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Another Identity for Complete Bell Polynomials Based on Ramanujan's Congruences

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Abstract

Let $p(n)$ be the number of partitions of a positive integer n. We derive a new identity for complete Bell polynomials based on a generating function of $p(7n+5)$ given by Ramanujan.

1 Introduction

Let $(x_1, x_2, \ldots)$ be a sequence of real numbers. The partial exponential Bell polynomials are polynomials given by

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{\pi(n,k)} \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}$$

where $\pi(n,k)$ is the positive integer sequence $(j_1, j_2, \ldots, j_{n-k+1})$ satisfying the following equations:

$$\begin{aligned} j_1 + j_2 + \cdots + j_{n-k+1} &= k, \\ j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} &= n. \end{aligned}$$

For $n \geq 1$, the n^{th}-complete exponential Bell polynomial $B_n(x_1, \ldots, x_n)$ is as follows:

$$B_n(x_1, \ldots, x_n) = \sum_{k=1}^{n} B_{n,k}(x_1, \ldots, x_{n-k+1}).$$

The complete exponential Bell polynomials can also be defined by power series expansion as follows:

$$\exp\Big(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\Big) = \sum_{n=0}^{\infty} B_n(x_1, \ldots, x_n) \frac{t^n}{n!}, \tag{1}$$

where $B_0 \equiv 1$. Bell polynomials were first introduced by Bell [2]. The books written by Comtet [4] and Riordan [6] serve as excellent references for the numerous applications of Bell polynomials in combinatorics.

Let $(a;q)_n$ be the q-Pochhammer symbol for $n \geq 1$. That is,

$$(a;q)_n := \prod_{k=0}^{n-1}(1 - aq^k) = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}).$$

Considered as a formal power series in q, the definition of q-Pochhammer symbol can be extended to an infinite product. That is,

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1 - aq^k).$$

We note that $(q;q)_\infty$ is the Euler's function. Let $p(n)$ be the number of partitions of n. The generating function of $p(n)$ can be written as

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty}.$$

Andrew's book [1] serves as an excellent reference to the theory of partitions.

Ramanujan's congruences are congruence properties for $p(n)$:

$$p(5k+4) \equiv 0 \pmod{5}; \quad p(7k+5) \equiv 0 \pmod{7}; \quad p(11k+6) \equiv 0 \pmod{11}.$$

In 1919, Ramanujan [5] proved the first two congruences by the following two identities:

$$\sum_{k=0}^{\infty} p(5k+4) q^k = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}, \tag{2}$$

$$\sum_{k=0}^{\infty} p(7k+5) x^k = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49x\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}. \tag{3}$$

Bouroubi and Benyahia-Tani [3] proved an identity for complete Bell polynomials based on (2). As an analogue to their result, we give an identity for complete Bell polynomials based on (3). In other words, we derive formulas that relate $p(7n+5)$ and certain complete Bell polynomials.

2 Main theorem

Let $\sigma(n)$ be the sum of divisors (including 1 and n) for n. It is well known that $\sigma(n)$ is a multiplicative function. That is, if n and m are coprime, then

$$\sigma(mn) = \sigma(m)\sigma(n). \tag{4}$$

If $m \geq 1$, then

$$\sigma(p^m) = 1 + p + \cdots + p^{m-1} + p^m = \frac{p^{m+1} - 1}{p - 1}. \tag{5}$$

Lemma 1. *Let $n = 7^m n'$ such that $m \geq 1$ and $\gcd(n, n') = 1$. Then,*

$$\sigma(n) = \frac{7^{m+1} - 1}{7^m - 1} \sigma\left(\frac{n}{7}\right).$$

Proof. By (4) and (5),

$$\sigma(n) = \sigma(7^m)\sigma(n') = \frac{7^{m+1} - 1}{6} \sigma\left(\frac{n}{7^m}\right), \tag{6}$$

$$\sigma\left(\frac{n}{7}\right) = \sigma\left(7^{m-1}\right)\sigma\left(\frac{n}{7^m}\right) = \frac{7^m - 1}{6} \sigma\left(\frac{n}{7^m}\right). \tag{7}$$

By a combination of (6) and (7), we get the desired result. □

Theorem 2. *Let $n \geq 1$. We write $n = 7^m n'$ where $m \geq 0$ and $\gcd(n, n') = 1$. Let d_n and e_n be the following sequences of numbers respectively:*

$$d_n = \frac{\sigma(n)}{n}\left(1 + \frac{18}{7^{m+1} - 1}\right),$$
$$e_n = \frac{\sigma(n)}{n}\left(1 + \frac{42}{7^{m+1} - 1}\right).$$

Then we have the following identity:

$$7B_n(1!d_1, 2!d_2, \ldots, n!d_n) + 49nB_{n-1}(1!e_1, 2!e_2, \ldots, (n-1)!e_{n-1}) = n!p(7n + 5).$$

Proof. Let $G(x)$ and $H(x)$ be the following functions:

$$G(x) = 7\frac{(x^7; x^7)^3_\infty}{(x; x)^4_\infty}, \tag{8}$$

$$H(x) = 49x\frac{(x^7; x^7)^7_\infty}{(x; x)^8_\infty}. \tag{9}$$

The functions $G(x)$ and $H(x)$ are well-defined on the interior of the unit disk in the complex plane by analytic continuation. We get the following two equations by (8) and (9),

$$\ln(G(x)) = \ln 7 + 3\sum_{i=1}^{\infty}\ln(1-x^{7i}) - 4\sum_{i=1}^{\infty}\ln(1-x^{i}),$$
$$\ln(H(x)) = \ln 49 + \ln x + 7\sum_{i=1}^{\infty}\ln(1-x^{7i}) - 8\sum_{i=1}^{\infty}\ln(1-x^{i}).$$

By using the power series expansion of $\ln(1-x)$, we get

$$\ln(G(x)) = \ln 7 - 3\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{x^{7ij}}{j} + 4\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{x^{ij}}{j}, \tag{10}$$
$$\ln(H(x)) = \ln 49 + \ln x - 7\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{x^{7ij}}{j} + 8\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{x^{ij}}{j}. \tag{11}$$

Let $f_1(x)$ and $f_2(x)$ be the following two functions:

$$f_1(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{x^{7ij}}{j}, \tag{12}$$
$$f_2(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{x^{ij}}{j}. \tag{13}$$

The function $f_1(x)$ has non-zero coefficients for x^m if and only if m is a multiple of 7. More precisely, let

$$f_1(x) = \sum_{i=1}^{\infty} a_i x^i.$$

Then,

$$a_i = \begin{cases} \frac{\sigma(i/7)}{i/7}, & \text{if } 7|i; \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

We write the function $f_2(x)$ as

$$f_2(x) = \sum_{i=1}^{\infty} b_i x^i$$

where

$$b_i = \frac{\sigma(i)}{i}. \tag{15}$$

By (10), (12), (13), (14), (15), we get

$$\ln(G(x)) = \ln 7 + \sum_{i=1}^{\infty} d_i x^i \tag{16}$$

where

$$d_i = \begin{cases} \frac{4\sigma(i)}{i} - \frac{3\sigma(i/7)}{i/7}, & \text{if } 7|i; \\ \frac{4\sigma(i)}{i}, & \text{otherwise.} \end{cases} \tag{17}$$

By Lemma 1, we write (17) as

$$d_i = \frac{\sigma(i)}{i}\Big(1 + \frac{18}{7^{m+1} - 1}\Big) \tag{18}$$

for $i = 7^m i'$, $m \geq 0$. Similarly, by (11), (12), (13), (14), (15), we get

$$\ln(H(x)) = \ln 49 + \ln x + \sum_{i=1}^{\infty} e_i x^i \tag{19}$$

where

$$e_i = \begin{cases} \frac{8\sigma(i)}{i} - \frac{7\sigma(i/7)}{i/7}, & \text{if } 7|i; \\ \frac{8\sigma(i)}{i}, & \text{otherwise.} \end{cases} \tag{20}$$

By Lemma 1, we write (20) as

$$e_i = \frac{\sigma(i)}{i}\Big(1 + \frac{42}{7^{m+1} - 1}\Big) \tag{21}$$

for $i = 7^m i'$, $m \geq 0$. By (1), (16), (18), we have

$$\begin{aligned} G(x) &= \exp(\ln G(x)) = 7\exp\Big(\sum_{n=1}^{\infty} d_n x^n\Big) = 7\exp\Big(\sum_{n=1}^{\infty} (n!d_n)\frac{x^n}{n!}\Big) \\ &= 7\Big(\sum_{n=0}^{\infty} B_n(1!d_1, 2!d_2, \dots, n!d_n)\frac{x^n}{n!}\Big). \end{aligned} \tag{22}$$

By (1), (19), (21), we have

$$\begin{aligned} H(x) &= \exp(\ln H(x)) = 49x\exp\Big(\sum_{n=1}^{\infty} e_n x^n\Big) = 49x\exp\Big(\sum_{n=1}^{\infty} (n!e_n)\frac{x^n}{n!}\Big) \\ &= 49x\Big(\sum_{n=0}^{\infty} B_n(1!e_1, 2!e_2, \dots, n!e_n)\frac{x^n}{n!}\Big) \\ &= 49\Big(\sum_{n=1}^{\infty} B_{n-1}(1!e_1, 2!e_2, \dots, (n-1)!e_{n-1})\frac{x^n}{(n-1)!}\Big). \end{aligned} \tag{23}$$

By (3), (8), (9), (22), (23), we have the following identity:

$$\sum_{n=1}^{\infty} p(7n+5)x^n = \sum_{n=1}^{\infty} \Big(7B_n(1!d_1, 2!d_2, \ldots, n!d_n) + 49nB_{n-1}(1!e_1, 2!e_2, \ldots, (n-1)!e_{n-1})\Big)\frac{x^n}{n!}$$

as desired. □

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