Journal of Integer Sequences, Vol. 20 (2017),
Article 17.4.8

# An Explicit Formula for Sums of Powers of Integers in Terms of Stirling Numbers

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**Abstract**

In this note we present an explicit formula for the sum of powers of the first $n$ terms of a general arithmetic sequence in terms of Stirling numbers. We then provide an algorithm for calculating the sum of consecutive powers of integers.

# 1 Introduction

This paper is concerned with sums of $p$th powers of the first $n$ terms of a general arithmetic sequence

$$S_{p,(a,d)}(n) = a^p + (a+d)^p + \cdots + (a+(n-1)d)^p,\ n \geq 1$$

where $p$ is a nonnegative integer, and $a, d$ are complex numbers with $d \neq 0$. In particular, we have

$$S_{p,(1,1)}(n) = 1^p + 2^p + 3^p + \cdots + n^p,$$

which has been studied extensively by many authors.

The properties of $S_{p,(a,d)}(n)$ were obtained by Howard [4] via the following generating function

$$\mathcal{B}_0(z) := \sum_{p\geq 0} S_{p,(a,d)}(n) \frac{z^p}{p!} = \sum_{k=0}^{n-1} e^{(a+kd)z}. \tag{1}$$

For recent articles on this subject, see [1, 6, 8].

In this note we establish a generalization of the well-known formula [3, 5, 9]

$$S_{p,(1,1)}(n) = \sum_{k=0}^{p} k! \begin{Bmatrix} p \\ k \end{Bmatrix} \binom{n+1}{k+1},\ p \geq 1.$$

for the sums $S_{p,(a,d)}(n)$. Also, we provide an algorithm for calculating the sum of the $p$th powers of the first $n$ terms of a general arithmetic sequence.

First we present some definitions and notations and some results that will be useful in the rest of the paper. For $x \in \mathbb{C}$, the falling factorial $(x)_n$ is defined by $(x)_0 = 1, (x)_n = x(x-1)\cdots(x-n+1)$ for $n > 0$. The generalized binomial coefficient is defined as follows:

$$\binom{x}{n} = \begin{cases} \frac{(x)_n}{n!}, & \text{if } n \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

The (signed) Stirling numbers of the first kind $s(n,k)$ are the coefficients in the expansion

$$(x)_n = \sum_{k=0}^{n} s(n,k) x^k$$

and satisfy the recurrence relation

$$s(n+1,k) = s(n,k-1) - ns(n,k) \text{ for } 1 \leq k \leq n. \tag{2}$$

The Stirling numbers of the second kind, denoted by $\left\{ {n \atop k} \right\}$, are the coefficients in the expansion

$$x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (x)_k.$$

These numbers count the number of ways to partition a set of $n$ elements into exactly $k$ nonempty subsets.

For any positive integer $r$, the $r$-Stirling numbers of the second kind $\left\{ {n \atop k} \right\}_r$ are obviously generalizations of Stirling numbers of the second kind. These numbers count the number of partitions of a set of $n$ objects into exactly $k$ nonempty, disjoint subsets, such that the first $r$ elements are in distinct subsets. The exponential generating function is given by

$$\sum_{n\geq k} \left\{ {n+r \atop k+r} \right\}_r \frac{z^n}{n!} = \frac{1}{k!} e^{rz} \left(e^z - 1\right)^k .$$

For more details about these numbers see [2, 3].

## 2 Main results

First, we obtain the following result via the generalized Stirling transform method.

**Theorem 1.** *For all $n \geq 1$, non-negative integer $p$ and complex numbers $a$ and $d$ ($d \neq 0$), we have*

$$S_{p,(a,d)}(n) = d^p \sum_{k=0}^{p} k! \left\{ {p \atop k} \right\} \left( \binom{n + \frac{a}{d}}{k+1} - \binom{\frac{a}{d}}{k+1} \right) . \tag{3}$$

*Proof.* Since

$$S_{p,(a,d)}(n) = d^p S_{p,\left(\frac{a}{d},1\right)}(n) ,$$

we can investigate the sums $S_{p,\left(\frac{a}{d},1\right)}(n)$. It follows from (1) and [7, Theorem 4] that

$$\begin{aligned} \mathcal{A}_0(z) &= \sum_{m\geq 0} a_{0,m} \frac{z^m}{m!} \\ &= \mathcal{B}_0\left(\ln(1+z)\right) \\ &= \sum_{k=0}^{n-1} (1+z)^{\frac{a}{d}+k} \\ &= \sum_{m\geq 0} \frac{z^m}{m!} \sum_{k=0}^{n-1} \left(\frac{a}{d} + k\right)_m . \end{aligned}$$

Since

$$\sum_{k=0}^{n} (x+k)_m := \frac{(x+n+1)_{m+1} - (x)_{m+1}}{m+1},$$

we can easily verify that

$$\begin{aligned} a_{0,m} &= [z^m]\, \mathcal{A}_0(z) &(4)\\ &= m! \left( \binom{n+\frac{a}{d}}{m+1} - \binom{\frac{a}{d}}{m+1} \right) &(5) \end{aligned}$$

where $[z^n]f(z) = f_n$ denotes the operation of extracting the coefficient of $z^n$ in the formal power series $f(z) = \sum_n f_n \frac{z^n}{n!}$.

Now, from [7, Corollary 1] we get

$$\sum_{k=0}^{m} \frac{1}{d^{p+k}} s(m,k)\, S_{p+k,(a,d)}(n) =$$
$$\sum_{k=0}^{p} \left\{ {p+m \atop k+m} \right\}_m (m+k)! \left( \binom{n+\frac{a}{d}}{m+k+1} - \binom{\frac{a}{d}}{m+k+1} \right). \quad (6)$$

and the proof is completed by letting $m = 0$ in the above identity. □

Setting $p = 0$ in (6), one obtains the following recursive formula for the sum of the $p$th powers of the first $n$ terms of a general arithmetic sequence involving the Stirling numbers of the first kind:

**Corollary 2.** *We have*

$$\sum_{k=0}^{m} \frac{1}{d^{k}} s(m,k)\, S_{k,(a,d)}(n) = m! \left( \binom{n+\frac{a}{d}}{m+1} - \binom{\frac{a}{d}}{m+1} \right).$$

Thus, for example, when $m = 0, 1, 2, 3$, we obtain

$$S_{0,(a,d)}(n) = n,$$
$$S_{1,(a,d)}(n) = d \left( \binom{n+\frac{a}{d}}{2} - \binom{\frac{a}{d}}{2} \right),$$
$$-\frac{1}{d} S_{1,(a,d)}(n) + \frac{1}{d^2} S_{2,(a,d)}(n) = 2 \left( \binom{n+\frac{a}{d}}{3} - \binom{\frac{a}{d}}{3} \right),$$
$$\frac{2}{d} S_{1,(a,d)}(n) - \frac{3}{d^2} S_{2,(a,d)}(n) + \frac{1}{d^3} S_{3,(a,d)}(n) = 6 \left( \binom{n+\frac{a}{d}}{4} - \binom{\frac{a}{d}}{4} \right).$$

In the next paragraph, we propose an algorithm based on a three-term recurrence relation for calculating the $p$th powers of the first $n$ terms of a general arithmetic sequence $S_{p,(a,d)}(n)$. It is convenient to introduce the following sequence $A_{p,m}^{(a,d)}(n)$ with two indices by

$$A_{p,m} := A_{p,m}^{(a,d)}(n) = \frac{S_{0,(a,d)}(n)}{a_{0,m}} \sum_{k=0}^{m} \frac{1}{d^k} s(m,k)\, S_{p+k,(a,d)}(n), \quad (7)$$

with $A_{0,m} = n$ and $A_{p,0} = S_{p,(a,d)}(n)$.

**Theorem 3.** *The sequence $A_{p,m}^{(a,d)}(n)$ satisfies the following three-term recurrence relation*

$$A_{p+1,m} = d \frac{a_{0,m+1}}{a_{0,m}} A_{p,m+1} + dm A_{p,m}, \quad (8)$$

*with the initial sequence $A_{0,m} = n$, and $a_{0,m}$ is defined by (5).*

*Proof.* From (7) and (2), we have

$$A_{p,m+1} = \frac{n+1}{a_{0,m+1}} \sum_{k=0}^{m+1} \frac{1}{d^k} s\left(m+1,k\right) S_{p+k,(a,d)}\left(n\right)$$
$$= \frac{n+1}{a_{0,m+1}} \sum_{k=1}^{m+1} \frac{1}{d^k} \left(s\left(m,k-1\right) - ms\left(m,k\right)\right) S_{p+k,(a,d)}\left(n\right).$$

After some rearrangement, we get

$$A_{p,m+1} = \frac{a_{0,m}}{da_{0,m+1}} A_{p+1,m} - \frac{a_{0,m}m}{a_{0,m+1}} A_{p,m}.$$

This completes the proof. □

As an immediate application of (8) we have the following algorithm for evaluating Faulhaber's formula $S_{p,(1,1)}\left(n\right)$. Starting with the sequence $R_{0,m} := n$, as the first row of the matrix $\left(R_{p,m}\right)_{p,m\geq 0}$, each entry is determined recursively by

$$R_{p+1,m} = mR_{p,m} + \frac{\left(m+1\right)\binom{n+1}{m+2}}{\binom{n+1}{m+1} - \delta_{0,m}} R_{p,m+1},$$

where $\delta_{i,j}$ denotes the Kronecker symbol.

Then $R_{p,0} := S_{p,(1,1)}\left(n\right)$ is Faulhaber's formula.

# 3 Acknowledgments

We thank the referee and the editor for helpful comments and suggestions.

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2010 *Mathematics Subject Classification*: Primary 11B73; Secondary 11B37.
*Keywords*: algorithm, explicit formula, Stirling number, sums of powers of integers.

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Received October 26 2016; revised version received February 24 2017. Published in *Journal of Integer Sequences*, February 24 2017.

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