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On an Integer Sequence Related to a Product of Trigonometric Functions, and its Combinatorial Relevance

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Abstract

In this paper it is shown that for $n \equiv 0$ or $3 \pmod 4$, the middle term $S(n)$ in the expansion of the polynomial $(1+x)(1+x^2)\cdots(1+x^n)$ occurs naturally when one analyzes when a discontinuous product of trigonometric functions is a derivative of a function. This number also represents the number of partitions of $T_n/2 = n(n+1)/4$, (where T_n is the nth triangular number) into distinct parts less than or equal to n. It is proved in a constructive way that $S(n) \geq 6S(n-4)$ for every $n \geq 8$, and an

asymptotic evaluation of $S(n)^{1/n}$ is obtained as a consequence of the unimodality of the coefficients of this polynomial. Also an integral expression of $S(n)$ is deduced.

1 Notation and preliminary results

In a paper of Andrica [3] the following necessary and sufficient condition that some product of derivatives is also a derivative is deduced:

Theorem 1.1 *Let $n_1, \ldots, n_k \geq 0$ be integers with $n_1 + \ldots + n_k \geq 1$ and let $\alpha_1, \ldots, \alpha_k$ be real numbers different from zero. The function $f_{n_1,\ldots,n_k}^{\alpha_1,\ldots,\alpha_k} : \mathbb{R} \to \mathbb{R}$, defined by*

$$f_{n_1,\ldots,n_k}^{\alpha_1,\ldots,\alpha_k}(x) = \begin{cases} \cos^{n_1}(\alpha_1/x) \cdots \cos^{n_k}(\alpha_k/x), & \text{if } x \neq 0; \\ \alpha, & \text{if } x = 0; \end{cases}$$

is a derivative if and only if

$$\alpha = \frac{1}{2^{n_1+\ldots+n_k}} S(n_1, \ldots, n_k; \alpha_1, \ldots, \alpha_k),$$

where $S(n_1, \ldots, n_k; \alpha_1, \ldots, \alpha_k)$ is the number of all choices of signs $+$ and $-$ such that

$$\underbrace{\pm\alpha_1 \pm \ldots \pm \alpha_1}_{n_1 \text{ times}} \underbrace{\pm\alpha_2 \pm \ldots \pm \alpha_2}_{n_2 \text{ times}} \pm \ldots \underbrace{\pm\alpha_k \pm \ldots \pm \alpha_k}_{n_k \text{ times}} = 0. \tag{1}$$

Note that this theorem extends one previously published in [2].

We shall present another combinatorial interpretations of the numbers

$$S(n_1, \ldots, n_k; \alpha_1, \ldots, \alpha_k)$$

and an integral representation, while the last section is devoted to the sequence $S(n) = S(\underbrace{1, \ldots, 1}_{n \text{ times}}; 1, 2, 3, \ldots, n)$ for $n \geq 1$.

Let M be a multiset of type $\alpha_1^{n_1}\alpha_2^{n_2}\ldots\alpha_k^{n_k}$, i.e., a multiset containing α_i with multiplicity n_i for every $1 \leq i \leq k$. It is clear that $S(n_1, \ldots, n_k; \alpha_1, \ldots \alpha_k)$ is the number of ordered partitions having equal sums of M, i.e., of ordered pairs (C_1, C_2) such that $C_1 \cup C_2 = M$, $C_1 \cap C_2 = \emptyset$ and $\sum_{x \in C_1} x = \sum_{y \in C_2} y = \frac{1}{2}\sum_{i=1}^k n_i\alpha_i$. Indeed, there exists a bijection between the set of all choices of $+$ or $-$ signs in (1) and the set of all ordered partitions with equal sums of M defined as follows: We put α_i from (1) in C_1 if its sign is $+$ and in C_2 otherwise.

It is also clear that $S(n_1, \ldots, n_k; \alpha_1 \ldots, \alpha_k)$ is the term not depending on z in the expansion

$$F(z) = \left(z^{\alpha_1} + \frac{1}{z^{\alpha_1}}\right)^{n_1} \left(z^{\alpha_2} + \frac{1}{z^{\alpha_2}}\right)^{n_2} \ldots \left(z^{\alpha_k} + \frac{1}{z^{\alpha_k}}\right)^{n_k}. \tag{2}$$

Wilf [10] outlines a proof that for $n_1 = n_2 = \ldots = n_k = 1$, the coefficient of z^n in $F(z)$ represents the number of ways of choosing + or − signs such that $\pm\alpha_1 \pm \alpha_2 \pm \ldots \pm \alpha_k = n$. If $\alpha_1, \ldots, \alpha_k$ are positive integers, from (2) one gets

$$F(z) = S(n_1, \ldots, n_k; \alpha_1, \ldots, \alpha_k) + \sum_{\alpha \neq 0} a_\alpha z^\alpha, \tag{3}$$

where the sum has only a finite number of terms and α and a_α are integers. By substituting $z = \cos t + i \sin t$, $t \in \mathbb{R}$ in (3) one deduces

$$2^{n_1+\ldots+n_k} \prod_{j=1}^{k} (\cos \alpha_j t)^{n_j} = S(n_1, \ldots, n_k; \alpha_1, \ldots, \alpha_k) + \sum_{\alpha \neq 0} a_\alpha (\cos \alpha t + i \sin \alpha t)$$

By integration on $[0, 2\pi]$ we find the following integral expression of $S(n_1, \ldots, n_k; \alpha_1, \ldots, \alpha_k)$:

$$S(n_1, \ldots, n_k; \alpha_1, \ldots, \alpha_k) = \frac{2^{n_1+\ldots+n_k}}{2\pi} \int_0^{2\pi} (\cos \alpha_1 t)^{n_1} \cdots (\cos \alpha_k t)^{n_k} dt.$$

2 A particular case and its connection with polynomial unimodality

An interesting particular case is obtained for $n_1 = n_2 = \ldots = n_k = 1$ and $\alpha_i = i$ for every $1 \leq i \leq k$. In this case $S(n)$ is the number of ways of choosing + and − signs such that $\pm 1 \pm 2 \pm \ldots \pm n = 0$. Since now $M = \{1, 2, \ldots, n\}$ has sum $T_n = n(n+1)/2$ and every class of an ordered bipartition of M must have sum $T_n/2$, it follows that $S(n) = 0$ for $n \equiv 1$ or $2 \pmod 4$ and $S(n) \neq 0$ for $n \equiv 0$ or $3 \pmod 4$. The following theorem proposes several equivalent definitions of the sequence $S(n)$ for $n \geq 1$.

Theorem 2.1 *For every $n \geq 1$ the following properties are equivalent:*

(i) $S(n)$ is the number of choices of + and − signs such that $\pm 1 \pm 2 \pm \ldots \pm n = 0$;

(ii) $S(n)$ is the number of ordered bipartitions into classes having equal sums of $\{1, 2, \ldots, n\}$;

(iii) $S(n)$ is the term not depending on x in the expansion of

$$\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right) \ldots \left(x^n + \frac{1}{x^n}\right);$$

(iv) $S(n)$ is the number of partitions of $T_n/2$ into distinct parts, less than or equal to n, if $n \equiv 0$ or $3 \pmod 4$, and $S(n) = 0$ otherwise;

(v) $S(n)$ is the number of distinct subsets of $\{1, \ldots, n\}$ whose elements sum to $T_n/2$ if $n \equiv 0$ or $3 \pmod 4$, and $S(n) = 0$ if $n \equiv 1$ or $2 \pmod 4$;

(vi) $S(n)$ is the coefficient of $x^{T_n/2}$ in the polynomial $G_n(x) = (1+x)(1+x^2)\ldots(1+x^n)$ when $n \equiv 0$ or $3 \pmod 4$, and $S(n) = 0$ otherwise;

(vii)

$$S(n) = \frac{2^{n-1}}{\pi} \int_0^{2\pi} \cos t \cos 2t \cdots \cos nt \, dt;$$

(viii) $S(n)/2^n$ is the unique real number α having the property that the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} \cos(1/x)\cos(2/x)\cdots\cos(n/x), & \text{if } x \neq 0; \\ \alpha, & \text{if } x = 0; \end{cases}$$

is a derivative.

Proof: Some equivalences are obvious or were shown in the general case. For example, the equivalence between (ii) and (v) is given by the bijection φ defined for every bipartition $M = C_1 \cup C_2$ such that $\sum_{x \in C_1} x = \sum_{y \in C_2} y$ by $\varphi(C_1 \cup C_2) = C_1 \subset M$. □

Let us denote

$$G_n(x) = (1+x)(1+x^2)\dots(1+x^n) = \sum_{i=0}^{T_n} G(n,i)x^i. \tag{4}$$

Note that the property that the coefficient of x^i in $G_n(x)$ is the number of distinct subsets of $\{1,\dots,n\}$ whose elements sum to i was used by Friedman and Keith [5] to deduce a necessary and sufficient condition for the existence of a basic (n,k) magic carpet. Stanley [9], using the "hard Lefschetz theorem" from algebraic geometry, proved that the posets $M(n)$ of all partitions of integers into distinct parts less than or equal to n are rank unimodal, by showing the existence of a chain decomposition for $M(n)$. This fact is equivalent to the unimodality of the polynomial $G_n(x)$, which implies that $S(n)$ is the maximum coefficient in the expansion of $G_n(x)$ for $n \equiv 0$ or 3 (mod 4). Stanley's proof was subsequently simplified by Proctor [6].

The property of symmetry of the coefficients in (4), namely $G(n,i) = G(n, T_n - i)$ for every $0 \le i \le T_n$ was pointed out by Friedman and Keith[5]; they also found the recurrence $G(n,i) = G(n-1,i) + G(n-1,i-n)$. This latter recurrence, which is a consequence of the identity $G_n(x) = G_{n-1}(x)(1+x^n)$, allows us to compute any finite submatrix of the numbers $G(n,i)$ and thus the numbers $S(n) = G(n, T_n/2)$.

Some values of $S(n)$, starting with $n = 3$, are given in the following table:

n	$S(n)$	n	$S(n)$	n	$S(n)$	n	$S(n)$
3	2	13	0	23	99,820	33	0
4	2	14	0	24	187,692	34	0
5	0	15	722	25	0	35	221,653,776
6	0	16	1,314	26	0	36	425,363,952
7	8	17	0	27	1,265,204	37	0
8	14	18	0	28	2,399,784	38	0
9	0	19	8,220	29	0	39	3,025,553,180
10	0	20	15,272	30	0	40	5,830,034,720
11	70	21	0	31	16,547,220	41	0
12	124	22	0	32	31,592,878	42	0

and thus the terms different from zero form a subsequence of the sequence A025591 in Sloane [7].

Another recurrence satisfied by the numbers $G(n,i)$ is the following:

Lemma 2.2 *We have* $G(n,i) = \sum_{j\geq 0} G(n-1-j, i-n+j)$.

Proof: Let $\mathcal{P}(k,i)$ denote the set of partitions of i into distinct parts such that the maximum part is equal to k. It is clear that

$$G(n,i) = \left|\bigcup_{j\geq 0} \mathcal{P}(n-j,i)\right| = \sum_{j\geq 0} |\mathcal{P}(n-j,i)| = \sum_{j\geq 0} G(n-1-j, i-n+j).$$

Indeed, there is a bijection between the set of partitions of i into distinct parts such that the maximum part equals $n-j$ and the set of partitions of $i-n+j$ into distinct parts less than or equal to $n-1-j$, defined by deleting the maximum part, equal to $n-j$, in any partition in $\mathcal{P}(n-j,i)$. Hence $|\mathcal{P}(n-j,i)| = G(n-1-j, i-n+j)$. □

Theorem 2.3 *For any* $n \geq 8$ *we have* $S(n) \geq 6S(n-4)$.

Proof: For $n \leq 11$ this inequality is verified by inspection.

For $n \geq 12$ we shall propose a constructive proof yielding for any ordered partition of $\{1,\ldots,n-4\}$ in two classes C_1 and C_2 with equal sums six ordered partitions of $\{1,\ldots,n\}$ in two classes $\mathtt{C}'_1$ and $\mathtt{C}'_2$ having equal sums and all partitions generated will be distinct. Indeed, for any ordered bipartition with equal sums $\{1,\ldots,n-4\} = C_1 \cup C_2$ we can generate six ordered bipartitions with equal sums $\{1,\ldots,n\} = \mathtt{C}'_1 \cup \mathtt{C}'_2$ as follows:

(a) $\mathtt{C}'_1 = C_1 \cup \{n-3, n\}$ and $\mathtt{C}'_2 = C_2 \cup \{n-2, n-1\}$;

(b) $\mathtt{C}'_1 = C_1 \cup \{n-2, n-1\}$ and $\mathtt{C}'_2 = C_2 \cup \{n-3, n\}$;

(c) Without loss of generality suppose $1 \in C_1$. We define $\mathtt{C}''_1 = C_1 \backslash \{1\}$, $\mathtt{C}''_2 = C_2 \cup \{1\}$, $\mathtt{C}'_1 = \mathtt{C}''_1 \cup \{n-2, n\}$ and $\mathtt{C}'_2 = \mathtt{C}''_2 \cup \{n-3, n-1\}$;

(d) Without loss of generality suppose $2 \in C_1$. Now $\mathtt{C}''_1 = C_1 \backslash \{2\}$, $\mathtt{C}''_2 = C_2 \cup \{2\}$, $\mathtt{C}'_1 = \mathtt{C}''_1 \cup \{n-1, n\}$, $\mathtt{C}'_2 = \mathtt{C}''_2 \cup \{n-3, n-2\}$.

Case (e) is a little more complicated, but we will be able to do it by combining two simple transformations.

(e) Suppose $1 \in C_1$. If $n-4$ belongs to the same class, we define $\mathtt{C}''_1 = C_1 \backslash \{1, n-4\}$, $\mathtt{C}''_2 = C_2 \cup \{1, n-4\}$, $\mathtt{C}'_1 = \mathtt{C}''_1 \cup \{n-3, n-2, n-1\}$ and $\mathtt{C}'_2 = \mathtt{C}''_2 \cup \{n\}$. This transformation resolves the imbalance of $2n-6$ between $\mathtt{C}''_1$ and $\mathtt{C}''_2$ and will be called of type A.

Otherwise $1 \in C_1$ and $n-4 \in C_2$. If $2 \in C_2$ one defines $\mathtt{C}''_2 = C_2 \backslash \{2, n-4\}$, $\mathtt{C}''_1 = C_1 \cup \{2, n-4\}$, $\mathtt{C}'_1 = \mathtt{C}''_1 \cup \{n-1\}$ and $\mathtt{C}'_2 = \mathtt{C}''_2 \cup \{n, n-2, n-3\}$. This transformation balances classes $\mathtt{C}''_1$ and $\mathtt{C}''_2$ by $2n-4$ and will be called of type B.

Otherwise $2 \in C_1$, hence $C_1 = \{1, 2, \ldots\}$ and $C_2 = \{n-4, \ldots\}$. If $n-5 \in C_1$ then $\mathtt{C}''_1 = C_1 \backslash \{2, n-5\}$, $\mathtt{C}''_2 = C_2 \cup \{2, n-5\}$, $\mathtt{C}'_1 = \mathtt{C}''_1 \cup \{n-3, n-2, n-1\}$ and $\mathtt{C}'_2 = \mathtt{C}''_2 \cup \{n\}$.

Otherwise $n-5 \in C_2$, hence $C_1 = \{1, 2, \ldots\}$, $C_2 = \{n-4, n-5, \ldots\}$. Now if $3 \in C_2$ we move 3 and $n-5$ into C_1 and apply a type B transformation.

Otherwise $3 \in C_1$ and if $n-6 \in C_1$, we add $n-6$ and 3 to C_2 and apply a type A transformation; otherwise $C_1 = \{1, 2, 3, \ldots\}$ and $C_2 = \{n-4, n-5, n-6, \ldots\}$ and so on.

Note that a transformation of type A or B can be applied to every partition $\pi = C_1 \cup C_2$ of $\{1, \ldots, n-4\}$ since otherwise π must have classes $C_1 = \{1, 2, 3, \ldots\}$ and $C_2 = \{n-4, n-5, n-6, \ldots\}$ such that for every $k \in C_1$ verifying $1 \leq k \leq (n-4)/2$, the number $n-k-3$ belongs to C_2. But this contradicts the property that C_1 and C_2 have the same sum for every $n \geq 8$.

If $1 \in C_2$ this algorithm runs similarly and all partitions generated in this way are pairwise distinct.

(f) Suppose $3 \in C_1$. If $n-4 \in C_1$, we move 3 and $n-4$ into C_2 and annihilate the imbalance equal to $2n-2$ by defining $\mathtt{C}_1' = \mathtt{C}_1'' \cup \{n, n-1, n-3\}$ and $\mathtt{C}_2' = \mathtt{C}_2'' \cup \{n-2\}$ (a type C transformation).

Otherwise $C_1 = \{3, \ldots\}$, $C_2 = \{n-4, \ldots\}$. If $4 \in C_2$ we move 4 and $n-4$ into C_1 which produces an imbalance equal to $2n$; then define $\mathtt{C}_1' = \mathtt{C}_1'' \cup \{n-3\}$ and $\mathtt{C}_2' = \mathtt{C}_2'' \cup \{n, n-1, n-2\}$ (a type D transformation).

Otherwise $C_1 = \{3, 4, \ldots\}$ and $C_2 = \{n-4, \ldots\}$. If $n-5 \in C_1$ we move 4 and $n-5$ into C_2 and apply a type C transformation; otherwise $C_1 = \{3, 4, \ldots\}$ and $C_2 = \{n-4, n-5, \ldots\}$. In this way we can apply a transformation of type C or D to every partition π of $\{1, \ldots, n-4\}$ since otherwise $C_1 = \{3, 4, 5, \ldots\}$, $C_2 = \{n-4, n-5, n-6, \ldots\}$ such that for every $k \in C_1$, $3 \leq k \leq (n-2)/2$, we have $n-k-1 \in C_2$. This is a contradiction, since in this case C_1 and C_2 cannot have the same sum for every $n \geq 12$. As in the previous cases all partitions produced in this way are distinct. □

This theorem has the following consequence:

Corollary 2.4 *We have*

$$S(n) > 6^{n/4} \approx 1.56508^n \tag{5}$$

for every $n \equiv 0$ *or* $3 \pmod 4$ *and* $n \geq 16$.

Proof: If $n = 4k$ one gets $S(4k) \geq 6^{n/4-4} S(16) > 6^{n/4}$ since $S(16) = 1,314$. Similarly, $S(4k+3) \geq 6^{k-3} S(15) = 6^{(n-15)/4} S(15) > 6^{n/4}$ because $S(15) = 722$. □

Note that in [5] the maximum coefficient in the polynomial $G_n(x)$, which coincides with $S(n)$ for $n \equiv 0$ or 3 (mod 4), is bounded below by $2(n+1)$ for every $n \geq 10$.

Although the lower bound (5) is exponential, its order of magnitude is far from being exact, as can be seen below.

Lemma 2.5

$$\lim_{n\to\infty} S(4n)^{1/(4n)} = \lim_{n\to\infty} S(4n+3)^{1/(4n+3)} = 2. \tag{6}$$

Proof: Since the sequence of coefficients $(G(n,i))_{i=0,\ldots,T_n}$ in $G_n(x)$ is unimodal ([6, 7]) and symmetric, and the first and last coefficient are equal to 1, it follows that for every $n \geq 5$, $n \equiv 0$ or 3 (mod 4),

$$S(n) > \frac{2^n - 2}{T_n - 1} > \frac{2^n}{T_n} = \frac{2^{n+1}}{n^2 + n}.$$

Indeed, $\sum_{i=0}^{T_n} G(n,i) = G_n(1) = 2^n$ and $T_n < 2^{n-1}$ for every $n \geq 5$. On the other hand, $S(n) < 2^n - 2$, the number of ordered partitions having two classes of $\{1, \ldots, n\}$, and these two inequalities imply (6). □

A better upper bound for $S(n)$ is $\binom{n}{\lfloor n/2 \rfloor} \le C_1 \frac{2^n}{\sqrt{n}}$ for some constant $C_1 > 0$. This follows from the following particular case of a result of Erdős (see [1] or [4]): Fix an interval of length 2 and consider the set of combinations $\sum_{i=1}^{n} \varepsilon_i i$, that lie within the interval, where $\varepsilon_i \in \{1, -1\}$ for every $1 \le i \le n$. The sets $\{i : \varepsilon_i = 1\}$ that correspond to these combinations form an antichain in the poset of subsets of $\{1, \ldots, n\}$ ordered by inclusion. By Sperner's theorem [8] the maximum number of elements in such an antichain is $\binom{n}{\lfloor n/2 \rfloor}$, which is an upper bound for the number of combinations $\sum_{i=1}^{n} \varepsilon_i i$ that sum to 0.

Conjecture 2.6 *For* $n \equiv 0$ or 3 (mod 4) *we have*

$$S(n) \sim \sqrt{6/\pi} \cdot \frac{2^n}{n\sqrt{n}},$$

where $f(n) \sim g(n)$ means that $\lim_{n\to\infty} f(n)/g(n) = 1$.

This behavior was verified by computer experiments up to $n = 100$.

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