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Some Formulas for Numbers of Restricted Words

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Abstract

For an arithmetic function f_0, we consider the number $c_m(n,k)$ of weighted compositions of n into k parts, where the weights are the values of the $(m-1)^{\text{th}}$ invert transform of f_0. We connect $c_m(n,k)$ with $c_1(n,k)$ via Pascal matrices. We then relate $c_m(n,k)$ to the number of certain restricted words over a finite alphabet. In addition, we develop a method which transfers some properties of restricted words over a finite alphabet to words over a larger alphabet.

Several examples illustrate our findings. Some examples concern binomial coefficients and Fibonacci numbers. Some examples also extend the classical results about weighted compositions of Hoggatt and Lind. In each example, we derive an explicit formula for $c_m(n,k)$.

1 Introduction

For a given initial arithmetic function f_0, Janjić [5] examined some properties of the function f_m, which is the m^{th} invert transform of f_0. In that paper, as well as in Birmajer et al. [2], some cases in which f_m counts the number of restricted words over a finite alphabet were considered. In the present paper, we consider the function $c_m(n,k)$, which is the number of weighted compositions of n into k parts, where the weights are $\{f_{m-1}(1), f_{m-1}(2), \ldots\}$. Note that, in Janjić [6], properties of $c_1(n,k)$ were investigated.

As mentioned by Birmajer at al. in a recent preprint [3], a formula for the number $c_1(n,k)$ was firstly given by Hoggatt and Lind [4], based on results by Moser and Whitney [7]. For a sequence of weights $\{f_{m-1}(1), f_{m-1}(2), \ldots\}$, the number of f_{m-1}-weighted compositions of n into k parts is

$$c_m(n,k) = \sum_{\pi_k(n)} \frac{k!}{k_1! \cdots k_n!} f_{m-1}(1)^{k_1} \cdots f_{m-1}(1)^{k_n},$$

where the sum runs over all solutions of

$$k_1 + 2k_2 + \cdots + nk_n = n \text{ such that } k_1 + \cdots + k_n = k \text{ and } k_j \in \mathbb{N}_0 \text{ for all } j.$$

In other words,

$$c_m(n,k) = \frac{k!}{n!} B_{n,k}(1!f_{m-1}(1), 2!f_{m-1}(2), \ldots), \tag{1}$$

where $B_{n,k}(x_1, x_2, \ldots)$ is the Bell partial polynomial. Moreover, as discussed in Hoggatt and Lind [4], this formula is equivalent to

$$c_m(n,k) = \sum_{\gamma_k(n)} f_{m-1}(a_1) \cdots f_{m-1}(a_k), \tag{2}$$

where $\gamma_k(n)$ indicates summations over k-part compositions $a_1 + a_2 + \cdots + a_k$ of n. As an immediate consequence, we obtain the following two results:

Corollary 1. *The function $c_m(n,k)$ satisfies the recurrence*

$$c_m(0,0) = 1,\ c_m(n,0) = 0, (n > 1),$$

and

$$c_m(n,k) = \sum_{i=1}^{n-k+1} f_{m-1}(i) c_m(n-i, k-1), (1 \le k \le n). \tag{3}$$

Corollary 2. *The following formula holds:*

$$f_m(n) = \sum_{k=1}^{n} c_m(n,k). \tag{4}$$

Note that, throughout the paper, letters m, n, k will have the meaning as in the definition of $c_m(n,k)$. Using Birmajer et al. [1, Corollary 10], we derive a formula connecting $c_m(n,k)$ with $c_{m-1}(n,k)$. The formula may be written in terms of the lower triangular Pascal matrices. We then extend this result to obtain a relation between $c_m(n,k)$ and $c_1(n,k)$.

For the particular case $f_0(1) = 1$, we develop a method which allows us to derive an interpretation of $c_m(n,k)$ in terms of restricted words, when we know the number of restricted words counted by f_{m-1}. We finish the paper with a number of examples illustrating our results. Some examples extend the classical results on weighted compositions given by Hoggatt and Lind [4].

It is important to note that quantities $f_m(n)$ and $c_m(n,k)$ depend only on the initial arithmetic function f_0.

2 A connection of $c_m(n,k)$ and $c_{m-1}(n,k)$

Let $C_m(n)$ be the lower triangular matrix of order n, whose (i,j) entry is $c_m(i,j),(i = 1,2,\dots,n; 1 \leq j \leq i)$. We let L_n denote the lower triangular Pascal matrix of order n. Hence, the (i,j) entry of L_n is $\binom{i-1}{j-1},(1 \leq j \leq i)$. First, we prove the following:

Proposition 3. *For each $m > 1$, we have*

$$C_m(n) = C_{m-1}(n) \cdot L_n.$$

Proof. It is easy to see that the statement is equivalent to the equation

$$c_m(n,k) = \sum_{i=k}^{n} \binom{i-1}{k-1} c_{m-1}(n,i). \tag{5}$$

In our terminology, Birmajer et al. [1, Corollary 10] may be written in the form

$$\sum_{j_1+j_2+\cdots+j_k=n} f_{m-1}(j_1)\cdots f_{m-1}(j_k) = \sum_{i=1}^{n} \binom{i+k-1}{i} c_{m-1}(n,i),$$

where the sum is taken over nonnegative $j_1,\dots,j_k$. Since at most $k-1$ of j_t may be equal 0, (2) yields

$$\sum_{j=0}^{k-1} \binom{k}{j} c_m(n,k-j) = \sum_{i=1}^{n} \binom{i+k-1}{i} c_{m-1}(n,i).$$

Replacing $k-j$ by t, and denoting $\sum_{i=1}^{n} \binom{i+k-1}{i} c_{m-1}(n,i) = a_k$, implies

$$\sum_{t=1}^{k} \binom{k}{t} c_m(n,t) = a_k, (k = 1,2,\dots,n).$$

Denoting $X = (c_m(n,1), c_m(n,2),\dots,c_m(n,n))^T$, and $A = (a_1,a_2,\dots,a_n)^T$, this system may be written in the matrix form

$$Q \cdot X = A,$$

where Q is obtained from the Pascal matrix L_{n+1} by omitting the first row and the first column. It follows that $X = Q^{-1} \cdot A$, where $Q^{-1} = \left((-1)^{i+j}\binom{i}{j}\right)_{n\times n}$. For $k = 1,2,\dots,n$, we obtain

$$c_m(n,k) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} (-1)^{j+k} \binom{k}{j} \binom{i+j-1}{i} \right) c_{m-1}(n,i). \tag{6}$$

Equation (6) holds for each $m > 1$, as well as for any arbitrary arithmetic function f_0. In particular, taking $f_0(1) = 1, f_0(i) = 0, (i > 1)$, we obviously have $c_1(n,n) = 1$, and

$c_1(n,k) = 0$ for $k < n$. Also, $f_1(n) = 1$ for all n. In this case, $c_2(n,k)$ is the number of compositions of n into k parts, that is, $c_2(n,k) = \binom{n-1}{k-1}$. Therefore, (6) becomes

$$\binom{n-1}{k-1} = \sum_{j=1}^{n} (-1)^{j+k} \binom{k}{j} \binom{n+j-1}{n}. \tag{7}$$

Hence the expression in the square brackets in (6) equals $\binom{i-1}{k-1}$, which proves (5). □

Remark 4. As a byproduct, we proved the binomial identity (7).

Remark 5. Replacing $i-k$ by t in (5), we obtain

$$c_m(n,k) = \sum_{t=0}^{n-k} \binom{k+t-1}{t} c_{m-1}(n,k+t). \tag{8}$$

From the equation $C_m(n) = C_{m-1}(n) \cdot L_n$ follows

$$C_m(n) = C_{m-1}(n) \cdot L_n = C_{m-2}(n) \cdot L_n^2 = \cdots = C_1(n) \cdot L_n^{m-1}.$$

We thus obtain,

Proposition 6. *The following matrix equation holds:*

$$C_m(n) = C_1(n) L_n^{m-1}.$$

Or, explicitly,

$$c_m(n,k) = \sum_{i=k}^{n} (m-1)^{i-k} \binom{i-1}{k-1} c_1(n,i), \ (1 \le k \le n). \tag{9}$$

Proof. The assertion is true since (i,k) entry of L_n^{m-1} is $(m-1)^{i-k}\binom{i-1}{k-1}$. □

Now, we derive a formula in which $f_m(n)$ is expressed in terms of $c_1(n,k)$.

Proposition 7. *The following formula holds:*

$$f_m(n) = \sum_{i=1}^{n} m^{i-1} c_1(n,i). \tag{10}$$

Proof. Equation (4) yields

$$f_m(n) = \sum_{k=1}^{n} \sum_{i=k}^{n} (m-1)^{i-k} \binom{i-1}{k-1} c_1(n,i).$$

Changing the order of summation gives

$$f_m(n) = \sum_{i=1}^{n} \left(\sum_{k=1}^{i} (m-1)^{i-k} \binom{i-1}{k-1} \right) c_1(n,i).$$

Using the binomial theorem, we obtain (10). □

Note 8. Equation (10) appears in Birmajer et al. [2] with a combinatorial proof based on the enumeration of certain restricted words.

As an immediate consequence of (1) and (9), we obtain the following identity for the partial Bell polynomials:

Identity 9. *If the sequence $y_1, y_2, \ldots$ is the invert transform of the sequence $x_1, x_2, \ldots$, then*

$$k!B_{n,k}(y_1, 2! \cdot y_2, 3! \cdot y_3, \ldots) = \sum_{i=k}^{n} \binom{i-1}{k-1} i!B_{n,i}(x_1, 2! \cdot x_2, 3! \cdot x_3, \ldots).$$

We next prove the following result.

Proposition 10. *Assume that $f_0(1) = 1$ and $m > 1$. Assume next that, for $n \geq 1$, $f_{m-1}(n)$ is the number of some words of length $n-1$ over a finite alphabet α. Let x be a letter which is not in α. Then $c_m(n,k)$ is the number of words of length $n-1$ over the alphabet $\alpha \cup \{x\}$, in which exactly $k-1$ letters equal x.*

Proof. Since $f_0(1) = 1$, it follows from Janjić [5, Corollary 2] that $f_{m-1}(1) = 1$. We use induction on k. For $k = 1$, (2) yields $c_m(n,1) = f_{m-1}(n)$. Since $f_{m-1}(n)$ is the number of words of length $n-1$ over α not containing x, we conclude that the statement is true for $k = 1$. Assume that the claim is true for $k-1$. Consider the first term $f_{m-1}(1)c_m(n-1,k-1)$ in (3). By the induction hypothesis, $c_m(n-1,k-1)$ is the number of words of length $n-2$ having $k-2$ letters equal to x. Adding x at the beginning of each such word, we obtain all the words of length $n-1$ over $\alpha \cup \{x\}$, having $k-1$ letters equal to x, and all beginning with x.

Consider now the term $f_{m-1}(i) \cdot c_m(n-i,k-1), (i > 1)$ in (3). By the induction hypothesis, $c_m(n-i,k-1)$ is the number of words of length $n-i-1$ with $k-2$ letters equal to x. We first insert x at the beginning of each such word. In front of x, we insert an arbitrary word of length $i-1$ over α, which are $f_{m-1}(i)$ in number. We thus obtain all words of length $n-1$ over $\alpha \cup \{x\}$, such that the first appearance of x is at the ith position. It follows that the right-hand side of (3) counts all the desired words. □

Remark 11. We stress the fact that the preceding method may be applied only when we know the number of $(n-1)$-length words counted by $f_{m-1}(n)$. This is always true when $f_0(1), f_0(2), \ldots$ is a binary sequence. Namely, there is a bijection between the compositions counted by $c_1(n,k)$ and the binary words of length $n-1$ with $k-1$ ones. This bijection is given by the correspondence

$$1 \to 1, 2 \to 10, 3 \to 100, \ldots \ . \tag{11}$$

In this way, the compositions of n into k parts designate binary words of length n and having k ones, all of which begin with 1. The converse is also true. Omitting the leading 1, we obtain the desired correspondence. Equation (4) implies that f_1 and $c_1(n,k)$ both count some binary words of length $n-1$. We thus may apply Proposition 10 to obtain a combinatorial interpretation of $c_m(n,k)$ as well as $f_m(n)$. Note that this method may be applied in non-binary cases also.

Remark 12.

1. Janjić [5, 6] derive several formulas concerning correspondence (11).

2. Also, Birmajer at al. [2, Theorem 3, Corollary 5] developed a method to obtain the words counted by f_{m-1} starting with the initial function f_0.

We now illustrate our method by a simple example.

Example 13. Assume that $f_0(i) = 1$ for $i = 1, 2, \ldots$. Then, we have $f_{m-1}(n) = m^{n-1}$. It yields that $f_{m-1}(n)$ is the number of all words of length $n-1$ over $\alpha = \{0, 1, \ldots, m-1\}$. Then, $c_1(n,k) = \binom{n-1}{k-1}$. This means that $C_1(n) = L_n$. It follows that $C_m(n) = L_n^m$. From the well-known formula for the terms of L_n^m, we obtain

$$c_m(n,k) = m^{n-k}\binom{n-1}{k-1}. \tag{12}$$

Equation (12) is in accordance with Proposition 10. Namely, according to Proposition 10, $c_m(n,k)$ is the number of words of length $n-1$ over $\{0, 1, \ldots, m\}$ with $k-1$ letters equal to m. These $k-1$ letters may be chosen in $\binom{n-1}{k-1}$ ways. The remaining letters may be arbitrary letters from $\{0, 1, \ldots, m-1\}$, which are m^{n-k} in number.

As a byproduct, using (9), we obtain the following binomial identity:

Identity 14. *For $m > 1$, we have*

$$m^{n-k}\binom{n-1}{k-1} = \sum_{j=0}^{n-k}(m-1)^j\binom{n-1}{k+j-1}\binom{k+j-1}{j}. \tag{13}$$

This simple case is related to the coefficients of the Tchebychev polynomials $U_n(x)$ of the second kind.

Corollary 15. *The number $|[x^{n-k}](U_{n+k-2}(x))|$ is the number of words of length $n-1$ over the alphabet $\{0, 1, 2\}$ having $k-1$ twos.*

Proof. It is a well known that $(-1)^k 2^{n-k}\binom{n-1}{k-1}$ is the coefficient of $U_{n+k-2}(x)$ by x^{n-k}. We thus obtain

$$[x^{n-k}](U_{n+k-2}(x)) = (-1)^{n-k}\sum_{j=0}^{n-k}\binom{n-1}{k+j-1}\binom{k+j-1}{j}.$$

This is the case when $m = 2$ in (13). $\square$

3 More examples

Firstly, we revise the result from Janjić [6, Corollary 9].

Example 16. We define $f_0(1) = f_0(2) = 1$, and $f_0(n) = 0$ otherwise. According to Janjić [5, Corollary 33], $f_{m-1}(n)$ is the number of words of length $n-1$ over the alphabet $\{0, 1, \ldots, m-1\}$ having all zeros isolated.

Corollary 17. *The number $c_m(n,k)$ is the number of words of length $n-1$ over the alphabet $\{0, 1, \ldots, m\}$, which have $k-1$ letters equal to m and all zeros isolated. Also,*

$$c_1(n,k) = \binom{k}{n-k},$$

and

$$c_m(n,k) = \sum_{j=\lceil \frac{n}{2} \rceil - k}^{n-k} (m-1)^j \binom{j+k-1}{k-1} \binom{j+k}{n-j-k}, \left(m > 1, \left\lceil \frac{n}{2} \right\rceil \geq k\right).$$

Proof. The first formula is Hoggatt and Lind [4, Case (ii)]. Since $c_1(n, k+j) = \binom{k+j}{n-k-j}$, $(j = 0, \ldots, n-k)$, we have $k + j \geq n - k - j$, which yields $2j \geq n - 2k$ and $n \geq 2k$. The second formula is true according to (9). □

The arrays A030528 and A154929 in Sloane [8] are related to Example 16.

Next, we reexamine the result in Janjić [5, Corollary 28].

Example 18. We define $f_0(n) = 1$ when n is odd, and $f_0(n) = 0$ otherwise. According to Janjić [5, Corollary 28], $f_{m-1}(n)$ is the number of words of length $n-1$ over the alphabet $\{0, 1, \ldots, m-1\}$, avoiding runs of zeros of odd lengths. From Janjić [6, Proposition 24], it follows that

$$c_1(n,k) = \begin{cases} \binom{\frac{n-k}{2}+k-1}{k-1}, & \text{if } n-k \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

The number $c_1(n,k)$ is the number of binary words of length $n-1$ with $k-1$ ones, and avoiding runs of zeros of odd lengths. This follows from bijection (11). We add a short combinatorial proof.

Proposition 19. *The number $c_1(n,k)$ is the number of binary words of length $n-1$ with $k-1$ ones, avoiding runs of zeros of odd lengths.*

Proof. Assume that n and k are of different parities. Since a word of length $n-1$ with $k-1$ ones must have $n-k$ zeros, and since $n-k$ is odd, we conclude that such a word must have an odd run of zeros. It follows that $c_1(n,k) = 0$. If n and k are of the same parity, then $n-k$ is even. This means that there are $\frac{n-k}{2}$ pairs of zeros. Of these $\frac{n-k}{2}$ pairs and $k-1$ ones, we may form $\binom{\frac{n-k}{2}+k-1}{k-1}$ words of length $n-1$ having $k-1$ ones and avoiding runs of zeros of odd lengths. □

Remark 20. The formula for $c_1(n,k)$ appears in Hoggatt and Lind [4, Case (iv)].

Using induction and Proposition 10, we obtain

Corollary 21. *The number $c_m(n,k)$ is the number of words of length $n-1$ over $\{0,1,\ldots,m\}$ with $k-1$ letters equal to m, avoiding runs of zeros of odd lengths.*

From (9), we obtain an explicit formula for $c_m(n,k)$.

The arrays A037027 and A054456 in Sloane [8] are related to Example 18.

Example 22. We define $f_0(i) = i, (i = 1,2,\ldots)$. According to Janjić [5, Corollary 37], $f_{m-1}(n)$ is the number of 01-avoiding words of length $n-1$ over the alphabet $\{0,1,\ldots,m\}$.

Applying Proposition 10 several times, we obtain

Corollary 23. *The number $c_m(n,k)$ is the number of words of length $n-1$ over $\{0,1,\ldots,m+1\}$ having $k-1$ letters equal to $m+1$ and avoiding 01.*

Corollary 24. *The following formula holds:*

$$c_1(n,k) = \binom{n+k-1}{2k-1}. \tag{14}$$

Also,

$$c_m(n,k) = \sum_{i=k}^{n}(m-1)^{i-k}\binom{i-1}{k-1}\binom{n+i-1}{2i-1}.$$

Proof. Formula (14) may be found in Hoggatt and Lind [4, Case (iii)]. □

Since $\binom{n+k-1}{2k-1}$ is obviously the number of binary words of length $n+k-1$ with $2k-1$ zeros, we obtain the following Euler-type identity:

Identity 25. *The number of binary words of length $n+k-1$ with $2k-1$ zeros equals the number of ternary words of length $n-1$, having $k-1$ letters equal to 2 and avoiding 01.*

The arrays A125662, A207823, and A207824 in Sloane [8] are related to Example 22.

The last two examples concern the case $f_0(1) = 0$. Note that in these examples, Proposition 10 can not be used. The first example is an extension of the result from Janjić [6, Proposition 13].

Example 26. We define $f_0(1) = 0$, and $f_0(n) = 1$ otherwise. It follows from Janjić [5, Corollary 24] that, for $n > 3$, $f_m(n)$ is the number of words of length $n-3$ over $\{0,1,\ldots,m\}$, where no two consecutive letters are nonzero. From Janjić [6, Proposition 13], we obtain $c_1(n,k) = \binom{n-k-1}{k-1}$ for $\left(1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor\right)$, and $c_1(n,k) = 0$ otherwise. Equation (5) implies that $c_m(n,k) = 0$ when $k > \left\lfloor \frac{n}{2} \right\rfloor$.

Remark 27. Note that the formula for $c_1(n,k)$ also appears in Hoggatt and Lind [4, Case (iii)].

Corollary 28. *For $n > 3$ and $1 \le k \le \lfloor \frac{n}{2} \rfloor$, the number $c_m(n,k)$ is the number of words of length $n-3$ over $\{0,1,\dots,m\}$ with $k-1$ ones, and all nonzero letters are isolated. An explicit formula for $c_m(n,k)$ is*

$$c_m(n,k) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - k} (m-1)^j \binom{j+k-1}{k-1} \binom{n-k-j-1}{k+j-1}, \left(1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor\right),$$

and $c_m(n,k) = 0$ when $k > \lfloor \frac{n}{2} \rfloor$.

Proof. We know that $c_1(n,k)$ is the number of compositions of n into k parts, each of which is greater than 1. Using the bijection (11), we conclude that, for $n > 3$, $c_1(n,k)$ is the number of binary words of length n beginning with 10, ending with 0 and all ones are isolated. Omitting 10 at the beginning, and 0 at the end of each word, we conclude that $c_1(n,k), (n > 3)$ is the number of binary words of length $n-3$ with $k-1$ ones, all of which are isolated. Hence, the statement is true for $m = 1$. Assume that the statement is true for $m-1$. In (8), by the induction hypothesis, $c_{m-1}(n,k+t)$ is the number of words of length $n-3$ with $k+t-1$ ones, in which all nonzero letters are isolated. Replacing t ones with m, we obtain the desired words. The number t may be chosen in $\binom{k+t-1}{t}$ ways. Hence, the right-hand side of (8) counts all the desired words. The formula follows from (9) and the fact that $n-k-j-1 \ge k+j-1$. $\square$

The arrays A037027, A249139, and A006130 in Sloane [8] are related to Example 26.

Example 29. Define f_0 in the following way: $f_0(2) = f_0(3) = 1$, and $f_0(n) = 0$ otherwise. Janjić [6, Proposition 5] proved that $c_1(n,k) = \binom{k}{n-2k}, \left(\lceil \frac{n}{3} \rceil \le k \le \lfloor \frac{n}{2} \rfloor\right)$ and $c_1(n,k) = 0$ otherwise.

We know that $c_1(n,k)$ is the number of compositions of n into k parts equal to either 2 or 3. In other words, for $n \ge 3$ and $1 \le k \le \lfloor \frac{n}{2} \rfloor$, $c_1(n,k)$ is the number of binary words of length $n-1$ with $k-1$ ones. These words begin with 0 and end with 0. Also, zero avoids a run of length greater than 2, and all ones are isolated.

Corollary 30. *For $n > 3$, the number $c_m(n,k)$ is the number of words of length $n-1$ over the alphabet $\{0,1,\dots,m\}$ with $k-1$ ones, which begin and end with 0. Also, 0 avoids a run of length greater than 2 and all nonzero letters are isolated.*

Proof. The statement holds for $m = 1$. Assume that the statement is true for $m-1$. Consider the term $\binom{k+t-1}{t} c_{m-1}(n,k+t)$ in (8). The number $c_{m-1}(n,k+t)$ is the number of the desired words of length $n-1$ over $\{0,1,\dots,m-1\}$ with $k+t-1$ ones. We replace t of $k+t-1$ ones with m and then sum over t to obtain $c_m(n,k)$. $\square$

From (5) follows that $c_m(n,k) = 0$ if $k > \lfloor \frac{n}{2} \rfloor$. Otherwise, from (9), we obtain

$$c_m(n,k) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (m-1)^j \binom{j+k-1}{k-1} \binom{k+j}{n-2k-2j}, \left(1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor\right).$$

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