \\ &= m^n \sum_{i=0}^{n} (-1)^{n-i} \begin{bmatrix} n \\ i \end{bmatrix}_q \left( \frac{\hat{z} - r}{m} \right)^i \\ &= m^n \sum_{i=0}^{n} (-1)^{n-i} \begin{bmatrix} n \\ i \end{bmatrix}_q \frac{1}{m^i} \sum_{k=0}^{i} \binom{i}{k} \hat{z}^k (-r)^{i-k} \\ &= \sum_{k=0}^{n} (-1)^{n-k} \left\{ \sum_{i=k}^{n} m^{n-i} \begin{bmatrix} n \\ i \end{bmatrix}_q \binom{i}{k} r^{i-k} \right\} \hat{z}^k, \end{aligned}$$

where $\hat{z} = ma^\dagger a + r$. Furthermore, comparing the coefficient of $\hat{z}^k$ with that in equation (15) yields equation (23).

To prove equation (24), we write

$$\begin{aligned} (ma^\dagger a + r)^n &= \sum_{i=0}^{n} \binom{n}{i} r^{n-i} m^i (a^\dagger a)^i \\ &= \sum_{i=0}^{n} \binom{n}{i} r^{n-i} m^i \sum_{k=0}^{i} \begin{Bmatrix} i \\ k \end{Bmatrix}_q (a^\dagger)^k a^k \\ &= \sum_{k=0}^{n} \left\{ \sum_{i=k}^{n} r^{n-i} m^i \begin{Bmatrix} i \\ k \end{Bmatrix}_q \binom{n}{i} \right\} (a^\dagger)^k a^k. \end{aligned}$$

Comparing the coefficient of $(a^\dagger)^k a^k$ with that in equation (16) gives us (24). □

*Remark* 4. (a) As $q \to 1$, we have

$$w_{m,r}(n,k) = \sum_{i=k}^{n} (-1)^{n-k} \binom{i}{k} r^{i-k} m^{n-i} \begin{bmatrix} n \\ i \end{bmatrix};$$

$$W_{m,r}(n,k) = \sum_{i=k}^{n} \binom{n}{i} r^{n-i} m^{i-k} \begin{Bmatrix} i \\ k \end{Bmatrix}.$$

(b) Note that of all the factors in equations (23) and (24) only the Stirling numbers are $q$-deformed.

The following corollary is a direct consequence of the previous theorem.

**Corollary 5.** *The $(q,r)$-Stirling numbers are given by*

$$\widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_{q,r} = \sum_{i=k}^{n} \binom{i}{k} r^{i-k} \begin{bmatrix} n \\ i \end{bmatrix}_q ; \tag{25}$$

$$\widehat{\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}}_{q,r} = \sum_{i=k}^{n} \binom{n}{i} r^{n-i} \begin{Bmatrix} i \\ k \end{Bmatrix}_q . \tag{26}$$

# 3 Some recurrence relations

In this section, we present some recurrence relations involving the $(q,r)$-Whitney numbers.

We recall the $q$-boson identities

$$[a, (a^{\dagger})^n]_{q^n} = [n]_q (a^{\dagger})^{n-1}$$

and

$$[a^n, a^{\dagger}]_{q^n} = [n]_q a^{n-1},$$

that can be easily established by induction. The latter can also be written in the form

$$a^{\dagger} a^n = q^{-n}(a^n a^{\dagger} - [n]_q a^{n-1}).$$

**Theorem 6.** *The $(q,r)$-Whitney numbers $w_{m,r,q}(n,k)$ and $W_{m,r,q}(n,k)$ satisfy the following triangular recurrence relations:*

$$w_{m,r,q}(n+1,k) = q^{-n}\Big(w_{m,r,q}(n,k-1) - (m[n]_q + r) w_{m,r,q}(n,k)\Big), \tag{27}$$

$$W_{m,r,q}(n+1,k) = q^{k-1} W_{m,r,q}(n,k-1) + (m[k]_q + r) W_{m,r,q}(n,k). \tag{28}$$

*Proof.* From equation (15), $\sum_{k=0}^{n+1} w_{m,r,q}(n+1,k)(ma^{\dagger}a + r)^k = m^{n+1}(a^{\dagger})^n (a^{\dagger} a^n) a$

$$\begin{aligned}
&= m^{n+1}(a^{\dagger})^n q^{-n}(a^n a^{\dagger} - [n]_q a^{n-1}) a \\
&= m^{n+1} q^{-n} \left((a^{\dagger})^n a^n\right)(a^{\dagger} a) - m^{n+1} q^{-n} [n]_q (a^{\dagger})^n a^n \\
&= q^{-n} \sum_{k=0}^{n} w_{m,r,q}(n,k)(ma^{\dagger}a + r)^k (ma^{\dagger}a + r - r) - m q^{-n} [n]_q \sum_{k=0}^{n} w_{m,r,q}(n,k)(ma^{\dagger}a + r)^k \\
&= q^{-n} \sum_{k=1}^{n+1} w_{m,r,q}(n,k-1)(ma^{\dagger}a + r)^k - q^{-n}(m[n]_q + r) \sum_{k=0}^{n} w_{m,r,q}(n,k)(ma^{\dagger}a + r)^k \\
&= q^{-n} \sum_{k=0}^{n+1} \{w_{m,r,q}(n,k-1) - (m[n]_q + r) w_{m,r,q}(n,k)\} (ma^{\dagger}a + r)^k.
\end{aligned}$$

Equating coefficients of $(ma^\dagger a + r)^k$ gives us (27) and a similar derivation yields equation (28). $\square$

Equations (27) and (28) are useful in computing the first few values of $w_{m,r,q}(n,k)$ and $W_{m,r,q}(n,k)$, using the initial values specified above.

*Remark* 7. (a) From (27) we obtain the explicit expression

$$w_{m,r,q}(n,0) = (-1)^n q^{-\frac{n(n-1)}{2}} \prod_{i=0}^{n-1} (m[i]_q + r).$$

On the other hand, the relation (23) yields

$$w_{m,r,q}(n,0) = (-1)^n \sum_{i=0}^{n} r^i m^{n-i} \begin{bmatrix} n \\ i \end{bmatrix}_q .$$

Equating these expressions and substituting $x = \frac{r}{m}$ we obtain

$$\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q x^i = q^{-\frac{n(n-1)}{2}} \prod_{i=0}^{n-1} ([i]_q + x).$$

This is a horizontal generating function for the $q$-Stirling numbers of the first kind in terms of a $q$-analogue of the rising factorial. Indeed, replacing $x$ by $-[s]_q$, and noting that

$$[s]_q - [i]_q = q^{-i}[s-i]_q$$

and

$$\prod_{i=0}^{n-1} q^i = q^{\binom{n}{2}},$$

we obtain

$$\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q (-1)^i [s]_q^i = (-1)^n \prod_{i=0}^{n-1} [s-i]_q .$$

(b) From (28) $W_{m,r,q}(n+1,0) = rW_{m,r,q}(n,0)$, hence $W_{m,r,q}(n,0) = r^n$. The same result is obtained from (24). That is,

$$W_{m,r,q}(n,0) = \sum_{i=0}^{n} \binom{n}{i} r^{n-i} m^i \delta_{i,0} = r^n .$$

(c) As $q \to 1$, we have

$$w_{m,r}(n+1,k) = w_{m,r}(n,k-1) - (mn+r) w_{m,r}(n,k);$$

$$W_{m,r}(n+1,k) = W_{m,r}(n,k-1) + (mk+r) W_{m,r}(n,k).$$

This confirms that $w_{m,r,q}(n,k)$ and $W_{m,r,q}(n,k)$ are proper $q$-analogues of $w_{m,r}(n,k)$ and $W_{m,r}(n,k)$, respectively.

As a consequence of the previous theorem, when $m = 1$ we have

**Corollary 8.** *The $(q, r)$-Stirling numbers satisfy the following triangular recurrence relations:*

$$\begin{aligned} \left[\widehat{\begin{matrix} n+1+r \\ k+r \end{matrix}}\right]_{q,r} &= q^{-n}\left[\widehat{\begin{matrix} n+r \\ k-1+r \end{matrix}}\right]_{q,r} + ([n]_q + r)q^{-n}\left[\widehat{\begin{matrix} n+r \\ k+r \end{matrix}}\right]_{q,r}, \\ \left\{\widehat{\begin{matrix} n+1+r \\ k+r \end{matrix}}\right\}_{q,r} &= q^{k-1}\left\{\widehat{\begin{matrix} n+r \\ k-1+r \end{matrix}}\right\}_{q,r} + ([k]_q + r)\left\{\widehat{\begin{matrix} n+r \\ k+r \end{matrix}}\right\}_{q,r}. \end{aligned}$$

We can use these recurrence relations to compute the first few values of the $(q, r)$-Stirling numbers of the first and second kind, respectively.

**Theorem 9.** *The $(q, r)$-Whitney numbers satisfy the following recurrence relations*

$$w_{m,r+1,q}(n, \ell) = \sum_{k=\ell}^{n} \binom{k}{\ell} (-1)^{k-\ell} w_{m,r,q}(n, k), \tag{29}$$

$$W_{m,r+1,q}(n, k) = \sum_{\ell=k}^{n} \binom{n}{\ell} W_{m,r,q}(\ell, k). \tag{30}$$

*Proof.* From equation (15), we have

$$\begin{aligned} m^n (a^\dagger)^n a^n &= \sum_{k=0}^{n} w_{m,r,q}(n, k)(ma^\dagger a + r)^k \\ &= \sum_{k=0}^{n} w_{m,r,q}(n, k)\Big((ma^\dagger a + r + 1) - 1\Big)^k \\ &= \sum_{k=0}^{n} w_{m,r,q}(n, k) \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{k-\ell} (ma^\dagger a + r + 1)^\ell \\ &= \sum_{\ell=0}^{n} (ma^\dagger a + r + 1)^\ell \sum_{k=\ell}^{n} \binom{k}{\ell} (-1)^{k-\ell} w_{m,r,q}(n, k). \end{aligned}$$

On the other hand,

$$m^n (a^\dagger)^n a^n = \sum_{\ell=0}^{n} w_{m,r+1,q}(n, \ell)(ma^\dagger a + r + 1)^\ell.$$

Hence, by comparing the coefficients of $(ma^\dagger a + r + 1)^\ell$ we obtain equation (29).

Similarly, from equation (16)

$$(ma^\dagger a + r + 1)^n = \sum_{k=0}^{n} m^k W_{m,r+1,q}(n, k)(a^\dagger)^k a^k,$$

and since

$$\begin{aligned}(ma^\dagger a+r+1)^n &= \sum_{\ell=0}^{n}\binom{n}{\ell}(ma^\dagger a+r)^\ell\\ &= \sum_{\ell=0}^{n}\binom{n}{\ell}\sum_{k=0}^{\ell}m^k W_{m,r,q}(\ell,k)(a^\dagger)^k a^k\\ &= \sum_{k=0}^{n}m^k(a^\dagger)^k a^k\sum_{\ell=k}^{n}\binom{n}{\ell}W_{m,r,q}(\ell,k),\end{aligned}$$

we obtain equation (30). □

When $m=1$, the theorem reduces to the recursion formulas for $(q,r)$-Stirling numbers. That is,

**Corollary 10.**

$$\begin{aligned}\widehat{\begin{bmatrix}n+r+1\\ l+r+1\end{bmatrix}}_{q,r+1} &= \sum_{k=l}^{n}(-1)^{l-k}\binom{k}{l}\widehat{\begin{bmatrix}n+r\\ k+r\end{bmatrix}}_{q,r},\\ \widehat{\begin{Bmatrix}n+r+1\\ k+r+1\end{Bmatrix}}_{q,r+1} &= \sum_{l=k}^{n}\binom{n}{l}\widehat{\begin{Bmatrix}l+r\\ k+r\end{Bmatrix}}_{q,r}.\end{aligned}$$

# 4 Orthogonality and inverse relations

**Theorem 11.** *The $(q,r)$-Whitney numbers $w_{m,r,q}(n,k)$ and $W_{m,r,q}(k,j)$ satisfy the following orthogonality relations:*

$$\sum_{k=j}^{n}W_{m,r,q}(n,k)w_{m,r,q}(k,j)=\delta_{jn}, \tag{31}$$

*and*

$$\sum_{k=j}^{n}w_{m,r,q}(n,k)W_{m,r,q}(k,j)=\delta_{jn}, \tag{32}$$

*where $\delta_{jn}$ is the Kronecker delta.*

*Proof.* Using equation (15) we substitute $m^k(a^\dagger)^k a^k$ in (16), obtaining

$$\begin{aligned}(ma^\dagger a+r)^n &= \sum_{k=0}^{n}W_{m,r,q}(n,k)\sum_{j=0}^{k}w_{m,r,q}(k,j)(ma^\dagger a+r)^j\\ &= \sum_{j=0}^{n}\left\{\sum_{k=j}^{n}W_{m,r,q}(n,k)w_{m,r,q}(k,j)\right\}(ma^\dagger a+r)^j.\end{aligned}$$

Comparing the coefficients of $(ma^\dagger a + r)^j$ yields equation (31). Equation (32) is obtained similarly. □

The classical *binomial inversion formula* given by

$$f_k = \sum_{j=0}^{k} \binom{k}{j} g_j \Leftrightarrow g_k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f_j \tag{33}$$

can be a useful tool in deriving the explicit formula of the classical Stirling numbers of the second kind. The $q$-analogue of (33) is given by [8]

$$f_k = \sum_{j=0}^{k} \binom{k}{j}_q g_j \Leftrightarrow g_k = \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \binom{k}{j}_q f_j, \tag{34}$$

The next theorem presents an inverse relation for the $(q,r)$-Whitney numbers $w_{m,r,q}(n,k)$ and $W_{m,r,q}(k,j)$.

**Theorem 12.** *The $(q,r)$-Whitney numbers $w_{m,r,q}(n,\ell)$ and $W_{m,r,q}(n,\ell)$ satisfy the following inverse relation:*

$$f_n = \sum_{\ell=0}^{n} w_{m,r,q}(n,\ell) g_\ell \Leftrightarrow g_n = \sum_{\ell=0}^{n} W_{m,r,q}(n,\ell) f_\ell. \tag{35}$$

*Proof.* By the hypothesis,

$$\begin{aligned}
\sum_{\ell=0}^{n} W_{m,r,q}(n,\ell) f_\ell &= \sum_{\ell=0}^{n} W_{m,r,q}(n,\ell) \sum_{k=0}^{\ell} w_{m,r,q}(\ell,k) g_k \\
&= \sum_{k=0}^{n} \left\{ \sum_{\ell=k}^{n} W_{m,r,q}(n,\ell) w_{m,r,q}(\ell,k) \right\} g_k \\
&= \sum_{k=0}^{n} \{\delta_{kn}\} g_k \\
&= g_n.
\end{aligned}$$

The converse can be shown similarly. □

The next theorem can be deduced in a similar way, from the orthogonality relations

**Theorem 13.** *The $(q,r)$-Whitney numbers $w_{m,r,q}(n,\ell)$ and $W_{m,r,q}(n,\ell)$ satisfy the following inverse relation:*

$$f_\ell = \sum_{n=\ell}^{\infty} w_{m,r,q}(n,\ell) g_n \Leftrightarrow g_\ell = \sum_{n=\ell}^{\infty} W_{m,r,q}(n,\ell) f_n. \tag{36}$$

## 5 $(q, r)$-Dowling polynomials and numbers

Cheon and Jung [7] defined the *r-Dowling polynomials*, denoted by $D_{m,r}(n, x)$, in terms of sums of $r$-Whitney numbers of the second kind. That is,

$$D_{m,r}(n, x) = \sum_{k=0}^{n} W_{m,r}(n, k)x^k. \tag{37}$$

When $x = 1$, we obtain the *r-Dowling numbers* $D_{m,r}(n) \equiv D_{m,r}(n, 1)$. The polynomials (37) are actually equivalent to the $(r, \beta)$-Bell polynomials $G_{n,\beta,r}(x)$ of R. B. Corcino and C. B. Corcino [13]. That is,

$$D_{\beta,r}(n, x) = G_{n,\beta,r}(x).$$

Moreover,

- when $m = 1$ and $r = 1$, we recover the classical *Dowling polynomials* $D(n, x) \equiv D_{1,1}(n, x)$;
- when $m = 1$ and $r = 0$, we recover the classical *Bell polynomials* $B_n(x) \equiv D_{1,0}(n, x)$;
- when $m = 1$, we recover Mező's [28] *r-Bell polynomials* $B_{n,r}(x)$. That is, $D_{1,r}(n, x) = B_{n,r}(x)$; and
- when $m = \alpha$ and $r = 0$, we recover the *translated Dowling polynomials* $\widetilde{D}_{(\alpha)}(n; x)$ by Mangontarum et al. [25]. That is, $D_{\alpha,0}(n, x) = \widetilde{D}_{(\alpha)}(n; x)$.

Taking these into consideration, the next definition seems to be natural.

**Definition 14.** For non-negative integers $n$ and $k$, and complex numbers $m$ and $r$, the $(q, r)$-Dowling polynomials, denoted by $D_{m,r,q}(n, x)$, are defined by

$$D_{m,r,q}(n, x) = \sum_{k=0}^{n} W_{m,r,q}(n, k)x^k \tag{38}$$

and the $(q, r)$-Dowling numbers, denoted by $D_{m,r,q}(n)$, are defined by

$$D_{m,r,q}(n) = D_{m,r,q}(n, 1). \tag{39}$$

The *coherent states*

$$|\gamma\rangle = \exp\left(-\frac{|\gamma|^2}{2}\right)\sum_{k\geq 0}\frac{\gamma^k}{\sqrt{k!}}|k\rangle, \tag{40}$$

where $\gamma$ is an arbitrary (complex-valued) constant, satisfy $a|\gamma\rangle = \gamma|\gamma\rangle$ and $\langle\gamma|\gamma\rangle = 1$. Katriel [23] gave an illustration on how (40) can be a very useful tool in the derivation of certain

Dobinski-type formulas. The *q-coherent states* corresponding to the $q$-Boson operators were defined as

$$|\gamma\rangle_q = \left(\widehat{e}_q(-|\gamma|^2)\right)^{\frac{1}{2}} \sum_{k\geq 0} \frac{\gamma^k}{\sqrt{[k]_q!}}|k\rangle \tag{41}$$

which satisfy $a|\gamma\rangle = \gamma|\gamma\rangle$. Here, $\widehat{e}_q(x)$ is the type 2 $q$-exponential function given by

$$\widehat{e}_q(x) = \prod_{i=1}^{\infty}(1+(1-q)q^{i-1}x) = \sum_{i\geq 0} q^{\binom{i}{2}} \frac{x^i}{[i]_q!}, \tag{42}$$

which is the inverse of the type 1 $q$-exponential function

$$e_q(x) = \prod_{i=1}^{\infty}(1-(1-q)q^{i-1}x)^{-1} = \sum_{i\geq 0} \frac{x^i}{[i]_q!}. \tag{43}$$

That is, $e_q(x)\widehat{e}_q(-x) = 1$.

Taking the expectation value of both sides of (16) with respect to $|\gamma\rangle$ yields

$$\langle\gamma|(ma^\dagger a + r)^n|\gamma\rangle = \sum_{k=0}^{n} m^k W_{m,r,q}(n,k)|\gamma|^{2k}. \tag{44}$$

The left-hand-side can be evaluated using the $q$-coherent states in (41), yielding

$$\langle\gamma|(ma^\dagger a + r)^n|\gamma\rangle = \widehat{e}_q\left(-|\gamma|^2\right) \sum_{k\geq 0} \frac{|\gamma|^{2k}}{[k]_q!}(m[k]_q + r)^n. \tag{45}$$

Defining $x = m|\gamma|^2$ we obtain

$$\sum_{k=0}^{n} W_{m,r,q}(n,k)x^k = \widehat{e}_q\left(-\frac{x}{m}\right) \sum_{k\geq 0} \left(\frac{x}{m}\right)^k \frac{(m[k]_q + r)^n}{[k]_q!}. \tag{46}$$

Using (38), the following theorem is easily observed.

**Theorem 15.** *The $(q,r)$-Dowling polynomials $D_{m,r,q}(n,x)$ and the $(q,r)$-Dowling numbers $D_{m,r,q}(n)$ have the following explicit formulas:*

$$D_{m,r,q}(n,x) = \widehat{e}_q\left(-\frac{x}{m}\right) \sum_{k\geq 0} \left(\frac{x}{m}\right)^k \frac{(m[k]_q + r)^n}{[k]_q!}, \tag{47}$$

*and*

$$D_{m,r,q}(n) = \widehat{e}_q\left(-m^{-1}\right) \sum_{k\geq 0} \frac{(m[k]_q + r)^n}{m^k[k]_q!}. \tag{48}$$

*Proof.* (48) can be obtained by letting $x = 1$ in (47). □

Katriel [23] defined the $q$-Bell polynomial as

$$\sum_{\ell=0}^{k} \left\{ {k \atop \ell} \right\}_q x^\ell = \widehat{e}_q(x) \sum_{m=1}^{\infty} x^m \frac{[m]_q^k}{[m]_q!}. \tag{49}$$

Expanding the right-hand side using (42) yields

$$\sum_{\ell=0}^{k} \left\{ {k \atop \ell} \right\}_q x^\ell = \sum_{\ell=0}^{\infty} \frac{x^\ell}{[\ell]_q!} \sum_{j=0}^{\ell} (-1)^{\ell-j} q^{\binom{\ell-j}{2}} \binom{\ell}{j}_q [j]_q^k. \tag{50}$$

Equating coefficients of equal powers of $x$ gives us

$$\left\{ {k \atop \ell} \right\}_q = \frac{1}{[\ell]_q!} \sum_{j=0}^{\ell} (-1)^{\ell-j} q^{\binom{\ell-j}{2}} \binom{\ell}{j}_q [j]_q^k. \tag{51}$$

Notice that as $q \to 1$, (51) reduces to the well-known explicit formula of $\left\{ {k \atop j} \right\}$. That is

$$\lim_{q\to 1} \left\{ {k \atop \ell} \right\}_q = \frac{1}{\ell!} \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} j^k. \tag{52}$$

In the following theorem, we will present an expression analogous to (51) for the $q$-analogue $W_{m,r,q}(n,k)$.

**Theorem 16.** *The $(q,r)$-Whitney numbers of the second kind, $W_{m,r,q}(n,k)$, have the following explicit formula:*

$$W_{m,r,q}(n,\ell) = \frac{1}{m^\ell [\ell]_q!} \sum_{k=0}^{\ell} (-1)^{\ell-k} q^{\binom{\ell-k}{2}} \binom{\ell}{k}_q (m[k]_q + r)^n. \tag{53}$$

*Proof.* Substituting $y = \frac{x}{m}$ in (47) gives us

$$\begin{aligned} \sum_{k=0}^{n} m^k W_{m,r,q}(n,k) y^k &= \sum_{i\geq 0} q^{\binom{i}{2}} \frac{(-y)^i}{[i]_q!} \sum_{k\geq 0} y^k \frac{(m[k]_q + r)^n}{[k]_q!} \\ &= \sum_{\ell\geq 0} \frac{y^\ell}{[\ell]_q!} \sum_{k=0}^{\ell} (-1)^{\ell-k} q^{\binom{\ell-k}{2}} \binom{\ell}{k}_q (m[k]_q + r)^n. \end{aligned}$$

Equating the coefficients of equal powers of $y$ on both sides of this equation we obtain equation (53). □

Note that as $q \to 1$, we have

$$\begin{aligned}\lim_{q\to 1} W_{m,r,q}(n,\ell) &= \frac{1}{m^\ell \ell!}\sum_{k=0}^{\ell}(-1)^{\ell-k}\binom{\ell}{k}(mk+r)^n \\ &= W_{m,r}(n,\ell).\end{aligned}$$

Furthermore,

$$\lim_{q\to 1} W_{m,1,q}(n,l) = W_m(n,l).$$

*Remark* 17. We can also prove (53) in the following manner: First, we write (16) as

$$\begin{aligned}(m[\ell]_q + r)^n &= \sum_{k=0}^{n} m^k W_{m,r,q}(n,k)[\ell]_{q,k} \\ &= \sum_{k=0}^{\ell}\binom{\ell}{k}_q \left\{\frac{m^k W_{m,r,q}(n,k)[\ell]_{q,k}}{\binom{\ell}{k}_q}\right\}.\end{aligned}$$

Next, we apply the $q$-binomial inversion formula in (34) which gives us

$$\frac{m^\ell W_{m,r,q}(n,\ell)[\ell]_{q,\ell}}{\binom{k}{k}_q} = \sum_{k=0}^{\ell}(-1)^{\ell-k}q^{\binom{\ell-k}{2}}\binom{l}{k}_q (m[k]_q + r)^n.$$

This is precisely the explicit formula obtained in the previous theorem.

Now, using (53),

$$\begin{aligned}\sum_{n\geq 0} W_{m,r,q}(n,k)\frac{t^n}{[n]_q!} &= \sum_{n\geq 0}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{m^k[k]_q!}q^{\binom{k-j}{2}}\binom{k}{j}_q (m[j]_q+r)^n\frac{t^n}{[n]_q!} \\ &= \frac{1}{m^k[k]_q!}\sum_{j=0}^{k}(-1)^{k-j}q^{\binom{k-j}{2}}\binom{k}{j}_q e_q\left[(m[j]_q+r)t\right],\end{aligned}$$

where $e_q(x)$ is the type 1 $q$-exponential function in (43). Thus, we have the following theorem.

**Theorem 18.** *The $(q,r)$-Whitney numbers of the second kind satisfy the following exponential generating function:*

$$\sum_{n\geq 0} W_{m,r,q}(n,k)\frac{t^n}{[n]_q!} = \frac{1}{m^k[k]_q!}\sum_{j=0}^{n}(-1)^{k-j}q^{\binom{k-j}{2}}\binom{k}{j}_q e_q\left[(m[j]_q+r)t\right]. \tag{54}$$

*Remark* 19. As $q \to 1$, we have

$$\lim_{q\to 1}\sum_{n\geq 0} W_{m,r,q}(n,k)\frac{t^n}{[n]_q!} = \frac{e^{rt}}{k!}\left(\frac{e^{mt}-1}{m}\right)^k,$$

which is the exponential generating function of the $r$-Whitney numbers of the second kind.

The $q$-difference operator [24] can be written in the form

$$\Delta_q^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \binom{k}{j}_q f(x+j). \tag{55}$$

We are now ready to state the next theorem.

**Theorem 20.** *The $(q,r)$-Whitney numbers of the second kind satisfy the following identity:*

$$\sum_{n\geq 0} W_{m,r,q}(n,k)\frac{t^n}{[n]_q!} = \left\{\Delta_q^k\left(\frac{e_q[(m[x]_q+r)t]}{m^k[k]_q!}\right)\right\}_{x=0}. \tag{56}$$

*Proof.* (56) follows directly from (54) and (55). □

The next corollary is easily verified.

**Corollary 21.** *The $(q,r)$-Whitney numbers of the second kind can be expressed explicitly as*

$$W_{m,r,q}(n,k) = \left\{\Delta_q^k\left(\frac{(m[x]_q+r)^n}{m^k[k]_q!}\right)\right\}_{x=0}. \tag{57}$$

# 6 Further identities for the $(q,r)$-Whitney numbers

Graham et al. [21] presented a useful set of Stirling number identities while Katriel [22] presented the $q$-analogues of all but two of them. Three of these identities are generalized to the $(q,r)$-Whitney numbers using appropriate modifications of the procedures presented by Katriel [22]. Their derivation requires the following.

**Lemma 22.** *For $f(x)$ a polynomial, the operator identity*

$$a^\dagger f(1+qa^\dagger a)a = a^\dagger a f(a^\dagger a), \tag{58}$$

*holds.*

*Proof.* We write the $q$-commutation relation, equation (12), in the form $aa^\dagger = 1+qa^\dagger a$. It follows that

$$(a^\dagger a)(a^\dagger a)^k = a^\dagger(aa^\dagger)^k a = a^\dagger(1+qa^\dagger a)^k a.$$

For $f(x)=\sum_k c_k x^k$ we obtain

$$\begin{aligned} a^\dagger a f(a^\dagger a) &= \sum_k c_k (a^\dagger a)(a^\dagger a)^k \\ &= \sum_k c_k a^\dagger (1+qa^\dagger a)^k a = a^\dagger\left(\sum_k c_k(1+qa^\dagger a)^k\right)a \\ &= a^\dagger f(1+qa^\dagger a)a. \end{aligned}$$

□

*Remark* 23. The lemma can also be written in the form

$$a^\dagger g(a^\dagger a)a = a^\dagger a g\left(\frac{1}{q}(a^\dagger a - 1)\right), \tag{59}$$

where $g(x)$ is a polynomial.

**Theorem 24** (Identity 1)**.** *The* $(q,r)$*-Whitney numbers of the second kind satisfy*

$$W_{m,r,q}(n+1,k) - rW_{m,r,q}(n,k) = \sum_{\ell=k-1}^{n} \binom{n}{\ell} q^\ell (m + r(1-q))^{n-\ell} W_{m,r,q}(\ell, k-1).$$

*Proof.* In terms of the identity (58) and with the aid of (16)

$$\begin{aligned}
a^\dagger\Big(m(1+qa^\dagger a)+r\Big)^n a &= a^\dagger a(ma^\dagger a + r)^n \\
&= \frac{1}{m}(ma^\dagger a + r - r)(ma^\dagger a + r)^n \\
&= \frac{1}{m}(ma^\dagger a + r)^{n+1} - \frac{r}{m}(ma^\dagger a + r)^n \\
&= \sum_{k=0}^{n+1} m^{k-1}(a^\dagger)^k a^k \Big(W_{m,r,q}(n+1,k) - rW_{m,r,q}(n,k)\Big).
\end{aligned}$$

On the other hand, defining $\alpha = m + r(1-q)$ (which will hold throught the present section),

$$\begin{aligned}
a^\dagger\Big(m(1+qa^\dagger a)+r\Big)^n a &= a^\dagger\Big(q(ma^\dagger a + r) + \alpha\Big)^n a \\
&= a^\dagger\left(\sum_{\ell=0}^{n}\binom{n}{\ell} q^\ell \alpha^{n-\ell}(ma^\dagger a + r)^\ell\right) a \\
&= \sum_{\ell=0}^{n}\binom{n}{\ell} q^\ell \alpha^{n-\ell} \sum_{k=0}^{\ell} m^k W_{m,r,q}(\ell,k)(a^\dagger)^{k+1}a^{k+1} \\
&= \sum_{k=1}^{n+1} m^{k-1}(a^\dagger)^k a^k \sum_{\ell=k-1}^{n}\binom{n}{\ell} q^\ell \alpha^{n-\ell} W_{m,r,q}(\ell,k-1).
\end{aligned}$$

Equating coefficients of $m^{k-1}(a^\dagger)^k a^k$ the theorem follows. □

For $r=0$ this identity reduces to the $q$-Stirling numbers identity [22, identity 1]

$$W_{m,0,q}(n+1,k) = \sum_{\ell=k-1}^{n}\binom{n}{\ell} q^\ell m^{n-\ell} W_{m,0,q}(\ell,k-1).$$

The following corollary is an immediate consequence of the previous theorem.

**Corollary 25.** *As $q \to 1$,*

$$W_{m,r}(n+1,k) - rW_{m,r}(n,k) = \sum_{\ell=k-1}^{n} \binom{n}{\ell} m^{n-\ell} W_{m,r}(\ell,k-1).$$

**Theorem 26** (Identity 2)**.** *The $(q,r)$-Whitney numbers of the first kind satisfy*

$$\begin{aligned} w_{m,r,q}(n+1,\ell) &= \sum_{k=\ell-1}^{n} \frac{1}{q^k} w_{m,r,q}(n,k)\Big(-(m+r(1-q)\Big)^{k-\ell} \\ &\quad \cdot \left(\binom{k}{\ell-1}(-(m+r(1-q))) - r\binom{k}{\ell}\right). \end{aligned}$$

*Proof.* We note that from (15),

$$m^{n+1}(a^\dagger)^{n+1}a^{n+1} = \sum_{\ell=0}^{n+1} w_{m,r,q}(n+1,\ell)(ma^\dagger a + r)^\ell.$$

On the other hand, using (59),

$$\begin{aligned} m^{n+1}(a^\dagger)^{n+1}a^{n+1} &= ma^\dagger\Big(m^n(a^\dagger)^n a^n\Big)a \\ &= ma^\dagger\Big(\sum_{k=0}^{n} w_{m,r,q}(n,k)(ma^\dagger a + r)^k\Big)a \\ &= ma^\dagger a \sum_{k=0}^{n} w_{m,r,q}(n,k)\Big(\frac{m}{q}(a^\dagger a - 1) + r\Big)^k \\ &= ma^\dagger a \sum_{k=0}^{n} w_{m,r,q}(n,k)\frac{1}{q^k}\Big((ma^\dagger a + r) - \alpha\Big)^k \\ &= ((ma^\dagger a + r) - r)\sum_{k=0}^{n} w_{m,r,q}(n,k)\frac{1}{q^k}\sum_{\ell=0}^{k}\binom{k}{\ell}(ma^\dagger a + r)^\ell(-\alpha)^{k-\ell} \\ &= \sum_{k=0}^{n} w_{m,r,q}(n,k)\frac{1}{q^k}\sum_{\ell=0}^{k}\binom{k}{\ell}(ma^\dagger a + r)^{\ell+1}(-\alpha)^{k-\ell} \\ &\quad -r\sum_{k=0}^{n} w_{m,r,q}(n,k)\frac{1}{q^k}\sum_{\ell=0}^{n}\binom{k}{\ell}(ma^\dagger a + r)^\ell(-\alpha)^{k-\ell} \\ &= \sum_{\ell=0}^{n+1}(ma^\dagger a + r)^\ell \sum_{k=\ell-1}^{n}\frac{1}{q^k}w_{m,r,q}(n,k)(-\alpha)^{k-\ell} \cdot \\ &\quad \cdot\left(\binom{k}{\ell-1}(-\alpha) - r\binom{k}{\ell}\right). \end{aligned}$$

Equating the coefficients of equal powers of $ma^\dagger a + r$ we obtain the theorem. □

For $r=0$, we recover the $q$-Stirling numbers identity [22, identity 2]

$$w_{m,0,q}(n+1,\ell)=\sum_{k=\ell-1}^{n}\frac{1}{q^k}w_{m,0,q}(n,k)(-m)^{k-\ell+1}\binom{k}{\ell-1},$$

Moreover, we have the following corollary:

**Corollary 27.** *As $q\to 1$,*

$$w_{m,r}(n+1,\ell)=-\sum_{k=\ell-1}^{n}w_{m,r}(n,k)(-m)^{k-\ell}\left(m\binom{k}{\ell-1}+r\binom{k}{\ell}\right).$$

**Theorem 28** (Identity 3)**.** *The $(q,r)$-Whitney numbers of the second kind satisfy*

$$W_{m,r,q}(n,k-1)=\frac{1}{q^n}\sum_{\ell=k}^{n+1}(-m-r(1-q))^{n-\ell}\left(\binom{n}{\ell-1}(-m-r(1-q))-\binom{n}{\ell}r\right)W_{m,r,q}(\ell,k).$$

*Proof.* Note that

$$\begin{aligned}a^\dagger(ma^\dagger a+r)^n a &= \sum_{k=0}^{n}m^k W_{m,r,q}(n,k)(a^\dagger)^{k+1}a^{k+1}\\ &= \sum_{k=1}^{n+1}m^{k-1}W_{m,r,q}(n,k-1)(a^\dagger)^k a^k,\end{aligned}$$

and on the other hand, using (59),

$$\begin{aligned}a^\dagger(ma^\dagger a+r)^n a &= a^\dagger a\left(\frac{m}{q}(a^\dagger a-1)+r\right)^n = a^\dagger a\frac{1}{q^n}(ma^{(\dagger}a+r)-\alpha)^n\\ &= \frac{1}{m}((ma^\dagger a+r)-r)\frac{1}{q^n}\sum_{\ell=0}^{n}\binom{n}{\ell}(ma^\dagger a+r)^\ell(-\alpha))^{n-\ell}\\ &= \frac{1}{mq^n}\sum_{\ell=1}^{n+1}(ma^\dagger a+r)^\ell(-\alpha)^{n-\ell}\cdot\left(\binom{n}{\ell-1}(-\alpha)-r\binom{n}{\ell}\right)\\ &= \frac{1}{mq^n}\sum_{k=0}^{n+1}m^k(a^\dagger)^k a^k\sum_{\ell=k}^{n+1}(-\alpha)^{n-\ell}\left(\binom{n}{\ell-1}(-\alpha)-\binom{n}{\ell}r\right)W_{m,r,q}(\ell,k).\end{aligned}$$

Equating the coefficients of $(a^\dagger)^k a^k$ we obtain the theorem. □

For $r=0$ this theorem reduces to

$$W_{m,0,q}(n,k-1)=\frac{1}{q^n}\sum_{\ell=k}^{n+1}(-m)^{n+1-\ell}\binom{n}{\ell-1}W_{m,0,q}(\ell,k).$$

Using equation (18), we can verify that this is just the $q$-Stirling numbers identity [22, identity 3]. The next corollary is easily verified.

**Corollary 29.** *As* $q \to 1$,

$$W_{m,r}(n,k-1) = \sum_{\ell=k}^{n+1}(-m)^{n-\ell}\left[\binom{n}{\ell-1}(-m) - \binom{n}{\ell}r\right]W_{m,r}(\ell,k).$$

Presently, much is yet to be learnt regarding the $(q,r)$-Whitney numbers. The classical $r$-Whitney and Stirling numbers are known to have various applications in different fields. It is tempting to explore applications for the $(q,r)$-Whitney numbers.

To close this section, Corcino and Hererra [17] defined the $q$-analogue of the limit of the differences of the generalized factorial $F_{\alpha,\gamma}(n,k)$ in (5), denoted by $\phi_{\alpha,\gamma}[n,k]_q$. $\phi_{\alpha,\gamma}[n,k]_q$ can be defined in terms of the relation

$$\sum_{k=0}^{n}\phi_{\alpha,\gamma}[n,k]_q t^k = \langle t+[\gamma]_q|[\alpha]_q\rangle_n^q\,, \tag{60}$$

where

$$\langle t+[\gamma]_q|[\alpha]_q\rangle_n^q = \prod_{j=0}^{n-1}\left(t+[\gamma]_q - [j\alpha]_q\right). \tag{61}$$

The numbers $\phi_{\alpha,\gamma}[n,k]_q$ are actually $q$-analogues of the numbers $w_{m,r}(n,k)$. Similarly, Corcino and Montero [18] defined the $q$-analogue $\sigma[n,k]_q^{\beta,r}$ of the Rucinski-Voigt numbers in terms of the reccurence relation

$$\sigma[n,k]_q^{\beta,r} = \sigma[n-1,k-1]_q^{\beta,r} + \left([k\beta]_q + [r]_q\right)\sigma[n-1,k]_q^{\beta,r}. \tag{62}$$

$\sigma[n,k]_q^{\beta,r}$ is also a $q$-analogue of the numbers $\left\langle {n \atop k} \right\rangle_{r,\beta}$ and $W_{m,r}(n,k)$. However, by comparing the defining relations for $\phi_{\alpha,\gamma}[n,k]_q$ and $\sigma[n,k]_q^{\beta,r}$ with those of the$(q,r)$-Whitney numbers $w_{m,r,q}(n,k)$ and $W_{m,r,q}(n,k)$, respectively, we note that they represent distinctly motivated $q$-analogues that cannot be simply related to one another.

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