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Some Notes on Alternating Power Sums of Arithmetic Progressions

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Abstract

We show that the alternating power sum

$$r^n - (m+r)^n + (2m+r)^n - \cdots + (-1)^{\ell-1}\left((\ell-1)\,m+r\right)^n$$

can be expressed in terms of Stirling numbers of the first kind and r-Whitney numbers of the second kind. We also prove a necessary and sufficient condition for the integrality of the coefficients of the polynomial extensions of the above alternating power sum.

1 Introduction

Power sums and alternating power sums of consecutive numbers are widely investigated objects in the literature of combinatorics and number theory. It is well known, among others, that the sum of the n-th power of the first $\ell-1$ positive integers

$$S_n(\ell) := 1^n + 2^n + \cdots + (\ell-1)^n$$

is closely connected to the Bernoulli polynomials $B_n(x)$ via the identity

$$S_n(\ell) = \frac{1}{n+1}\left(B_{n+1}(\ell) - B_{n+1}\right),$$

where the polynomials $B_n(x)$ are defined by the generating series

$$\frac{te^{tx}}{e^t-1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}$$

and $B_{n+1} = B_{n+1}(0)$.

It is also well known that the alternating power sum

$$T_n(\ell) := -1^n + 2^n - \cdots + (-1)^{\ell-1}(\ell-1)^n$$

can be expressed by means of the classical Euler polynomials $E_n(x)$ via:

$$T_n(\ell) = \frac{E_n(0) + (-1)^{\ell-1}E_n(\ell)}{2},$$

where the classical Euler polynomials $E_n(x)$ are usually defined by the generating function

$$\frac{2e^{xt}}{e^t+1} = \sum_{k=0}^{\infty} E_k(x)\frac{t^k}{k!} \quad (|t| < \pi).$$

For the properties of Bernoulli and Euler polynomials which will be often used in this paper, sometimes without special reference, we refer to the paper of Brillhart [9] and the book of Abramowitz and Stegun [1].

Let $\ell > 1, m \neq 0, r$ be integers with $\gcd(m,r) = 1$ and consider the following sums

$$S_{m,r}^n(\ell) = r^n + (m+r)^n + (2m+r)^n + \cdots + ((\ell-1)m + r)^n,$$

$$T_{m,r}^n(\ell) := r^n - (m+r)^n + (2m+r)^n - \cdots + (-1)^{\ell-1}((\ell-1)m + r)^n.$$

Bazsó et al. [5] showed that $S_{m,r}^n(\ell)$ can be extended to the following polynomial in x:

$$\mathtt{S}_{m,r}^n(x) = \frac{m^n}{n+1}\left(B_{n+1}\left(x + \frac{r}{m}\right) - B_{n+1}\left(\frac{r}{m}\right)\right). \tag{1}$$

Using a different approach, Howard [11] also obtained relation (1) together with its analogue for $T_{m,r}^{n}(\ell)$:

$$T_{m,r}^{n}(\ell) = \frac{m^n}{2}\left(E_n\left(\frac{r}{m}\right) + (-1)^{\ell-1}E_n\left(\ell+\frac{r}{m}\right)\right), \tag{2}$$

whence, the following polynomial extensions arise for $T_{m,r}^{n}(\ell)$:

$$\mathtt{T}_{m,r}^{n+}(x) = \frac{m^n}{2}\left(E_n\left(\frac{r}{m}\right) + E_n\left(x+\frac{r}{m}\right)\right), \tag{3}$$

$$\mathtt{T}_{m,r}^{n-}(x) = \frac{m^n}{2}\left(E_n\left(\frac{r}{m}\right) - E_n\left(x+\frac{r}{m}\right)\right). \tag{4}$$

Clearly, for positive integer values x, we have $\mathtt{T}_{m,r}^{n+}(x) = T_{m,r}^{n}(x)$ if x is odd, and $\mathtt{T}_{m,r}^{n-}(x) = T_{m,r}^{n}(x)$ if x is even.

For related diophantine results on the polynomials $\mathtt{S}_{m,r}^{n}(x), \mathtt{T}_{m,r}^{n+}(x)$, and $\mathtt{T}_{m,r}^{n-}(x)$ see [3, 6, 7, 8, 10, 12, 15] and the references given there. For results on the decomposition of these polynomials we refer to the papers [2, 5, 8].

In a recent paper [4], the present authors investigated the coefficients of $\mathtt{S}_{m,r}^{n}(x)$. We showed that these coefficients can be given in terms of the Stirling numbers of the first kind and r-Whitney numbers of the second kind. Moreover, we proved that $\mathtt{S}_{m,r}^{n}(x) \in \mathbb{Z}[x]$ if and only if m is divisible by $F(n)$, where $F(n)$ is the sequence with first few terms

$$2, 6, 2, 30, 6, 42, 6, 30, 10, 66, 6, 2730, \ldots$$

(cf. A144845 in Sloane's OEIS [14]). We [4] also gave an implicit formula for $F(n)$.

The aim of this note is to give analogues of our results [4] on $\mathtt{S}_{m,r}^{n}(x)$ for the alternating case.

2 An explicit formula for the alternating sum $T_{m,r}^{n}(\ell)$

From (2) we know that the alternating sum $T_{m,r}^{n}(\ell)$ can be expressed in terms of ℓ and the Euler polynomials. We give an explicit formula for $T_{m,r}^{n}(\ell)$ without the Euler polynomials included. To do this we need the following lemma.

Lemma 1. *For all $\ell \geq 1$ and $k \geq 0$ we have that*

$$\sum_{x=0}^{\ell-1}(-1)^x x^{\underline{k}} = k!\frac{(-1)^k}{2^{k+1}}\left(1+(-1)^{\ell+1}\sum_{i=0}^{k}\binom{\ell}{i}(-2)^i\right).$$

Here $x^{\underline{k}} = x(x-1)\cdots(x-k+1)$ is the falling factorial.

Proof. The idea we use here is due to Felix Marin. We found out about his idea on the Mathematics Stack Exchange forum [17].

First note that

$$\sum_{x=0}^{\ell-1}(-1)^x x^{\underline{k}} = k!\sum_{x=0}^{\ell-1}(-1)^x\binom{x}{k}. \tag{5}$$

Then we use the integral representation

$$\binom{x}{k} = \oint_{|z|<1}\frac{(1+z)^x}{z^{k+1}}\frac{dz}{2\pi i}.$$

Substituting this into (5) the summation can already be done. We have the intermediate result that

$$\sum_{x=0}^{\ell-1}(-1)^x x^{\underline{k}} = k!\oint_{|z|<1}\frac{1}{z^{k+1}}\frac{(-1)^{\ell+1}(1+z)^\ell+1}{2+z}\frac{dz}{2\pi i}. \tag{6}$$

The path integral on the right can be calculated by Cauchy's residue theorem:

$$\oint_{|z|<1}\frac{1}{z^{k+1}}\frac{(-1)^{\ell+1}(1+z)^\ell+1}{2+z}\frac{dz}{2\pi i} =$$

$$\mathrm{Res}_{z=0}\frac{1}{(2+z)z^{k+1}} + (-1)^{\ell+1}\mathrm{Res}_{z=0}\frac{(1+z)^\ell}{(2+z)z^{k+1}}.$$

The first residue is easy to determine:

$$\mathrm{Res}_{z=0}\frac{1}{(2+z)z^{k+1}} = \frac{(-1)^k}{2^{k+1}} \quad (k\ge 0). \tag{7}$$

The calculation of the second residue can be traced back to the first one by expanding $(1+z)^\ell$ by the binomial theorem:

$$\mathrm{Res}_{z=0}\frac{(1+z)^\ell}{(2+z)z^{k+1}} = \sum_{i=0}^{\ell}\binom{\ell}{i}\mathrm{Res}_{z=0}\frac{z^i}{(2+z)z^{k+1}}.$$

Note that if $i\ge k+1$ the function becomes analytic at $z=0$ and the residue disappears. Hence, recalling (7),

$$\mathrm{Res}_{z=0}\frac{(1+z)^\ell}{(2+z)z^{k+1}} = \sum_{i=0}^{k}\binom{\ell}{i}\frac{(-1)^{k-i}}{2^{k+1-i}} = \frac{(-1)^k}{2^{k+1}}\sum_{i=0}^{k}\binom{\ell}{i}(-2)^i.$$

This and (7) together gives the result. □

We recall the definition of the r-Whitney numbers $W_{m,r}(n,k)$ given by Mező [13]:

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k)x^{\underline{k}}. \tag{8}$$

Theorem 2. *If $\ell \geq 1$ then the sum $T^n_{m,r}(\ell)$ can be expressed as follows:*

$$T^n_{m,r}(\ell) = (1 + (-1)^{\ell+1}) \sum_{k=0}^{n} C_{m,r,n,k} +$$

$$\sum_{j=1}^{n} \ell^j \left((-1)^{\ell+1} \sum_{k=0}^{n} C_{m,r,n,k} \sum_{i=j}^{k} \frac{(-2)^i}{i!} S_1(i,j) \right).$$

Here

$$C_{m,r,n,k} = k! \frac{(-1)^k}{2^{k+1}} m^k W_{m,r}(n,k),$$

and $W_{m,r}(n,k)$ is an r-Whitney number.

Proof. We can see that it is enough to multiply both sides of (8) by $(-1)^x$ and sum from $x = 0, 1, \ldots, \ell - 1$ to get back $T^n_{m,r}(\ell)$. Hence

$$T^n_{m,r}(\ell) = \sum_{k=0}^{n} m^k W_{m,r}(n,k) \sum_{x=0}^{\ell-1} (-1)^x x^{\underline{k}}.$$

By the previous lemma we now have that

$$T^n_{m,r}(\ell) = \sum_{k=0}^{n} m^k W_{m,r}(n,k) k! \frac{(-1)^k}{2^{k+1}} \left(1 + (-1)^{\ell+1} \sum_{i=0}^{k} \binom{\ell}{i} (-2)^i \right). \tag{9}$$

Our original goal is to find the coefficients of ℓ in this expression. It is immediate that the constant term equals to (when $i = 0$)

$$C := (1 + (-1)^{\ell+1}) \sum_{k=0}^{n} m^k W_{m,r}(n,k) k! \frac{(-1)^k}{2^{k+1}}. \tag{10}$$

The other coefficients of ℓ can be determined by expanding the $\binom{\ell}{i}$ binomial coefficients with the aid of the Stirling numbers of the first kind:

$$\binom{\ell}{i} = \frac{1}{i!} \sum_{j=0}^{i} S_1(i,j) \ell^j.$$

Here, the index of the sum runs to n (this is the maximal value j can ever attain), because $S_1(i,j) = 0$ if $j > i$. So we can factor out ℓ^j in (9):

$$T^n_{m,r}(\ell) = C + \sum_{j=0}^{n} \ell^j \sum_{k=0}^{n} m^k W_{m,r}(n,k) k! \frac{(-1)^k}{2^{k+1}} (-1)^{\ell+1} \sum_{i=0}^{k} \frac{(-2)^i}{i!} S_1(i,j).$$

Noting that i runs from j, and recalling the definition of the constants $C_{m,r,n,k}$ we finish the proof. $\square$

3 The integrality of the coefficients of the polynomials $\mathtt{T}^{n+}_{m,r}(x)$ and $\mathtt{T}^{n-}_{m,r}(x)$

In this section, let m, r, n be integers with $m \neq 0, r$ coprime and $n > 0$.

Theorem 3. *For all m, r and n, both $2(n+1)\mathtt{T}^{n+}_{m,r}(x)$ and $2(n+1)\mathtt{T}^{n-}_{m,r}(x)$ are in $\mathbb{Z}[x]$.*

Our Theorem 3 follows from the following result.

Lemma 4. *For all m and n we have $(n+1)m^n E_n\left(\frac{x}{m}\right) \in \mathbb{Z}[x]$.*

Proof. This is part of a result of Sun [16, Lemma 2.2]. □

Proof of Theorem 3. By (3) we have

$$\mathtt{T}^{n+}_{m,r}(x) = \frac{m^n}{2}\left(E_n\left(\frac{r}{m}\right) + E_n\left(x + \frac{r}{m}\right)\right) = \\ = \frac{1}{2(n+1)}\left[(n+1)m^n\left(E_n\left(\frac{r}{m}\right) + E_n\left(\frac{mx+r}{m}\right)\right)\right]. \quad (11)$$

The expression in square brackets is a sum of two polynomials with integer coefficients by Lemma 4, whence $2(n+1)\mathtt{T}^{n+}_{m,r}(x) \in \mathbb{Z}[x]$. For $2(n+1)\mathtt{T}^{n-}_{m,r}(x)$, the proof is essentialy the same. □

By the *denominator* of a polynomial $P(x) \in \mathbb{Q}[x]$ we mean the smallest positive integer d such that $dP(x) \in \mathbb{Z}[x]$. An immediate consequence of Theorem 3 is that the denominators of $\mathtt{T}^{n+}_{m,r}(x)$ and $\mathtt{T}^{n-}_{m,r}(x)$ respectively, are divisors of $2(n+1)$. In the sequel, we give a more precise description of these denominators.

Remark 5. It is well known (see, e.g., the paper of Brillhart [9] p. 46) that an Euler polynomial of even index has only integer coefficients, and that the denominator of an odd index Euler polynomial is a power of 2. By Lemma 4 with choice $m = 1$, the denominator of the n-th Euler polynomial is either 1 or a power of 2 which divides $n + 1$.

Now we state the main result of this section.

Theorem 6. *All coefficients of the polynomials $\mathtt{T}^{n+}_{m,r}(x)$ and $\mathtt{T}^{n-}_{m,r}(x)$ are integers if and only if m is even.*

Proof. First we consider the integrality of the coefficients of the polynomial $\mathtt{T}^{n+}_{m,r}(x)$. By (3), we observe that all these coefficients are integers if and only if m^n is divisible by the denominator of $\mathtt{T}^{n+}_{m,r}(x)$. Let this denominator be denoted by D. Clearly, D is the product of 2 and the denominator of $E_n(x)$.

If n is even, then $D = 2$ by the above remark, and thus for even m all the coefficients of $\mathtt{T}^{n+}_{m,r}(x)$ are integers.

For odd n, by the same remark, the denominator of $E_n(x)$ is a power of 2 which divides $n+1$, say 2^q. Thus we have $D = 2^{q+1}$. Since $2^q \leq n+1 < 2^n$ for $n > 1$, it follows that $n \geq q+1$, thus D divides 2^n for $n > 1$. For $n = 1$, we have

$$\mathtt{T}^{1+}_{m,r}(x) = \frac{mx}{2} - \frac{m}{2} + r.$$

Hence for even m, we have $\mathtt{T}^{n+}_{m,r}(x) \in \mathbb{Z}[x]$.

The equivalence of the integrality of the coefficients of $\mathtt{T}^{n-}_{m,r}(x)$ and that m is even follows from a similar argument and from (4). This completes the proof. □

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