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Beukers' integrals and Apéry's recurrences

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Abstract

We give a new and completely elementary proof of the fact that the rational approximations to π^2 obtained by Apéry in his famous proof of the irrationality of certain values of the Riemann zeta function are identical to those obtained by Beukers in one of his alternative proofs of Apéry's result.

1 Introduction

Apéry's famous proof ([3], [10]) of the irrationality of $\zeta(3)$ makes ingenious use of certain identities and specific recurrences. The proof gives explicit rational approximations to $\zeta(3)$ which converge fast enough to prove its irrationality. Apéry's proof also produces analogous rational approximations to $\zeta(2) = \pi^2/6$, whose irrationality (in fact, transcendence) is well-known. Specifically, in the case of $\zeta(2)$, Apéry considers the recurrence relation

$$(n+1)^2 u_{n+1} - (11n^2 + 11n + 3)u_n - n^2 u_{n-1} = 0.$$

Let $\{a_n\}$ be the sequence solving the above recurrence with initial values $a_0 = 0$ and $a_1 = 5$. Also let $\{b_n\}$ be the sequence solving the recurrence with initial values $b_0 = 1$ and $b_1 = 3$. Then the sequence $\{a_n/b_n\}$ converges to $\zeta(2)$. Explicit formulas for a_n and b_n are given in [3] and [10]. We also note that $\{b_n\}$ is the sequence A005258 in the *On-Line Encyclopedia of Integer Sequences*. Apéry also gives analogous arguments for $\zeta(3)$.

Shortly after Apéry announced his proof, Beukers ([6]) produced an elegant and entirely different proof of the irrationality of $\zeta(2)$ and $\zeta(3)$. In the case of $\zeta(2)$, Beukers considers the double integrals I_n defined by

$$I_n = \int_0^1 \int_0^1 \frac{(1-y)^n P_n(x)}{1-xy} dxdy,$$

where $n \in \mathbb{N}$ and $P_n(x)$ is the Legendre-type polynomial given by

$$P_n(x) = \frac{1}{n!} \frac{d^n}{dx^n} (x^n(1-x)^n).$$

He then shows that

$$I_n = \beta_n \zeta(2) - \alpha_n,$$

where α_n, $\beta_n \in \mathbb{Q}$, for all n. An estimation of the latter linear form shows that it tends to 0 (as n approaches infinity) fast enough to yield the irrationality of $\zeta(2)$. Beukers also gives an analogous argument for the case of $\zeta(3)$ by using a triple integral instead. It is worth noting that an even more striking proof of Apéry's result was given by Beukers in [4] using modular forms.

It is rather remarkable (and apparently known, as explained below) that the rational approximations to $\zeta(2)$ and $\zeta(3)$ obtained by Beukers are identical to those obtained by Apéry. We list below some places we were able to find in the literature where proofs of this fact (or of related facts) are given:
For the 1-dimensional analogue of Beukers' method (i.e., appropriate single-variable integrals), a related fact was established by Alladi and Robinson in [1], using properties of values of Legendre polynomials. The method is also discussed independently by Beukers in [5]. For the 2-dimensional and 3-dimensional analogues of Beukers' method (i.e., Beukers' double and triple integrals), the fact is verified by Nesterenko in [9], by using rather advanced arguments involving, among other things, contour integrals of Barnes type and transformation properties of hypergeometric series (see also the recent preprint by Zudilin ([12]). The reader may also consult the article by Fischler ([7]) for a complete survey of the subject.

The purpose of this paper is to give a new, short and completely elementary proof of the fact mentioned above for $\zeta(2)$.

Theorem 1.1 *With notation as above, we have*

$$a_n = \alpha_n, \qquad\qquad b_n = \beta_n,$$

for all $n \in \mathbb{N}$.

As the referee of an earlier version of this paper pointed out, another elementary proof of the same fact can be given by combining clever manipulations with Zeilberger's powerful program Ekhad. Our proof is along different lines.

Our attempts to apply similar elementary arguments to the case of $\zeta(3)$ have invariably (and not surprisingly) led us to certain expressions involving special values of generalized hypergeometric series, which are notoriously difficult to compute. We do not address the case of $\zeta(3)$ any further in this paper.

The above theorem easily implies a recurrence relation between special values of certain generalized hypergeometric series (see the corollary below). It is not unlikely that this may also follow from the contiguous relations of Kummer (which were generalized by Wilson in [11]); we have not attempted to verify this. The reader may also consult the books by Andrews, Askey and Roy ([2]) or by Magnus, Oberhettinger and Soni ([8]) for a wealth of information regarding special functions of hypergeometric type.

If a is a positive integer, let ${}_3F_2(a,a,a;2a,2a;1)$ denote the value of the generalized hypergeometric series

$$ {}_3F_2(a,a,a;2a,2a;x) = 1 + \sum_{k=1}^{\infty} \frac{(a\ldots(a+k-1))^3}{((2a)\ldots(2a+k-1))^2}\frac{x^k}{k!} $$

at $x=1$. Then

Corollary 1.1 *For every integer a such that $a \geq 2$, we have*

$$ \begin{aligned} &{}_3F_2(a+1,a+1,a+1;2a+2,2a+2;1) \\ &= -\frac{176a^4-84a^2+4a+12}{a^4}\;{}_3F_2(a,a,a;2a,2a;1)+ \\ &+\frac{256a^4-128a^2+16}{a^4}\;{}_3F_2(a-1,a-1,a-1;2a-2,2a-2;1). \end{aligned} $$

2 The Proof

We first point out that some of the integrals below are improper; their use can be justified by replacing $\int_0^1$ by $\int_\epsilon^{1-\epsilon}$ and letting ϵ tend to 0. Also, in what follows, our manipulations of the series involved are valid because of their absolute and/or uniform convergence.

First note that

$$ I_0 = \int_0^1\int_0^1 \frac{1}{1-xy}dxdy = \sum_{k=0}^{\infty}\int_0^1\int_0^1 x^ky^kdxdy = \zeta(2), $$

$$ I_1 = \int_0^1\int_0^1 \frac{(1-y)(1-2x)}{1-xy}dxdy $$

$$= \sum_{k=0}^{\infty} \int_0^1 \int_0^1 (x^k y^k - x^k y^{k+1} - 2x^{k+1} y^k + 2x^{k+1} y^{k+1}) dxdy = -5 + 3\zeta(2),$$

so the theorem is true for $n = 0$ and $n = 1$. As Beukers shows in [6], we have

$$I_n = (-1)^n \int_0^1 \int_0^1 \frac{x^n y^n (1-x)^n (1-y)^n}{(1-xy)^{n+1}} dxdy,$$

for all n. Now, n-fold differentiation of the geometric series identity

$$\frac{1}{1-u} = \sum_{k=0}^{\infty} u^k$$

gives the following formal identity for $n \in \mathbb{N}$ and an indeterminate u:

$$\frac{1}{(1-u)^{n+1}} = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!\ n!} u^k.$$

Therefore,

$$I_n = (-1)^n \sum_{k=0}^{\infty} \frac{(n+k)!}{k!\ n!} \int_0^1 \int_0^1 x^{n+k} y^{n+k} (1-x)^n (1-y)^n dxdy =$$

$$(-1)^n \sum_{k=0}^{\infty} \frac{(n+k)!}{k!\ n!} (B(n+k+1, n+1))^2,$$

where $B(\cdot, \cdot)$ denotes Euler's beta function. Therefore,

$$I_n = (-1)^n \sum_{k=0}^{\infty} \frac{(n+k)!^3\ n!}{(2n+k+1)!^2\ k!}.$$

Since $\zeta(2)$ is irrational and $\{a_n\}$, $\{b_n\}$ satisfy the same recurrence relation (with different initial conditions), it follows that, in order to prove the theorem, it suffices to show that the sequence $\{I_n\}$ satisfies Apéry's recurrence relation, i.e., we need to show that

$$(n+1)^2 I_{n+1} - (11n^2 + 11n + 3) I_n - n^2 I_{n-1} = 0,$$

for all $n \geq 1$. Fix such an n. It suffices to show that

$$\sum_{k=0}^{\infty} ((n+1)^2 \frac{(n+k+1)!^3\ (n+1)!}{(2n+k+3)!^2\ k!} + (11n^2 + 11n + 3) \frac{(n+k)!^3\ n!}{(2n+k+1)!^2\ k!}$$

$$-n \frac{(n+k-1)!^3\ n!}{(2n+k-1)!^2\ k!}) = 0.$$

Let $S_{k,n}$ denote the expression inside the infinite sum on the left-hand side of the above equality. A tedious calculation shows that

$$S_{k,n} = \frac{(n+k-1)!^3\ n!}{(2n+k+3)!^2\ k!} (-nk^8 + (3 - n - 5n^2) k^7 + (10n^3 + 66n^2 + 88n + 31) k^6 +$$

$$+(119n^4+563n^3+881n^2+548n+114)k^5+$$
$$+(314n^5+1749n^4+3479n^3+3128n^2+1268n+183)k^4+$$
$$+(340n^6+2412n^5+6121n^4+7310n^3+4330n^2+1179n+109)k^3+$$
$$+(71n^7+1119n^6+4172n^5+6737n^4+5364n^3+2043n^2+291n)k^2+$$
$$+(-138n^8-503n^7-349n^6+782n^5+1468n^4+885n^3+183n^2)k$$
$$-(79n^9+474n^8+1141n^7+1400n^6+913n^5+294n^4+35n^3)).$$

Let $T_{m,n}$ denote the m-th partial sum of the latter infinite series, i.e.,

$$T_{m,n}=\sum_{k=0}^{m} S_{k,n}.$$

We claim that $T_{m,n}$ is given by the following closed formula:

$$T_{m,n}=\frac{(n+m)!^3\ n!}{(2n+m+3)!^2\ m!}(m^6+(4n+9)m^5-(13n^2-7n-26)m^4$$
$$-(102n^3+228n^2+112n-15)m^3-(225n^4+822n^3+1025n^2+479n+52)m^2$$
$$-(217n^5+1057n^4+1957n^3+1691n^2+658n+84)m$$
$$-(79n^6+474n^5+1141n^4+1400n^3+913n^2+294n+35)).$$

Although this formula is difficult to guess, its proof is a tedious but straightforward induction argument (on m), using the explicit formula for $S_{k,n}$ given above. The authors suspected the existence of such a closed formula for $T_{m,n}$ after explicitly computing it for the first few values of m. It should be pointed out that there is an algorithm (due to Gosper, and lying at the heart of the Ekhad program) that, given a hypergeometric summand S_k, determines whether or not $\sum_{k=0}^{m} S_k$ has a hypergeometric closed form. Also, one may try to use the Wilf-Zeilberger algorithm of creative telescoping to compute $T_{m,n}$; we have not attempted to use any of these algorithms.

It now remains to show that $T_{m,n}$ tends to 0 as m approaches infinity. By Stirling's formula, we see that

$$\lim_{m\to\infty} T_{m,n}=\lim_{m\to\infty}\frac{(\frac{n+m}{e})^{3n+3m}}{(\frac{m}{e})^m(\frac{2n+m+3}{e})^{4n+2m+6}}\frac{\sqrt{8\pi^3(n+m)^3}}{2\pi(2n+m+3)\sqrt{2\pi m}}m^6$$
$$=\lim_{m\to\infty} m^{-n}e^{n+6}=0,$$

and this completes the proof of the theorem.

Now note that the formula for I_n given in our proof of the theorem may be restated as follows:

$$I_n=(-1)^n\frac{n!^4}{(2n+1)!^2}\ {}_3F_2(n+1,n+1,n+1;2n+2,2n+2;1).$$

Since I_n satisfies Apéry's recurrence, the corollary follows from an easy calculation.

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