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Remark on the Complexified Iwasawa Decomposition

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Abstract. Let $G_{\mathbb{R}}$ be a real form of a complex semisimple Lie group $G_{\mathbb{C}}$. We identify the complexification $K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}} \subset G_{\mathbb{C}}$ of an Iwasawa decomposition $G_{\mathbb{R}} = K_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$ as $\{g \in G_{\mathbb{C}} \mid gB \in \Omega\}$ where $B \subset G_{\mathbb{C}}$ is a Borel subgroup of $G_{\mathbb{C}}$ that contains $A_{\mathbb{R}}N_{\mathbb{R}}$ and Ω is the open $K_{\mathbb{C}}$–orbit on $G_{\mathbb{C}}/B$. This is done in the context of subsets $K_{\mathbb{C}}R_{\mathbb{C}} \subset G_{\mathbb{C}}$, where $R_{\mathbb{C}}$ is a parabolic subgroup of $G_{\mathbb{C}}$ defined over $\mathbb{R}$, and the open $K_{\mathbb{C}}$–orbits on complex flag manifolds $G_{\mathbb{C}}/Q$.

1. Details

Let $G_{\mathbb{R}}$ be a real form of a complex semisimple Lie group $G_{\mathbb{C}}$. For simplicity we assume that $G_{\mathbb{C}}$ is connected and that the adjoint representation of $G_{\mathbb{R}}$ maps $G_{\mathbb{R}}$ into the group of inner automorphisms of the Lie algebra of $G_{\mathbb{C}}$. Choose an Iwasawa decomposition $G_{\mathbb{R}} = K_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$ and a corresponding minimal parabolic subgroup $P_{\mathbb{R}} = M_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$ of $G_{\mathbb{R}}$. Every parabolic subgroup of $G_{\mathbb{R}}$ is conjugate to one that contains $P_{\mathbb{R}}$. Let $R_{\mathbb{C}} \subset G_{\mathbb{C}}$ be the complexification of a (real) parabolic subgroup $R_{\mathbb{R}} \subset G_{\mathbb{R}}$. Passing to a $G_{\mathbb{R}}$–conjugate we may, and do, assume $P_{\mathbb{R}} \subset R_{\mathbb{R}}$. Let $K_{\mathbb{C}} \subset G_{\mathbb{C}}$ be the complexification of the maximal compactly embedded subgroup $K_{\mathbb{R}} \subset G_{\mathbb{R}}$. Then $G_{\mathbb{R}} = K_{\mathbb{R}}R_{\mathbb{R}}$. In particular the Lie algebras satisfy $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{r}_{\mathbb{R}}$, so their complexifications satisfy $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{r}_{\mathbb{C}}$, and thus $K_{\mathbb{C}}R_{\mathbb{C}}$ is open in $G_{\mathbb{C}}$. Since the projection $G_{\mathbb{C}} \to G_{\mathbb{C}}/R_{\mathbb{C}}$ is an open map, $\Omega_X := K_{\mathbb{C}}(1R_{\mathbb{C}})$ is an open $K_{\mathbb{C}}$–orbit in $X = G_{\mathbb{C}}/R_{\mathbb{C}}$. If $g \in G_{\mathbb{C}}$ then $g \in K_{\mathbb{C}}R_{\mathbb{C}}$ if and only if $gR_{\mathbb{C}} \in K_{\mathbb{C}}(1R_{\mathbb{C}}) \subset G_{\mathbb{C}}/R_{\mathbb{C}}$. Now

Lemma 1.1. *An element $g \in G_{\mathbb{C}}$ is contained in $K_{\mathbb{C}}R_{\mathbb{C}}$ if and only if $gR_{\mathbb{C}} \in \Omega_X$.*

Let Q be a parabolic subgroup of $G_{\mathbb{C}}$ and write Ω_Z for the open orbit $K_{\mathbb{C}}(1Q)$ on $Z = G_{\mathbb{C}}/Q$. The argument of Lemma 1.1 shows:

Lemma 1.2. *Suppose $K_{\mathbb{C}}Q = K_{\mathbb{C}}R_{\mathbb{C}}$. Then $g \in K_{\mathbb{C}}R_{\mathbb{C}}$ if and only if $gQ \in \Omega_Z$.*

There is a unique closed $G_{\mathbb{R}}$–orbit on X, and the other $G_{\mathbb{R}}$–orbits are finite in number and higher in dimension. See [4]. By $(G_{\mathbb{R}}, K_{\mathbb{C}})$–duality ([3]; see [2] for a geometric proof), there is a unique open $K_{\mathbb{C}}$–orbit on X (which thus must be Ω_X), and the other $K_{\mathbb{C}}$–orbits are finite in number and lower in dimension. Thus Ω_X is a dense open subset of X. Now Lemma 1.1 gives us

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Lemma 1.3. *$K_{\mathbb{C}}R_{\mathbb{C}}$ is a dense open subset of $G_{\mathbb{C}}$.*

Let $M_{\mathbb{C}}$, $A_{\mathbb{C}}$ and $N_{\mathbb{C}}$ denote the respective complexifications of $M_{\mathbb{R}}$, $A_{\mathbb{R}}$ and $N_{\mathbb{R}}$. Here $M_{\mathbb{C}}$ is the centralizer of $A_{\mathbb{C}}$ in $K_{\mathbb{C}}$. Let B denote a Borel subgroup of $G_{\mathbb{C}}$ that contains $A_{\mathbb{R}}N_{\mathbb{R}}$; in other words $B = B_M A_{\mathbb{C}} N_{\mathbb{C}}$ where B_M is a Borel subgroup of $M_{\mathbb{C}}$. The phenomenon $K_{\mathbb{C}}Q = K_{\mathbb{C}}R_{\mathbb{C}}$ of Lemma 1.2 occurs, for example, when $R_{\mathbb{R}}$ is the minimal parabolic subgroup $P_{\mathbb{R}}$ of $G_{\mathbb{R}}$ and $Q = B$. In that case $M_{\mathbb{R}} \subset K_{\mathbb{R}}$, so $B_M \subset M_{\mathbb{C}} \subset K_{\mathbb{C}}$, and thus $K_{\mathbb{C}} = K_{\mathbb{C}}B_M$. As $B = B_M A_{\mathbb{C}} N_{\mathbb{C}}$ now $K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}} = K_{\mathbb{C}}B$. We have proved

Proposition 1.4. *Let B be a Borel subgroup of $G_{\mathbb{C}}$ that contains $A_{\mathbb{R}}N_{\mathbb{R}}$. Let $g \in G_{\mathbb{C}}$. Then g is contained in $K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$ if and only if gB belongs to the open $K_{\mathbb{C}}$–orbit on $G_{\mathbb{C}}/B$. In particular $K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$ is a dense open subset of $G_{\mathbb{C}}$.*

Proposition 1.4 describes the complexified Iwasawa decomposition set $K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$ in terms of the open $K_{\mathbb{C}}$–orbit on the full flag manifold $G_{\mathbb{C}}/B$. Usually it is easy to decide whether a given group element belongs to $K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$, and that then describes the open $K_{\mathbb{C}}$–orbit on $G_{\mathbb{C}}/B$:

Example 1.5. Let $G_{\mathbb{R}} = SL_n(\mathbb{R})$ and $G_{\mathbb{C}} = SL_n(\mathbb{C})$. For $g \in G_{\mathbb{C}}$, let s_k be the $k \times k$ submatrix in the upper left corner of ${}^t gg$. Then $g \in K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$ if and only if $\det(s_k) \neq 0$ for $1 \leqq k \leqq n$.

Added in proof. After this note was accepted for publication, we learned that there is a nontrivial overlap with material in [1, Appendix B], specifically [1, Theorem B.1.2] and some of the immediately preceding discussion.

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