Journal of Integer Sequences, Vol. 9 (2006),
Article 06.3.5

Converting Between Generalized Bell, Lah, Stirling, and Tanh Numbers

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Abstract

We define a Riordan triangle for generalized Bell numbers and we establish general identities connecting Lah, Stirling, Tanh and the generalized Bell numbers. Several familiar inverse relations converting between special sequences are shown to be particular cases of the general identities.

1 Introduction

We consider, in the context of the (exponential) Riordan group $(\mathcal{R};\ \star)$, inverse relations

$$c(n,m) = \sum_i a(n,i)b(i,m) \Leftrightarrow b(n,m) = \sum_i A(n,i)c(i,m), \quad n, m \geq 0 \tag{1.1}$$

converting between arrays of numbers. The concept of Riordan group was introduced by Shapiro et al. [15] as a generalization of a study of Rogers [12] on *renewal arrays* and used in several applications, including [16, 17, 14, 19, 1, 10]. The last reference, pointed out to us by the referee, contains an application of the concept of a Riordan array to the Akiyama-Tanigawa transformation analogous to the formalism developed in the present paper. An element $R \in (\mathcal{R};\ \star)$ is denoted by $R = (q(u),\ R(u))$ where $q(u)$ is an exponential generating function such that $q(0) = 1$ and $R(u)$ is an exponential generating function such that $R(0) = 0$ and $R'(0) = 1$. The numbers $R(n, m)$, defined recursively by

$\frac{1}{m!}q(u)R^m(u) = \sum_{n\geq 0} R(n,m)u^n/n!,\ m = 0, 1, 2, \cdots$, are the entries of an *invertible* infinite lower triangular matrix $(R(n,m)_{n,m\geq 0})$, called a *Riordan array*. The Riordan group product $\star$ is defined by

$$c = a \star b = (g(u), a(u)) \star (h(u), b(u)) = (g(u)h(a(u)),\ b(a(u))). \tag{1.2}$$

This notation and the matrix notation $c(n,m) = \sum_i a(n,i)b(i,m)$ are isomorphic. The unit element is $I = (1,\ u)$ and the inverse r of R is $r = (g(u),\ R(u))^{-1} = (1/g(\overline{R}(u)),\ \overline{R}(u))$ where $\overline{R}(u)$ is the compositional inverse of $R(u) : R(\overline{R}(u)) = \overline{R}(R(u)) = u$.

In Section 2, we consider pairs of Riordan arrays $\{a, A\}$, $\{b, B\}$ and $\{c, C\}$ corresponding to numbers that are inverse of each other and from an identity $c = a \star b$, we derive new identities $C = B \star A$, $A = b \star C$, etc.. If numbers $R(n,m)$ can be extended to all integers n and m, then we will use *dual* elements $\widetilde{R}(n,m) = R(-m,-n)$ and dual identities defined by

$$\begin{aligned}\widetilde{c}(n,m) &= \sum_i a(-m,i)b(i,-n) = \sum_i a(-m,-i)b(-i,-n)\\ &= \sum_i \widetilde{b}(n,i)\widetilde{a}(i,m),\end{aligned}$$

that is, $\widetilde{c} = \widetilde{b} \star \widetilde{a}$. The change of i into $-i$ is legal because, conventionally, all numbers are null when $n < 0 \leq m$ and the summation ranges are unrestricted. We will also consider inverse relations that are duals ("$\sim$") of each other:

$$c = a \star b \Leftrightarrow b = A \star c \quad \sim \quad \widetilde{c} = \widetilde{b} \star \widetilde{a} \Leftrightarrow \widetilde{b} = \widetilde{c} \star \widetilde{A}.$$

In Section 3, we will discuss inverse relations converting between *sequences* that are special cases $m = 1,\ 2$ of inverse relations (1.1) converting between *arrays*.

2 Converting between Stirling, Lah, Tanh and Bell numbers

The Riordan group elements considered in this paper belong to the so-called *associated* subgroup of $(\mathcal{R};\ \star)$, i.e., they are of the form $R = (1,\ R(u))$. In this case the expression (1.2) of the product of two Riordan arrays $c = a \star b$ reduces to

$$\begin{aligned}c_m(u) &= \sum_{n\geq 0} c(n,m)\frac{u^n}{n!} = \sum_{n\geq 0}\sum_i a(n,i)b(i,m)\frac{u^n}{n!} = \sum_i b(i,m)\frac{a^i(u)}{i!}\\ &= b_m(a(u)).\end{aligned}$$

The referee suggested the use of the *negation* rule "-" : for any $R = (1,\ R(u))$, $-R = (1,\ -R(-u))$ and $-(\Psi \star \Omega) = -\Psi \star -\Omega$; the rule replaces $R(n,m)$ in the formulas by $(-1)^{n-m}R$ and this allows a more elegant formulation of our results. We shall also use a *scaling* rule "μ" : for any $R = (1,\ R(u))$, $\mu R = (1,\ \mu R(u/\mu))$ and $\mu(\Psi \star \Omega) = \mu\Psi \star \mu\Omega$ for all real or complex numbers μ; the rule replaces $R(n,m)$ in the formulas by $R(n,m)/\mu^{n-m}$. Negation is a scaling with $\mu = -1$ and $\lim_{\mu\to\pm\infty} \mu(1,\ R(u)) = (1,\ u) = I$.

Stirling numbers. The Stirling numbers of the first and second kind are denoted $s(n,m)$ and $S(n,m)$, respectively, [4]. The *unsigned* numbers are $\left[{n \atop m}\right] = (-1)^{n-m}s(n,m)$ and $\left\{{n \atop m}\right\} = S(n,m)$. Formal properties of the Stirling numbers [5] include

$$s = (1,\ s(u) = \ln(1+u)), \quad S = (1,\ S(u) = e^u - 1),$$
$$s \star S = S \star s = I, \quad \widetilde{S} = -s \quad \widetilde{s} = -S.$$

Lah numbers We define Lah numbers of the first and second kind $\lambda(n,m)$ and $\Lambda(n,m)$ as follows:

$$\lambda(n,m) = \frac{(-1)^m}{2^{n-m}} L(n,m), \quad \Lambda(n,m) = \frac{(-1)^n}{2^{n-m}} L(n,m),$$

where instead of the familiar explicit expression

$$L(n,m) = (-1)^n \frac{n!}{m!}\binom{n-1}{m-1}, \quad n \ge m > 0,$$

we will use

$$L(n,m) = (-1)^n (n-m)!\binom{n}{n-m}\binom{n-1}{n-m}, \quad \text{all integers } n, m,$$

that has the advantage of extending $L(n,m)$ to all integers n and m, and providing a definition for Lah dual numbers:

$$\widetilde{L}(n,m) = L(-m,-n) = (-1)^m (n-m)!\binom{-m}{n-m}\binom{-m-1}{n-m} = \tag{2.1}$$
$$(-1)^m (n-m)!\binom{n-1}{n-m}\binom{n}{n-m} = (-1)^{n-m} L(n,m).$$

From the exponential generating function of $L(n,m)$ (see [11, Problem 16 (b), p. 44])

$$L_m(u) = \sum_n L(n,m)\frac{u^n}{n!} = \frac{1}{m!}\left(\frac{-u}{1+u}\right)^m,$$

we derive the exponential generating functions

$$\lambda_m(u) = \frac{1}{m!}\left(\frac{u}{1+u/2}\right)^m, \quad \Lambda_m(u) = \frac{1}{m!}\left(\frac{u}{1-u/2}\right)^m.$$

These exponential generating functions, the fact that $\lambda(u) = -\Lambda(-u)$ and (2.1) yield the formal properties:

$$\lambda = \left(1,\ \lambda(u) = \frac{u}{1+u/2}\right), \quad \Lambda = \left(1,\ \Lambda(u) = \frac{u}{1+u/2}\right),$$
$$\Lambda(u) = \overline{\lambda}(u) \quad \lambda \star \Lambda = \Lambda \star \lambda = I, \quad \widetilde{\Lambda} = \Lambda = -\lambda \quad \widetilde{\lambda} = \lambda = -\Lambda.$$

Tanh numbers. Let $t = (1,\ t(u) = \arctan(u))$ and $T = (1,\ T(u) = \tan(u))$ be the Riordan arrays of the arctan $t(n,m)$ and tangent $T(n,m)$ numbers, respectively. The recurrence relations (see [4, pp. 259–260])

$$\begin{aligned} &t(n+1,m) = t(n,m-1) - n(n-1)t(n-1,m),\\ &T(n+1,m) = T(n,m-1) + m(m+1)T(n,m+1),\\ &t(0,m) = T(0,m) = [m=0], \quad t(n,0) = T(n=0) = [n=0], \end{aligned}$$

imply $\widetilde{T}(n,m) = T(-m,-n) = t(n,m)$ and $\widetilde{t}(n,m) = t(-m,-n) = T(n,m)$.

Our first and second kind Tanh numbers $\theta(n,m)$ and $\Theta(n,m)$ are

$$\theta(n,m) = \frac{(-1)^{(n-m)/2}}{2^{n-m}} t(n,m), \quad \Theta(n,m) = \frac{(-1)^{(n-m)/2}}{2^{n-m}} T(n,m),$$

that is, $\theta = \mu t$ and $\Theta = \mu T$ with $\mu = 2/\imath,\ \imath^2 = -1$. Summarizing:

$$\begin{aligned} &\theta = \left(1,\ \theta(u) = \ln\frac{1+u/2}{1-u/2}\right), \quad \Theta = \left(1,\ \Theta(u) = 2\frac{e^u-1}{e^u+1}\right),\\ &\Theta(u) = \overline{\theta}(u) \quad \theta\star\Theta = \Theta\star\theta = I, \quad \widetilde{\Theta} = \theta = -\theta \quad \widetilde{\theta} = \Theta = -\Theta, \end{aligned}$$

where $\theta = -\theta$ and $\Theta = -\Theta$ because $\theta(u)$ and $\Theta(u)$ are odd functions of u.

Theorem 2.1. *Numbers in each pair $\{s, S\}$, $\{\lambda, \Lambda\}$ and $\{\theta, \Theta\}$ convert between numbers in the other two pairs.*

Proof. With $S(u) = e^u - 1$, $\lambda(u) = u/(1+u/2)$ and $\Theta(u) = 2(e^u-1)/(e^u+1)$,

$$\begin{aligned} &\lambda(S(u)) = \lambda(e^u - 1) = \frac{e^u-1}{1+(e^u-1)/2} = \Theta(u), \quad \text{therefore}\\ &\Theta = S\star\lambda \quad \text{and in matrix notation} \quad \Theta(n,m) = \sum_i S(n,i)\lambda(i,m). \end{aligned}$$

The identities that can be derived from $\Theta = S\star\lambda$ are

$\theta = \Lambda\star s$	Riordan group inversion
$S = \Theta\star\Lambda$	right multiplication by Λ
$s = \lambda\star\theta$	left multiplication by s/right multiplication by θ
$\Lambda = \theta\star S$	left multiplication by θ/right multiplication by Λ
$\lambda = s\star\Theta$	left multiplication by s

and the inverse relations

$\Theta = S\star\lambda \Leftrightarrow \lambda = s\star\Theta$	$\sim$	$\theta = \Lambda\star s \Leftrightarrow \Lambda = \theta\star S$
$\theta = \Lambda\star s \Leftrightarrow s = \lambda\star\theta$	$\sim$	$\Theta = S\star\lambda \Leftrightarrow S = \Theta\star\Lambda$
$S = \Theta\star\Lambda \Leftrightarrow \Lambda = \theta\star S$	$\sim$	$s = \lambda\star\theta \Leftrightarrow \lambda = s\star\Theta.$

The above identities prove the theorem. In deriving dual inverse relations, we tacitly used duality and negation rules. □

Corollary 2.2. *Tanh numbers are represented by linear combinations of Stirling numbers, and conversely:*

$$\theta(n,m) = n!\sum_{i=m}^{n}\binom{n-1}{i-1}\frac{1}{2^{n-i}i!}s(i,m),$$
$$\Theta(n,m) = \frac{1}{m!}\sum_{i=m}^{n}\binom{i-1}{m-1}\frac{(-1)^{i-m}i!}{2^{i-m}}S(n,i);$$
$$s(n,m) = n!\sum_{i=m}^{n}\binom{n-1}{i-1}\frac{(-1^{n-i}}{2^{n-i}}\theta(i,m),$$
$$S(n,m) = \frac{1}{m!}\sum_{i=m}^{n}\binom{i-1}{m-1}\frac{i!}{2^{i-m}}\Theta(n,i).$$

Proof. Write explicitly λ and Λ in $\theta = \Lambda \star s$, $\Theta = S \star \lambda$, $s = \lambda \star \theta$ and $S = \Theta \star \Lambda$. □

Bell numbers. Bell numbers A_n, $n \geq 1$, can be defined by $\mathcal{B}(u) = \exp(e^u - 1) - 1 = \sum_{n\geq 1} A_n u^n/n!$, [2]. We define generalized Bell numbers of the second kind $\mathcal{B}(n,m)$ by

$$\mathcal{B}_m(u) = \frac{1}{m!}(\exp(e^u - 1) - 1)^m = \sum_{n\geq 1}\mathcal{B}(n,m)\frac{u^n}{n!}.$$

With $S(u) = e^u - 1$ and $s(u) = \overline{S}(u) = \ln(1+u)$, we have

$$\mathcal{B}(u) = S(S(u)) \quad \text{and} \quad \beta(u) = \overline{\mathcal{B}}(u) = s(s(u)) = \ln(1 + \ln(1+u)).$$

Thus, generalized Bell numbers of the first kind $\beta(n,m)$ can be defined by

$$\beta_m(u) = \frac{1}{m!}(\ln(1 + \ln(1+u)))^m = \sum_{n\geq 1}\beta(n,m)\frac{u^n}{n!}.$$

Theorem 2.3. *Bell numbers $\{\beta, \mathcal{B}\}$ can be represented as squares (group multiplication) of Stirling numbers: $\beta = s \star s$ and $\mathcal{B} = S \star S$.*

Proof. The Riordan array products $\beta = s \star s$ and $\mathcal{B} = S \star S$ hold in view of the functional relations $\beta(u) = s(s(u))$ and $\mathcal{B}(u) = S(S(u))$. □

Summarizing:

$$\beta = (1,\ \beta(u) = \ln(1 + \ln(1+u))),$$
$$\mathcal{B} = (1,\ \mathcal{B}(u) = \exp(e^u - 1) - 1)$$
$$\beta \star \mathcal{B} = \mathcal{B} \star \beta = I, \quad \widetilde{\mathcal{B}} = -\beta, \quad \widetilde{\beta} = -\mathcal{B}$$

where the dual relations follow from Stirling number dual relations.

Theorem 2.4. *Bell numbers in $\{\beta, \mathcal{B}\}$ and Stirling numbers in $\{s, S\}$ convert between each other.*

Proof. The identities that can be derived from $\mathcal{B} = S \star S$ are

$\beta = s \star s$ Riordan group inversion

$S = s \star \mathcal{B}$ left multiplication by s

$s = S \star \beta$ left multiplication by s and right multiplication by β,

and the inverse relations

$\mathcal{B} = S \star S \Leftrightarrow S = s \star \mathcal{B} \quad \sim \quad \beta = s \star s \Leftrightarrow s = \beta \star S$

$\beta = s \star s \Leftrightarrow s = S \star \beta \quad \sim \quad \mathcal{B} = S \star S \Leftrightarrow S = \mathcal{B} \star s.$

The above identities prove the theorem. In deriving dual inverse relations, we tacitly used duality and negation rules. □

Corollary 2.5. *Bell numbers $\{\beta, \mathcal{B}\}$ and Stirling numbers $\{s, S\}$ satisfy the commutation relations $S \star \beta = \beta \star S = s$ and $s \star \mathcal{B} = \mathcal{B} \star s = S$.*

Proof. Left multiplication by S and right multiplication by S of $\beta = s \star s$ give $S \star \beta = s$ and $\beta \star S = s$, respectively. By inversion, we get $\mathcal{B} \star s = s \star \mathcal{B} = S$. □

The Riordan triangle $(\mathcal{B}(n,m)_{n,m\geq 0})$ is different from the Pascal type triangle proposed by Shallit [13]. This is obtained by flipping and reorienting a *rectangular* array given by Cohn et al. [3] for an efficient calculation of the Bell numbers. The reader may find it quite interesting to see the nice properties of Shallit's triangle.

3 Special cases

The following examples are mainly special cases $m = 1, 2$ of the above relations and they will be compared with identities commonly found in the literature.

1) When $m = 1$, $\Theta = S \star \lambda \Leftrightarrow \lambda = s \star \Theta$ gives

$$\begin{aligned} \frac{C_n}{2^n} &= (-1)^{(n+1)/2}\frac{T_n}{2^n} + [n=0] = 2(1-2^{n+1})\frac{B_{n+1}}{n+1} \\ &= \sum_{i=0}^{n} \left\{ {n \atop i} \right\} \frac{(-1)^i i!}{2^i} \\ \Leftrightarrow \quad & \frac{n!}{2^n} = \sum_{i=0}^{n} \left[{n \atop i} \right] (-1)^i \frac{C_i}{2^i} = \sum_{i=0}^{n} \left[{n \atop i} \right] 2(1-2^{i+1})\frac{B_{i+1}}{i+1}. \end{aligned}$$

The first identity appears in ([7, p. 585]) and ([5, Exercise 6.76]). It can also be found in Sprugnoli ([16, p. 288]) as a result of the theory of Riordan arrays and in Chen [18] by application of the Akiyama-Tanigawa algorithm [6].

2) Since $\theta(n,m) = \Theta(n,m) = 0$ when $n-m$ is odd, Corollary 2.2 yields

$$\sum_{i=m}^{n} \binom{n-1}{i-1} \frac{2^i}{i!} s(i,m) = \sum_{i=m}^{n} \binom{i-1}{m-1} \frac{(-1)^i i!}{2^i} S(n,i) = 0, \quad n-m \text{ odd}.$$

For $m=1$, the first identity leads to a trivial result, whereas the second gives the identity $\sum_{i=1}^{n} \left\{ {2n \atop i} \right\} (-1)^i i!/2^i = 0,\ n > 0$ found by Lengyel ([9, p. 7]) in his study of the series $\sum_{k=0}^{\infty} k^n x^k$.
For $m=2$ with $s(n,2) = (-1)^n (n-1)! H_{n-1}$ where H_n are harmonic numbers, we find

$$\sum_{i=2}^{2n+1} \binom{2n}{i-1} \frac{2^i}{i!} s(i,2) = \sum_{i=1}^{2n} \binom{2n+1}{i+1} (-2)^i H_i = 0,\ n > 0.$$

3) For $m=1$, $S = \Theta \star \Lambda \Leftrightarrow \Lambda = \theta \star S$ gives

$$1 + [n=0] = \sum_{i=0}^{n} \Theta(n,i) \frac{i!}{2^{i-1}} \ \Leftrightarrow\ \frac{n!}{2^{n-1}} = \sum_{i=0}^{n} \theta(n,i) \left(1 + [n=0]\right).$$

4) For $m=1$, two identities in Lah's original paper ([8, Eqs. (34) and (43)]): $L(n,m) = \sum_{i=0}^{n} s(n,i)(-1)^i S(i,m) \Leftrightarrow (-1)^n S(n,m) = \sum_{i=0}^{n} S(n,i) L(i,m)$, yield the simple inverse relation

$$\sum_{i=0}^{n} \left[{n \atop i} \right] = n! \Leftrightarrow \sum_{i=0}^{n} \left\{ {n \atop i} \right\} (-1)^i i! = (-1)^n$$

.

5) The inverse of

$$\sum_{i} \left\{ {n \atop i} \right\} \left[{i+1 \atop m} \right] \frac{(-1)^{i+1-m}}{i+1} = \binom{n+1}{m} \frac{B_{n+1-m}}{n+1}$$

that appears in ([5, Section 6.5, Eq. (6.99)]) is

$$\sum_{i} \left[{n \atop i} \right] (-1)^{i+1-m} \binom{i+1}{m} \frac{B_{i+1-m}}{i+1} = \frac{\left[{n+1 \atop m} \right]}{n+1}.$$

For $m=1$, we find the well-known inverse relations ([4, pp. 220–221])

$$\sum_{i} \left\{ {n \atop i} \right\} \frac{(-1)^i i!}{i+1} = B_n \Leftrightarrow \sum_{i} \left[{n \atop i} \right] (-1)^i B_i = \frac{n!}{n+1}.$$

6) From $\mathcal{B} = S \star S$ and $S = \Theta \star \lambda$ we obtain $\mathcal{B} = S \star \Theta \star \Lambda$, which in matrix notation is $\mathcal{B}(n,m) = \sum_i \sum_j S(n,i) \Theta(i,j) \Lambda(j,m)$. To hold down the size of this paper, we give only one example of triple factorization.

4 Main results and Conclusion

Group theory provides a rapid and systematic way to derive from a given identity connecting Riordan arrays, further identities and inverse relations. Applying this approach to $\Theta(n,m) = \sum_i S(n,i)\lambda(i,m)$, we found that special cases $m = 1,\ 2$ of the general identities are results obtained in the literature by (often more complex) procedures which seem at first to be unrelated to each other, whereas here they have a common origin and are obtained by a unifying procedure.

The identities $\beta = s \star s$ and $\mathcal{B} = S \star S$ connect Bell, Stirling, Lah, and Tanh numbers through, for instance, the identity $\mathcal{B} = S \star \Theta \star \Lambda$. Triple factorizations $d = a \star b \star c$ may lead in special cases $m = 1,\ 2$ to interesting identities connecting three Riordan arrays; we hope to investigate in this direction in a future publication.

5 Acknowledgement

The author wishes to thank the referee for valuable comments and suggestions which have greatly improved the clarity of this paper.

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2000 *Mathematics Subject Classification*: Primary 05A19; Secondary 05A15.
Keywords: Riordan group, conversions, Bell triangle.

Received February 2 2006; revised version received July 21 2006. Published in *Journal of Integer Sequences*, July 21 2006.

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