Journal of Integer Sequences, Vol. 18 (2015),
Article 15.4.2

New Congruences for Partitions where the Odd Parts are Distinct

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Abstract

Let $\mathrm{pod}(n)$ denote the number of partitions of n wherein odd parts are distinct (and even parts are unrestricted). We find some new interesting congruences for $\mathrm{pod}(n)$ modulo 3, 5 and 9.

1 Introduction and Main Results

Let $\psi(q)$ be one of Ramanujan's theta functions, namely

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}.$$

We let $\mathrm{pod}(n)$ (see A006950) denote the number of partitions of n wherein the odd parts are distinct (and even parts are unrestricted). For example, $\mathrm{pod}(4) = 3$ since there are 3 different partitions of 3 such that the odd parts are distinct, namely $4 = 3 + 1 = 2 + 2$. The generating function of $\mathrm{pod}(n)$ is given by

$$\sum_{n=0}^{\infty} \mathrm{pod}(n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} = \frac{1}{\psi(-q)}. \tag{1}$$

The arithmetic properties of pod(n) were first studied by Hirschhorn and Sellers [4] in 2010. They obtained some interesting congruences involving the following infinite family of Ramanujan-type congruences: for any integers $\alpha \geq 0$ and $n \geq 0$,

$$\mathrm{pod}\Big(3^{2\alpha+3}n + \frac{23 \times 3^{2\alpha+2}+1}{8}\Big) \equiv 0 \pmod{3}.$$

Later on Radu and Sellers [7] obtained other deep congruences for pod(n) modulo 5 and 7, such as

$$\mathrm{pod}(135n+8) \equiv \mathrm{pod}(135n+107) \equiv \mathrm{pod}(135n+116) \equiv 0 \pmod{5}, \quad \text{and}$$

$$\mathrm{pod}(567n+260) \equiv \mathrm{pod}(567n+449) \equiv 0 \pmod{7}.$$

For nonnegative integers n and k, let $r_k(n)$ (resp., $t_k(n)$) denote the number of representations of n as sum of k squares (resp., triangular numbers). In 2011, based on the generating function of pod($3n+2$) found in [4], Lovejoy and Osburn [6] discovered the following arithmetic relation:

$$\mathrm{pod}(3n+2) \equiv (-1)^n r_5(8n+5) \pmod{3}. \tag{2}$$

Following their steps, we will present some new congruences modulo 5 and 9 for pod(n).

Firstly, we find that (2) can be improved to a congruence modulo 9.

Theorem 1. *For any integer $n \geq 0$, we have*

$$\mathrm{pod}(3n+2) \equiv 2(-1)^{n+1} r_5(8n+5) \pmod{9}.$$

The following result will be a consequence of Theorem 1 upon invoking some properties of $r_5(n)$.

Theorem 2. *Let $p \geq 3$ be a prime, and N be a positive integer such that $pN \equiv 5 \pmod{8}$. Let α be any nonnegative integer.*
(1) If $p \equiv 1 \pmod{3}$, then

$$\mathrm{pod}\Big(\frac{3p^{6\alpha+5}N+1}{8}\Big) \equiv 0 \pmod{3},$$

and

$$\mathrm{pod}\Big(\frac{3p^{18\alpha+17}N+1}{8}\Big) \equiv 0 \pmod{9}.$$

(2) If $p \equiv 2 \pmod{3}$, then

$$\mathrm{pod}\Big(\frac{3p^{4\alpha+3}N+1}{8}\Big) \equiv 0 \pmod{9}.$$

Secondly, with the same method used in proving Theorem 1, we can establish a similar congruence for pod(n) modulo 5.

Theorem 3. *For any integer $n \geq 0$, we have*

$$\mathrm{pod}(5n+2) \equiv 2(-1)^n r_3(8n+3) \pmod{5}.$$

Some miscellaneous congruences can be deduced from this theorem.

Theorem 4. *For any integers $n \geq 0$ and $\alpha \geq 1$, we have*

$$\mathrm{pod}\Big(5^{2\alpha+2}n + \frac{11 \cdot 5^{2\alpha+1}+1}{8}\Big) \equiv 0 \pmod{5},$$

and

$$\mathrm{pod}\Big(5^{2\alpha+2}n + \frac{19 \cdot 5^{2\alpha+1}+1}{8}\Big) \equiv 0 \pmod{5}.$$

Theorem 5. *Let $p \equiv 4 \pmod{5}$ be a prime, and N be a positive integer which is coprime to p such that $pN \equiv 3 \pmod{8}$. We have*

$$\mathrm{pod}\Big(\frac{5p^3N+1}{8}\Big) \equiv 0 \pmod{5}.$$

For example, let $p = 19$ and $N = 8n+1$ where $n \geq 0$ and $n \not\equiv 7 \pmod{19}$. We have

$$\mathrm{pod}(34295n + 4287) \equiv 0 \pmod{5}.$$

Theorem 6. *Let $p \geq 3$ be a prime, and N be a positive integer which is not divisible by p such that $pN \equiv 3 \pmod{8}$. Let α be any nonnegative integer.*
(1) If $p \equiv 1 \pmod{5}$, we have

$$\mathrm{pod}\Big(\frac{5p^{10\alpha+9}N+1}{8}\Big) \equiv 0 \pmod{5}.$$

(2) If $p \equiv 2, 3, 4 \pmod{5}$, we have

$$\mathrm{pod}\Big(\frac{5p^{8\alpha+7}N+1}{8}\Big) \equiv 0 \pmod{5}.$$

2 Preliminaries

Lemma 7. *(Cf. [7, Lemma 1.2].) Let p be a prime and α be a positive integer. Then*

$$(q;q)_\infty^{p^\alpha} \equiv (q^p;q^p)_\infty^{p^{\alpha-1}} \pmod{p^\alpha}.$$

Lemma 8. *For any prime $p \geq 3$, we have*

$$t_4\Big(pn + \frac{p-1}{2}\Big) \equiv t_4(n) \pmod{p}, \quad t_8(pn+p-1) \equiv t_8(n) \pmod{p^3}.$$

Proof. By [2, Theorem 3.6.3], we know $t_4(n) = \sigma(2n+1)$. For any positive integer N, we have

$$\sigma(N) = \sum_{d|N,\, p|d} d + \sum_{d|N,\, p\nmid d} d \equiv \sum_{d|N,\, p\nmid d} d \pmod p.$$

Let $N = 2n+1$ and $N = p(2n+1)$, respectively. It is easy to deduce that $\sigma(p(2n+1)) \equiv \sigma(2n+1) \pmod p$. This clearly implies the first congruence.

From [2, Eq.(3.8.3), page 81], we know

$$t_8(n) = \sum_{\substack{d|(n+1) \\ d\,\text{odd}}} \left(\frac{n+1}{d}\right)^3.$$

By a similar argument we can prove the second congruence. □

Lemma 9. *(Cf. [1].) For $1 \le k \le 7$, we have*

$$r_k(8n+k) = 2^k\left(1 + \frac{1}{2}\binom{k}{4}\right)t_k(n).$$

Lemma 10. *(Cf. [3].) Let $p \geq 3$ be a prime and n be a positive integer such that $p^2 \nmid n$. For any integer $\alpha \geq 0$, we have*

$$r_5(p^{2\alpha}n) = \left(\frac{p^{3\alpha+3}-1}{p^3-1} - p\left(\frac{n}{p}\right)\frac{p^{3\alpha}-1}{p^3-1}\right)r_5(n),$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

Lemma 11. *(Cf. [5].) Let $p \geq 3$ be a prime. For any integers $n \geq 1$ and $\alpha \geq 0$, we have*

$$r_3(p^{2\alpha}n) = \left(\frac{p^{\alpha+1}-1}{p-1} - \left(\frac{-n}{p}\right)\frac{p^{\alpha}-1}{p-1}\right)r_3(n) - p\frac{p^{\alpha}-1}{p-1}r_3(n/p^2),$$

where we take $r_3(n/p^2) = 0$ unless $p^2|n$.

3 Proofs of the Theorems

Proof of Theorem 1. Let $p = 3$ in Lemma 8. We deduce that $t_8(3n+2) \equiv t_8(n) \pmod 9$. By (1) we have

$$\psi(q)^9 \sum_{n=0}^{\infty} \mathrm{pod}(n)(-q)^n = \psi(q)^8 = \sum_{n=0}^{\infty} t_8(n)q^n.$$

By Lemma 7 we obtain $\psi(q)^9 \equiv \psi(q^3)^3 \pmod 9$. Hence

$$\psi(q^3)^3 \sum_{n=0}^{\infty} \mathrm{pod}(n)(-q)^n \equiv \sum_{n=0}^{\infty} t_8(n)q^n \pmod 9.$$

If we extract those terms of the form q^{3n+2} on both sides, we obtain

$$\psi(q^3)^3 \sum_{n=0}^{\infty} \mathrm{pod}(3n+2)(-q)^{3n+2} \equiv \sum_{n=0}^{\infty} t_8(3n+2) q^{3n+2} \pmod 9.$$

Dividing both sides by q^2, then replacing q^3 by q, we get

$$\psi(q)^3 \sum_{n=0}^{\infty} \mathrm{pod}(3n+2)(-q)^n \equiv \sum_{n=0}^{\infty} t_8(3n+2) q^n \equiv \sum_{n=0}^{\infty} t_8(n) q^n = \psi(q)^8 \pmod 9.$$

Hence

$$\sum_{n=0}^{\infty} \mathrm{pod}(3n+2)(-q)^n \equiv \psi(q)^5 \equiv \sum_{n=0}^{\infty} t_5(n) q^n \pmod 9.$$

Comparing the coefficients of q^n on both sides, we deduce that $\mathrm{pod}(3n+2) \equiv (-1)^n t_5(n) \pmod 9$.

Let $k = 5$ in Lemma 9. We obtain $t_5(n) = r_5(8n+5)/112$, and from this the theorem follows. □

Proof of Theorem 2. (1) Let $n = pN$ in Lemma 10, and then we replace α by $3\alpha + 2$. Since

$$\frac{p^{9\alpha+9} - 1}{p^3 - 1} = 1 + p^3 + \cdots + p^{3(3\alpha+2)} \equiv 0 \pmod 3,$$

we deduce that $r_5(p^{6\alpha+5}N) \equiv 0 \pmod 3$.

Let $n = \frac{p^{6\alpha+5}N-5}{8}$ in Theorem 1. We deduce that $\mathrm{pod}(\frac{3p^{6\alpha+5}N+1}{8}) \equiv 0 \pmod 3$.

Similarly, let $n = pN$ in Lemma 10 and we replace α by $9\alpha + 8$. Since $p \equiv 1 \pmod 3$ implies $p^3 \equiv 1 \pmod 9$, we have

$$\frac{p^{27\alpha+27} - 1}{p^3 - 1} = 1 + p^3 + \cdots + p^{3(9\alpha+8)} \equiv 0 \pmod 9.$$

Hence $r_5(p^{18\alpha+17}N) \equiv 0 \pmod 9$.

Let $n = \frac{p^{18\alpha+17}N-5}{8}$ in Theorem 1. We deduce that $\mathrm{pod}(\frac{3p^{18\alpha+17}N+1}{8}) \equiv 0 \pmod 9$.

(2) Let $n = pN$ in Lemma 10, and then we replace α by $2\alpha + 1$. Note that $p \equiv 2 \pmod 3$ implies $p^3 \equiv -1 \pmod 9$. Since $p^{6\alpha+6} \equiv 1 \pmod 9$, we have $r_5(p^{4\alpha+3}N) \equiv 0 \pmod 9$.

Let $n = \frac{p^{4\alpha+3}N-5}{8}$ in Theorem 1. We complete our proof. □

Proof of Theorem 3. Let $p = 5$ in Lemma 8. We deduce that $t_4(5n+2) \equiv t_4(n) \pmod 5$.

By (1) we have

$$\psi(q)^5 \sum_{n=0}^{\infty} \mathrm{pod}(n)(-q)^n = \psi(q)^4 = \sum_{n=0}^{\infty} t_4(n) q^n.$$

By Lemma 7 we obtain $\psi(q)^5 \equiv \psi(q^5) \pmod 5$. Hence

$$\psi(q^5)\sum_{n=0}^{\infty} \mathrm{pod}(n)(-q)^n \equiv \sum_{n=0}^{\infty} t_4(n)q^n \pmod 5.$$

If we extract those terms of the form q^{5n+2} on both sides, we obtain

$$\psi(q^5)\sum_{n=0}^{\infty} \mathrm{pod}(5n+2)(-q)^{5n+2} \equiv \sum_{n=0}^{\infty} t_4(5n+2)q^{5n+2} \pmod 5.$$

Dividing both sides by q^2, and then replacing q^5 by q, we get

$$\psi(q)\sum_{n=-\infty}^{\infty} \mathrm{pod}(5n+2)(-q)^n \equiv \sum_{n=0}^{\infty} t_4(5n+2)q^n \equiv \sum_{n=0}^{\infty} t_4(n)q^n = \psi(q)^4 \pmod 5.$$

Hence we have

$$\sum_{n=0}^{\infty} \mathrm{pod}(5n+2)(-q)^n \equiv \psi(q)^3 = \sum_{n=0}^{\infty} t_3(n)q^n \pmod 5.$$

Comparing the coefficients of q^n on both sides, we deduce that $\mathrm{pod}(5n+2) \equiv (-1)^n t_3(n) \pmod 5$.

Let $k = 3$ in Lemma 9. We obtain $t_3(n) = r_3(8n+3)/8$, from which the theorem follows. □

Proof of Theorem 4. Let $p = 5$ and $n = 5m + r$ ($r \in \{1, 4\}$) in Lemma 11. Since $\left(\frac{-r}{5}\right) = 1$, we deduce that $r_3(5^{2\alpha}(5m+r)) \equiv 0 \pmod 5$ for any integer $\alpha \geq 1$.

Let $n = \frac{5^{2\alpha}(40m+a)-3}{8}$ ($a \in \{11, 19\}$). By Theorem 3, we have

$$r_3(8n+3) = r_3(5^{2\alpha}(40m+a)) \equiv 0 \pmod 5.$$

Hence

$$\mathrm{pod}\Big(5^{2\alpha+2}m + \frac{a\cdot 5^{2\alpha+1}+1}{8}\Big) = \mathrm{pod}(5n+2) \equiv 2(-1)^n r_3(8n+3) \equiv 0 \pmod 5.$$

□

Proof of Theorem 5. Let $\alpha = 1$ and $n = pN$ in Lemma 11. We have

$$r_3(p^3N) = (1+p)r_3(pN) \equiv 0 \pmod 5.$$

Let $n = \frac{p^3N-3}{8}$ in Theorem 3. We have

$$\mathrm{pod}\Big(\frac{5p^3N+1}{8}\Big) = \mathrm{pod}(5n+2) \equiv 2(-1)^n r_3(8n+3) = 2(-1)^n r_3(p^3N) \equiv 0 \pmod 5.$$

□

Proof of Theorem 6. (1) Let $n = pN$ in Lemma 11, and then we replace α by $5\alpha + 4$. We have

$$\frac{p^{5\alpha+5}-1}{p-1} = 1 + p + \cdots + p^{5\alpha+4} \equiv 0 \pmod{5}.$$

Hence $r_3(p^{10\alpha+9}N) \equiv 0 \pmod{5}$. Let $n = \frac{p^{10\alpha+9}N-3}{8}$ in Theorem 3. We have

$$\mathrm{pod}\Big(\frac{5p^{10\alpha+9}N+1}{8}\Big) = \mathrm{pod}(5n+2) \equiv 2(-1)^n r_3(p^{10\alpha+9}N) \equiv 0 \pmod{5}.$$

(2) Let $n = pN$ in Lemma 11, and then replace α by $4\alpha + 3$. Since $p^{4\alpha+4} \equiv 1 \pmod{5}$, we deduce that $r_3(p^{8\alpha+7}N) \equiv 0 \pmod{5}$. Let $n = \frac{p^{8\alpha+7}N-3}{8}$ in Theorem 3. We have

$$\mathrm{pod}\Big(\frac{5p^{8\alpha+7}N+1}{8}\Big) = \mathrm{pod}(5n+2) \equiv 2(-1)^n r_3(p^{8\alpha+7}N) \equiv 0 \pmod{5}.$$

□

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2010 *Mathematics Subject Classification*: Primary 05A17; Secondary 11P83.
Keywords: congruence, partition, distinct odd parts, theta function, sum of squares.

(Concerned with sequences A000041 and A006950.)

Received January 8 2015; revised version received February 22 2015. Published in *Journal of Integer Sequences*, May 12 2015.

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