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On Generalized Fibonacci Polynomials and Bernoulli Numbers[1]

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Abstract

In this paper we use elementary methods to study the relationship between the generalized Fibonacci polynomials and the famous Bernoulli numbers, and give several interesting identities involving them.

1 Introduction and results

As usual, the famous Fibonacci polynomials $F(x) = \{F_n(x)\}$ are defined by the second-order linear recurrence

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x) \tag{1}$$

for $n \geq 0$ and $F_0(x) = 0$, $F_1(x) = 1$. These polynomials are of great importance in the study of many subjects such as algebra, geometry, and number theory itself. Obviously, they

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have a deep relationship with the famous Fibonacci numbers $(F_n)_{n\geq 0}$. That is, $F_n(1) = F_n$. Many scholars have studied numerous properties of the Fibonacci numbers. For example, R. L. Duncan [1] and L. Kuipers [2] proved that $(\log F_n)$ is uniformly distributed mod 1. N. Robbins [3] studied the Fibonacci numbers of the forms $px^2 \pm 1$, $px^3 \pm 1$, where p is a prime. Wenpeng Zhang [4] and Fengzhen Zhao [5] obtained some identities involving the Fibonacci numbers. Moreover, Yuan Yi and Wenpeng Zhang [6] studied the calculation on the summation involving the Fibonacci polynomials, and obtained the following

Proposition 1 *Let $F(x) = \{F_n(x)\}$ be defined by (1). Then for all positive integers k and n, we have the formula*

$$\sum_{a_1+a_2+\cdots+a_k=n} F_{a_1+1}(x)F_{a_2+1}(x)\cdots F_{a_k+1}(x) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+k-1-m}{m}\binom{n+k-1-2m}{k-1} x^{n-2m}$$

where the summation is over all k-dimension nonnegative integer coordinates $(a_1, a_2, \ldots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$, and $\lfloor z \rfloor$ denotes the greatest integer not exceeding z, and $\binom{m}{n} = \frac{m!}{n!(m-n)!}$.

On the other hand, the famous Bernoulli numbers are defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad \mid x \mid < 2\pi. \tag{2}$$

A recursion formula involving the Bernoulli numbers is

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k$$

for $n \geq 2$ and $B_0 = 1, B_1 = -\frac{1}{2}$, which successively yields the values

$$B_2 = \frac{1}{6}, \quad B_{2k+1} = 0, (k = 1, 2, \cdots), \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42},$$

$$B_8 = -\frac{1}{2}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}, \quad B_{14} = \frac{7}{6}, \quad \cdots$$

Moreover, the Bernoulli numbers B_{2k} alternate in sign, and are related to $\zeta(2k)$ as follows:

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2(2k)!}.$$

Other important results involving the Bernoulli numbers can be found in references [7, 8, 9].

Now we consider the polynomial sequence $H(x) = \{H_n(x)\}$ defined by $H_0(x) = 0$, $H_1(x) = 1$, and

$$H_{n+2}(x) = P(x)H_{n+1}(x) + Q(x)H_n(x), \tag{3}$$

where $P(x)$ and $Q(x)$ are polynomials with $\Delta(x) = P^2(x) + 4Q(x) > 0$. It is easy to see that (3) is a generalization of (1).

It is well known that the Fibonacci numbers and Lucas numbers are closely related to the Chebyshev polynomials. Yuankui Ma and the first author [10] studied the relationships between the Chebyshev polynomials of the first kind and the famous Euler numbers, and obtained an interesting identity involving them. But no similar relationship between the generalized Fibonacci polynomials and the Bernoulli numbers was previously known. In this paper, we use elementary methods to study the relationship between the generalized Fibonacci polynomials and the famous Bernoulli numbers, and give several interesting identities involving them. That is, we shall prove the following

Theorem 1 *For all positive integers k and n with $k \leq n$, we have the formula*

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{H_{a_1}(x)}{a_1!} \cdots \frac{H_{a_k}(x)}{a_k!} \frac{B_{b_1}}{b_1!} \cdots \frac{B_{b_k}}{b_k!} \left(\sqrt{\Delta(x)}\right)^{b_1+\cdots+b_k} = \frac{(k\beta)^{n-k}}{(n-k)!},$$

where $\beta = \frac{P(x)-\sqrt{\Delta(x)}}{2}$.

If we take $P(x) = x$ and $Q(x) = 1$ in Theorem 1, then we have

Corollary 1 *For all positive integers k and n with $k \leq n$, we have*

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{F_{a_1}(x)}{a_1!} \cdots \frac{F_{a_k}(x)}{a_k!} \frac{B_{b_1}}{b_1!} \cdots \frac{B_{b_k}}{b_k!} \left(\sqrt{x^2+4}\right)^{b_1+\cdots+b_k} = \frac{(k\beta(x))^{n-k}}{(n-k)!},$$

where $\beta(x) = \frac{x-\sqrt{x^2+4}}{2}$.

If $P(x)$ and nonzero $Q(x)$ in Theorem 1 are integers with $P^2 + 4Q > 0$, then we immediately obtain the following

Corollary 2 *For all positive integers k and n with $k \leq n$, we have the identity*

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{H_{a_1}}{a_1!} \cdots \frac{H_{a_k}}{a_k!} \frac{B_{b_1}}{b_1!} \cdots \frac{B_{b_k}}{b_k!} \left(\sqrt{P^2+4Q}\right)^{b_1+\cdots+b_k} = \frac{(k\beta')^{n-k}}{(n-k)!},$$

where $\beta' = \frac{P-\sqrt{P^2+4Q}}{2}$.

Taking $x = 1$ in Corollary 1, or $P = Q = 1$ in Corollary 2, we immediately deduce the following

Corollary 3 *For all positive integers k and n with $k \leq n$, we have the identity*

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{F_{a_1}}{a_1!} \cdots \frac{F_{a_k}}{a_k!} \frac{B_{b_1}}{b_1!} \cdots \frac{B_{b_k}}{b_k!} \left(\sqrt{5}\right)^{b_1+\cdots+b_k} = \frac{(k\beta(1))^{n-k}}{(n-k)!},$$

where $\beta(1) = \frac{1-\sqrt{5}}{2}$.

In particular, taking $k = 1, 2, 3$ in Corollary 3, we easily get

Corollary 4 *For all positive integers n, we have*

$$\sum_{a+b=n} \frac{F_a}{a!} \frac{B_b}{b!} \left(\sqrt{5}\right)^b = \frac{(\beta(1))^{n-1}}{(n-1)!}.$$

Corollary 5 *For all positive integers $n \geq 2$, we have*

$$\sum_{a+b+c+d=n} \frac{F_a}{a!} \frac{F_b}{b!} \frac{B_c}{c!} \frac{B_d}{d!} \left(\sqrt{5}\right)^{c+d} = \frac{(2\beta(1))^{n-2}}{(n-2)!}.$$

Corollary 6 *For all positive integers $n \geq 3$, we have*

$$\sum_{a+b+c+d+e+f=n} \frac{F_a}{a!} \frac{F_b}{b!} \frac{F_c}{c!} \frac{B_d}{d!} \frac{B_e}{e!} \frac{B_f}{f!} \left(\sqrt{5}\right)^{d+e+f} = \frac{(3\beta(1))^{n-3}}{(n-3)!}.$$

2 Proof of Theorem

In this section, we shall complete the proof of Theorem. First we let $\alpha = \frac{P(x)+\sqrt{\Delta(x)}}{2}$ and $\beta = \frac{P(x)+\sqrt{\Delta(x)}}{2}$ denote the roots of characteristic polynomial $\lambda^2 - P(x)\lambda - Q(x)$ of the generalized Fibonacci polynomial sequence $H(x)$, then the terms of the sequence $H(x)$ can be expressed as (see [11, 12])

$$H_n(x) = \frac{1}{\sqrt{\Delta(x)}} \left(\left(\frac{P(x) + \sqrt{\Delta(x)}}{2} \right)^n - \left(\frac{P(x) - \sqrt{\Delta(x)}}{2} \right)^n \right).$$

Then we easily deduce that the generating function of $H(t, x)$ is

$$H(t,x) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{\alpha^n - \beta^n}{(\alpha - \beta) n!} t^n. \tag{4}$$

That is,

$$H(t,x) = \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} = \frac{e^{\beta t} \left(e^{t\sqrt{\Delta(x)}} - 1 \right)}{\sqrt{\Delta(x)}}.$$

Therefore, we have

$$e^{\beta t} = \frac{H(t,x)}{t} \cdot \frac{t\sqrt{\Delta(x)}}{e^{t\sqrt{\Delta(x)}} - 1}.$$

Then from (2) and (4), we have

$$e^{\beta t} = \left(\sum_{m=0}^{\infty} \frac{H_m(x)}{m!} t^{m-1} \right) \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} \left(t\sqrt{\Delta(x)} \right)^n \right). \tag{5}$$

Note that for two absolutely convergent power series $\sum_{n=0}^{\infty} a_n t^n$ and $\sum_{n=0}^{\infty} b_n t^n$, we have

$$\left(\sum_{n=0}^{\infty} a_n t^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n t^n\right) = \sum_{n=0}^{\infty} \left(\sum_{u+v=n} a_u b_v\right) t^n,$$

so k times on the both sides of formula (5), we have

$$LHS = \left(e^{\beta t}\right)^k = e^{k\beta t} = \sum_{n=0}^{\infty} \frac{(k\beta)^n}{n!} t^n,$$

$$RHS = \sum_{n=0}^{\infty} \sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{H_{a_1}(x)}{a_1!} \cdots \frac{H_{a_k}(x)}{a_k!} \frac{B_{b_1}}{b_1!} \cdots \frac{B_{b_k}}{b_k!} \left(\sqrt{\Delta(x)}\right)^{b_1+\cdots+b_k} t^{n-k}.$$

Comparing the coefficients of t^{n-k} on the above, we immediately obtain the following identity

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{H_{a_1}(x)}{a_1!} \cdots \frac{H_{a_k}(x)}{a_k!} \frac{B_{b_1}}{b_1!} \cdots \frac{B_{b_k}}{b_k!} \left(\sqrt{\Delta(x)}\right)^{b_1+\cdots+b_k} = \frac{(k\beta)^{n-k}}{(n-k)!}.$$

This completes the proof of Theorem 1.

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References

[1] R. L. Duncan, Application of uniform distribution to the Fibonacci numbers, *Fibonacci Quart.* **5** (1967), 137–140.

[2] L. Kuipers, Remark on a paper by R. L. Duncan concerning the uniform distrubution mod 1 of the sequence of the logarithms of the Fibonacci numbers, *Fibonacci Quart.* **7** (1969), 465–466.

[3] N. Robbins, Fibonacci numbers of the forms $px^2 \pm 1, px^3 \pm 1$, where p is prime, in *Applications of Fibonacci Numbers*, Kluwer Academic, 1986, pp. 77–88.

[4] Wenpeng Zhang, Some identities involving the Fibonacci numbers, *Fibonacci Quart.* **35** (1997), 225–229.

[5] Fengzhen Zhao and Tianming Wang, Generalizations of some identities involving the Fibonacci polynomials, *Fibonacci Quart.* **39** (2001), 165–167.

[6] Yuan Yi and Wenpeng Zhang, Some identities involving the Fibonacci polynomials, *Fibonacci Quart.* **40** (2002), 314–318.

[7] Gi-Sang Cheon, A note on the Bernoulli and Euler polynomials, *Appl. Math. Lett.* **16** (2003), 365–368.

[8] H. M. Srivastava and Á. Pintér, Remarks on some relationships between the Bernoulli and Euler polynomials, *Appl. Math. Lett.* **17** (2004), 375–380.

[9] M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976.

[10] Yuankui Ma and Tianping Zhang, An identity involving the first-kind Chebyshev polynomials and the Euler numbers, *J. Ningxia University*, to appear.

[11] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 4th edition, Oxford Universty Press, 1962.

[12] Peter Borwein and Tamás Erdélyi, *Polynomials and Polynomial Inequalities*, Springer-Verlag, 1995.

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