Journal of Lie Theory
Volume **14** (2004) 287–316

The Structure of Parabolic Subgroups

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Communicated by E. B. Vinberg

Abstract. Suppose G is a real connected simple noncompact Lie group with (using standard notation) Iwasawa decomposition $G = KAN$. If $M = Z(A) \cap K$, the group $B = MAN$ is a minimal parabolic subgroup of G. Since A is a vector group and N is a simply connected nilpotent group, the topological structure of B is determined by the structure of M. When G is a linear group the structure of M is well known. However, if G is not a linear group there is very little available information about M. Our purpose here is to give a description of the group M for any connected, simply connected, nonlinear simple group G.

1. Introduction and Notation

Let $\mathbf{g}$ be a real simple Lie algebra with $\mathbf{g_C}$ its complexification. Suppose G is a real connected Lie group with Lie algebra $\mathbf{g}$. If G has finite center, G has an Iwasawa decomposition $G = KAN$. That is, K is a maximal compact subgroup of G, A is a maximal vector subgroup of G with $Ad(A)$ consisting of semisimple elements, and N is a maximal nilpotent group normalized by A. If G does not have a finite center, G is a covering of a group with finite center; the Iwasawa decomposition still holds if K is the inverse image of the maximal compact subgroup by the covering map $Ad : G \longrightarrow Ad(G)$. If $M = Z(A) \cap K$, the group MAN is a minimal parabolic subgroup of G. Unlike the case when G is a linear group, the structure of M is not known when G is not a linear group.

Let $\mathbf{k}, \mathbf{m}, \mathbf{a}$, and $\mathbf{n}$ be the respective Lie algebras of K, M, A, and N. When G is a linear group $M = Z_1 \cdot M_0 \cong Z_1 \times M_0$ where M_0 is the connected component of the identity of M and Z_1 is a subgroup of $(\exp i\mathbf{a}) \cap K$ isomorphic to $\mathbf{Z}_2^r$ for some $r \leq \dim \mathbf{a}$ (Satake [12], Helgason [7]). If $G \subset G_\mathbf{C}$ where $G_\mathbf{C}$ is a connected and simply connected Lie group with Lie algebra $\mathbf{g_C}$, it is known that r is the number of white dots in the Satake diagram of G that are neither attached to another white dot by an arrow nor adjacent to a black dot (Johnson [10]). If $\pi : \tilde{G} \longrightarrow G$ is a nontrivial covering of G with $\tilde{K} = \pi^{-1}(K)$, then $\tilde{G} = \tilde{K}AN$ where A and N are as above. If $\tilde{M} = \pi^{-1}(M)$, then $\tilde{M} = Z(A) \cap \tilde{K}$ and $\tilde{M}AN$ is a minimal parabolic subgroup of $\tilde{G}$. Again $\tilde{M} = \tilde{Z}_1 \cdot \tilde{M}_0$ where $\tilde{M}_0$ is the connected component of the identity of $\tilde{M}$ and $\tilde{Z}_1$ is a discrete subgroup.

ISSN 0949–5932 / \$2.50

A priori, the group $\tilde{Z}_1$ need not be either finite or abelian. Indeed we will show that $\tilde{Z}_1$ is in general not abelian. However, it is true that the group $\tilde{Z}_1$ is finite provided G/K is not a tube type domain.

Our main goal in this paper is to give a complete description of the structure of $\tilde{M}$ for any simply connected group $\tilde{G}$. To establish our footing we first prove two results that should be well known.

Proposition 1.1. *The Group G is a linear group Lie group if and only if it is isomorphic to a subgroup of $G_{\mathbf{C}}$.*

Proof. If $G \subset G_{\mathbf{C}}$, G is linear since $G_{\mathbf{C}}$ is already linear. On the other hand, if G is linear, we may assume G is a subgroup of $GL(n,\mathbf{C})$ for some n. Then G is a subgroup of the analytic subgroup of $GL(n,\mathbf{C})$ whose Lie algebra is $\mathbf{g_C}$. ■

Theorem 1.2. *Suppose $G \subset G_{\mathbf{C}}$ with $G_{\mathbf{C}}$ simply connected. Then no nontrivial covering group of G is linear.*

Proof. Suppose $\tilde{G}$ is a nontrivial covering group of G. Then $\tilde{G}/F \cong G$ for some discrete group F with $|F| > 1$. Since any finite dimensional representation σ of $\tilde{G}$ extends to a representation of $G_{\mathbf{C}}$, $\sigma(F) = I$. ■

In particular, note that the double cover of $SL(n,\mathbf{R})$ is not a linear group.

This paper begins with a brief review of the Clifford algebra and spinors followed by an examination of the finite group D_n and its representations. The bulk of the paper deals with the case by case study of the exceptional groups. Moreover, since every real simple exceptional simple Lie algebra may be realized as a subalgra of a real form of $\mathbf{e}_8$, we devote several sections to explicitly describe a real form of the Lie algebra $\mathbf{e}_8$. Sections 2, 3, and 4 are devoted to recalling results about the Clifford algebra, the group D_n and spinors. In sections 5 and 6, we explicitly construct the Lie algebra $\mathbf{e}_{8,\mathbf{C}}$. In sections 7 and 8, we obtain the split real forms of $\mathbf{e}_{8,\mathbf{C}}, \mathbf{e}_{7,\mathbf{C}}, \mathbf{e}_{6,\mathbf{C}}, \mathbf{f}_{4,\mathbf{C}}$ and the corresponding simply connected groups, and in section 9, we construct the corresponding groups $\tilde{M}$. The split group $\tilde{G}_{2(2)}$ is considered in section 10. The remaining nonsplit exceptional groups are examined in sections 11 through 16, and the classical groups are dealt with in section 17. The final results describing the structure of $\tilde{M}$ are summarized in section 18.

This paper grew out of joint work with Adam Korányi and Martin Reimann. I am particularly indebted to Adam Korányi for comments and suggestions. I am also grateful to David Benson who acquianted me with the properties of the group D_n.

Notation:

1. In the subsequent sections $E_{6,\mathbf{C}}$, $E_{7,\mathbf{C}}$, $E_{8,\mathbf{C}}$, $F_{4,\mathbf{C}}$, and $G_{2,\mathbf{C}}$ will all denote complex simply connected simple exceptional Lie groups. We will use the notation from [6] to denote specific real exceptional groups; the groups $E_{6(6)}, E_{6(2)}, E_{6(-14)}, E_{6(-26)}$ denote connected noncompact real forms of $E_{6,\mathbf{C}}$; the groups $E_{7(7)}, E_{7(-5)}, E_{7(-25)}$ denote connected noncompact real forms of $E_{7,\mathbf{C}}$; the groups $E_{8(8)}, E_{8(-24)}$ denote connected noncompact real forms of

$E_{8,\mathbf{C}}$; the groups $F_{4(4)}, F_{4(-20)}$ denote connected noncompact real forms of $F_{4,\mathbf{C}}$; and, the group $G_{2(2)}$ denotes a connected noncompact real form of $G_{2,\mathbf{C}}$.

2. The corresponding simply connected covering groups will be denoted by placing tilde over the letter E, F, or G.

3. The corresponding exceptional Lie algebras will be denoted by boldface lower case letters.

4. The symbol $\langle x_1, \ldots, x_n \rangle$ will denote either the complex or real linear span of $x_1, \ldots, x_n$ or the group generated by these terms. The precise meaning will be clear from the context.

2. The Clifford Algebra

Let $(,)$ denote the standard inner product on $\mathbf{R}^n$. Extend $(,)$ to be bilinear on $\mathbf{C}^n \times \mathbf{C}^n$. The respective tensor algebras of $\mathbf{R}^n$ and $\mathbf{C}^n$ will be denoted by $T(\mathbf{R}^n)$ and $T(\mathbf{C}^n)$. If $I(\mathbf{R}^n)$ is the two sided ideal of $T(\mathbf{R}^n)$ generated by all elements of the form $x \otimes x + (x,x)1$ $(x \in \mathbf{R}^n)$ and $I(\mathbf{C}^n)$ is the two sided ideal of $T(\mathbf{C}^n)$ generated by all elements of the form $z \otimes z + (z,z)1$ $(z \in \mathbf{C}^n)$, then $C(\mathbf{R}^n) = T(\mathbf{R}^n)/I(\mathbf{R}^n)$ is the Clifford algebra of $\mathbf{R}^n$ and $C(\mathbf{C}^n) = T(\mathbf{C}^n)/I(\mathbf{C}^n)$ is the Clifford algebra of $\mathbf{C}^n$. Since $T(\mathbf{C}^n) = T(\mathbf{R}^n) \otimes_{\mathbf{R}} \mathbf{C}$ and $I(\mathbf{C}^n) = I(\mathbf{R}^n) \otimes_{\mathbf{R}} \mathbf{C}$, we have $C(\mathbf{C}^n) = C(\mathbf{R}^n) \otimes_{\mathbf{R}} \mathbf{C}$. If $a, b \in C(\mathbf{V})$ denote their product in $C(\mathbf{V})$ by $a \cdot b$. An element of an irreducible $C(\mathbf{V})$-module is called a spinor. We now use one of the many well known constructions construct irreducible $C(\mathbf{R}^n)$ and $C(\mathbf{C}^n)$-modules. The one we are using may be found in Cartan [5].

Let $\{e_j : 1 \leq j \leq n\}$ be the standard basis for $\mathbf{R}^n$. Suppose first that $n = 2k$. Set $f_j = (e_{2j-1} - ie_{2j})/\sqrt{2}$, and $g_j = (e_{2j-1} + ie_{2j})/\sqrt{2}$ for $j \leq k$. Note that $(f_j, g_l) = \delta_{jl}$. The spaces $W = \langle f_1, \cdots, f_k \rangle$ and $\bar{W} = \langle g_1, \cdots, g_k \rangle$ are maximally isotropic subspaces and are dual to each other with respect to the inner product.

Set

$$\bigwedge^{*} W = \bigoplus_{j=0}^{k} \bigwedge^{j} W, \bigwedge^{e} W = \bigoplus_{j=0}^{[k/2]} \bigwedge^{2j} W,$$

and

$$\bigwedge^{o} W = \bigoplus_{j=0}^{[(k-1)/2]} \bigwedge^{2j+1} W$$

where, as usual, $\bigwedge^0 W = \mathbf{C}$. Consider the map

$$\gamma : \{f_1, \cdots, f_k, g_1, \cdots, g_k\} \to End \bigwedge^{*} W$$

defined by setting $\gamma(f_j)\omega = f_j \wedge \omega$ and $\gamma(g_j)\omega = -2\iota(g_j)\omega$. Now γ extends to a linear map of $\mathbf{C}^n$ into $End \bigwedge^* W$ and hence to an algebra map of $T(\mathbf{C}^{\mathbf{n}})$ into $End \bigwedge^* W$. Since γ maps any $z \otimes z + (z,z)1$ to zero, $I(\mathbf{C}^n)$ is in the kernel of γ. Hence γ induces an algebra map of $C(\mathbf{C}^n)$ into $End \bigwedge^* W$. Since $C(\mathbf{C}^n) = \langle e_{i_1} \cdots e_{i_r} : i_1 < \cdots < i_r \rangle$, we have

$dim C(\mathbf{C}^n) \le 2^n = 2^{2k} = dim End \bigwedge^* W$.

Notation: Set $S = \{1, \cdots, k\}$ and for $I \subset S$ set $f_I = f_{i_1} \wedge \cdots \wedge f_{i_r}$ where $I = \{i_1, \cdots, i_r\}$ and $i_1 < \cdots < i_r$.

Theorem 2.1. *The space $\bigwedge^* W$ is an irreducible $C(\mathbf{C}^n)$-module.*

Proof. If U is a $C(\mathbf{C}^n)$-submodule and $0 \neq \omega \in U$ by applying appropriate $\gamma(g_j)'s$, we see that $f_\phi = 1$ is in U. Similarly, by applying appropriate $\gamma(f_j)'s$, we see that $U = \langle f_I : I \subset S \rangle$. Hence $U = \bigwedge^* W$. ■

By Burnsides' theorem (see [8]) $\gamma(C(\mathbf{C}^n)) = End \bigwedge^* W$, and by dimension we have that γ is an isomorphism and $dim C(\mathbf{C}^n) = 2^n$.

If $n = 2k+1$, define W and $\bar{W}$ as before. Also, define γ on $\mathbf{C}^{n-1}$ as above. We extend γ to all of $\mathbf{C}^n$ by setting $\gamma(e_{2k+1}) = i$ on $\bigwedge^e W$ and $= -i$ on $\bigwedge^o W$. Thus $C(\mathbf{C}^{2k})$ is a simple associative algebra.

Note that if $z \epsilon \mathbf{C}^{2k}$, $\gamma(z)$ interchanges $\bigwedge^e W$ and $\bigwedge^o W$, but $\gamma(e_{2k+1})$ leaves both spaces invariant.

Since $C(\mathbf{V})$ is an associative algebra, it has the natural structure of a Lie algebra where $[a,b] = a \cdot b - b \cdot a$.

Let

$$\mathbf{g}(n) = \langle x \cdot y : (x,y) = 0, x, y \epsilon \mathbf{R}^n \rangle \quad \text{and} \quad \mathbf{g}(n)_{\mathbf{C}} = \langle u \cdot v : (u,v) = 0, u, v \epsilon \mathbf{C}^n \rangle.$$

Proposition 2.2. $\mathbf{g}(n)$ *and* $\mathbf{g}(n)_{\mathbf{C}}$ *are Lie algebras.*

The proposition is well known and its proof follows from a direct calculation.

Note that

$$[u \cdot v, w] = 2(u,w)v - 2(v,w)u(u, v, w \epsilon \mathbf{C}^n).$$

The map $u \cdot v \to 2(v \otimes u^t - v \otimes u^t)$ is a Lie algebra map of $\mathbf{g}(n)_{\mathbf{C}}$ onto $\mathbf{on}(n, \mathbf{C})$ that restricts to a map of $\mathbf{g}(n)$ onto $\mathbf{o}(n)$. We then obtain the following result.

Proposition 2.3. $\mathbf{g}(n) \cong \mathbf{o}(n)$, $\mathbf{g}(n)_{\mathbf{C}} \cong \mathbf{on}(n, \mathbf{C})$, *and* $\mathbf{g}(n)_{\mathbf{C}} \cong \mathbf{g}(n) \otimes_{\mathbf{R}} \mathbf{C}$.

The group $G(n)$ will be the group of all even products of the form $v_1 \cdots v_{2l}$ with $v_1, \cdots, v_{2l}$ unit vectors in $\mathbf{R}^n$.

Proposition 2.4. *$G(n)$ is an analytic group with Lie algebra $\mathbf{g}(n)$.*

Proof. Suppose $x \cdot y \in \mathbf{g}(n)$ with x and y both unit vectors. Since $(x \cdot y)^2 = -1$, $e^{tx \cdot y} = \cos t + \sin tx \cdot y = x \cdot (-\cos tx + \sin ty)$.

Also, if $|v_1| = |v_2| = 1$, $v_1 \cdot v_2 = v_1 \cdot (-\cos \psi v_1 + \sin \psi w)$ where $(v_1, w) = 0$. The proof now follows. ■

Let $G(n)_{\mathbf{C}}$ be the analytic subgroup of $C(\mathbf{C}^n)$ having Lie algebra $\mathbf{g}(n)_{\mathbf{C}}$. The bracket operations on $C(\mathbf{C}^n)$ and $C(\mathbf{R}^n)$ turn $\mathbf{C}^n$ and $\mathbf{R}^n$ into respective $\mathbf{g}(n)_{\mathbf{C}}$ and $\mathbf{g}(n)$-modules. Let $\tau : \mathbf{g}(n)_{\mathbf{C}} \to End \mathbf{C}^n$, $\tau : \mathbf{g}(n) \to End \mathbf{R}^n$ denote the representations induced by the bracket operation. Then τ induces representations (also denoted by τ) of $G(n)_{\mathbf{C}}$ on $\mathbf{C}^n$ and of $G(n)$ on $\mathbf{R}^n$ where $\tau(g)v = g{\cdot}v{\cdot}g^{-1}$.

As τ is a Lie algebra isomorphism, we see that $\tau : G(n) \to SO(n)$ is a covering homomorphism. Hence $\tau : G_{\mathbf{C}} \to SO(n,\mathbf{C})$ is also a covering homomorphism. As $\tau((e_1,e_2)^2) = \tau(-I) = 1$, we see that $G(n) = Spin(n)$ and so $G(n)_{\mathbf{C}} = Spin(n,\mathbf{C})$ for $n \geq 3$

If $n = 2k+1$, $Z(SO(n))$ is trivial and so $Z(Spin(n)) = \{\pm I\}$. Now

$$(e_1 \cdots e_n) \cdot e_j = (-1)^{n+1} e_j \cdot (e_1 \cdots e_n).$$

Hence, if n is even, $e_1 \cdots e_n \in Z(Spin(n))$. A simple calculation yields

$$(e_1 \cdots e_{2j})^2 = (-I)^j.$$

Thus, if n is even, $Z(SO(n)) = \{I, -I\}$ and $Z(Spin(n)) = \{\pm I, \pm e_1 \cdots e_n\}$. If $n \equiv 0 \bmod 4$, $(e_1 \cdots e_n)^2 = 1$ and so $Z(Spin(n)) = \mathbf{Z}_2 \times \mathbf{Z}_2$. If $n \equiv 2 \bmod 4$, $(e_1 \cdots e_n)^2 = (-1)$ and $Z(Spin(n)) = \mathbf{Z}_4$.

As $\gamma : C(\mathbf{C}^n) \to End \bigwedge^* W$ is a Lie algebra homomorphism, it restricts to a Lie algebra homomorphism of $\mathbf{g}(n)_{\mathbf{C}}$. Hence γ induces representations of $Spin(n,\mathbf{C})$ and of $Spin(n)$ on $\bigwedge^* W$. Note that for n even $\bigwedge^e W$ and $\bigwedge^o W$ are both $Spin(n)$-modules.

Proposition 2.5. *If n is odd, $\bigwedge^* W$ is an irreducible $Spin(n)$-module.*

If n is even, $\bigwedge^e W$ and $\bigwedge^o W$ are irreducible and inequivalent $Spin(n)$-modules.

We will not prove this result here but will prove a much stronger result in the next section.

Remark. Note that for $n = 2k+1$,

$$p_{\pm} = \frac{1 \pm i^{k+1}(e_1 \cdots e_n)}{\sqrt{2}}$$

are commuting central idempotents of $C(\mathbf{C}^n)$, and so

$$C(\mathbf{C}^n) \cong C(\mathbf{C}^{2k}) \oplus C(\mathbf{C}^{2k})$$

as an associative algebra.

3. The group D_n

Let $D_n = \{e_{i_1} \cdots e_{i_{2l}} : 1 \leq i_1, \ldots, i_{2l} \leq n\}$. Now D_n is a subgroup of $Spin(n)$ and

$$D_n/\{\pm I\} \equiv \{diag(\epsilon_1, \ldots, \epsilon_n) : \epsilon_1 \cdots \epsilon_n = 1, \epsilon_j = \pm 1\}$$

is the centralizer in $SO(n)$ of the diagonal matrices of $SL(n,\mathbf{R})$.

The following result was told to me by David Benson.

Theorem 3.1. *If n is even, the modules $\bigwedge^e W$ and $\bigwedge^o W$ are inequivalent irreducible D_n-modules. If n is odd, $\bigwedge^* W$ is an irreducible D_n-module.*

Proof. If $I \subset S$, $\gamma(e_{2j-1} \cdot e_{2j})f_I = if_I$ whenever $j \in I$ and $\gamma(e_{2j-1} \cdot e_{2j})f_I = -if_I$ whenever $j \notin I$. Suppose $n = 2k$ and U is a D_n-submodule of $\bigwedge^e W$. If $f \in \bigwedge^e W$, write $f = \sum a_I(f) f_I$ where the sum is taken over all subsets of S of even order. Set $\ell(f) = |\{I : a_I(f) \neq 0\}|$. Select $f \in U$ such that $\ell(f) = inf\{\ell(F) : 0 \neq F \in U\}$. Suppose $\ell(f) > 1$ with $a_I(f) \neq 0$ and $a_J(f) \neq 0$ for sets I and J. Without loss of generality, we may assume there is a $j \in I \sim J$. Then $(-i\gamma(e_{2j-1} \cdot e_{2j})(f) - f) = F \in U$ and $\ell(F) < \ell(f)$. This is a contradiction. Hence $\ell(f) = 1$. So $f = f_I$ for some set I. For $J \subset S$ set $e_J = e_{2j_1 - 1} \cdots e_{2j_{2l}-1}$ where $J = \{j_1, \ldots, j_{2l}\}$ and $j_1 < \cdots < j_{2l}$. Then for $J \subset S$, we have $\gamma(e_J)\gamma(e_I)f_I = cf_J$ for some $c \neq 0$. Thus we have that $f_J \in U$ for any $J \subset S$. Therefore $\bigwedge^e W$ is an irreducible D_n-module.

Since $\gamma(e_{2j-1} \cdot e_{2k-1})f_k = cf_j$ for some $c \neq 0$, we see that $\bigwedge^o W$ is also an irreducible D_n-module. To see that these two modules are inequivalent note that an intertwining operator maps eigenvectors of the elements $\gamma(e_{2j-1} \cdot e_{2j})$ to eigenvectors with the same eigenvalue. Since f_S is the only element of $\bigwedge^* W$ that is an eigenvector with eigenvalue i for each $\gamma(e_{2j-1} \cdot e_{2j})$, we see that the two modules are inequivalent.

Similarly, it is easy to see that if n is odd, $\bigwedge^* W$ is an irreducible D_n-module. ■

If $n = 2k$,

$$|D_n| = 2^{2k} = 2^{2k-1} + (2^{k-1})^2 + (2^{k-1})^2 =$$

$$|D_n/\{\pm I\}| + \dim End \overset{e}{\bigwedge} W + \dim End \overset{o}{\bigwedge} W.$$

Hence $\bigwedge^e W$ and $\bigwedge^o W$ are –up to equivalence– the only irreducible D_n-modules of dimension > 1.

Similarly, if $n = 2k+1$,

$$|D_n| = 2^{2k} + 2^{2k} = |D_n/\{\pm I\}| + \dim End \overset{*}{\bigwedge} W.$$

Hence $\bigwedge^* W$ is the only irreducible D_n-module of dimension > 1.

4. The Lie Algebra $\mathbf{g}(n)_{\mathbf{C}}$

If $n = 2k$ set $h_j = -i/2 e_{2j-1} \cdot e_{2j} = -1/4(f_j \cdot g_j - g_j \cdot f_j)$ and let $\mathbf{h} = \langle h_j : j \leq k \rangle$. Note that $[h_j, f_l] = \delta_{j,l} f_l$ and $[h_j, g_l] = -\delta_{j,l} g_l$. Set $\mathbf{n} = \langle f_j \cdot f_l, f_j \cdot g_l : j < l \leq k \rangle$ and $\overline{\mathbf{n}} = \langle g_j \cdot g_l, g_j \cdot f_l : j < l \leq k \rangle$. If $n = 2k$, $\mathbf{n}$ and $\overline{\mathbf{n}}$ are maximal nilpotent Lie subalgebras of $\mathbf{g}(n)_{\mathbf{C}}$ normalized by $\mathbf{h}$, and $\mathbf{g}(n)_{\mathbf{C}} = \mathbf{n} + \mathbf{h} + \overline{\mathbf{n}}$. In this case we see that $\mathbf{h}$ is a Cartan subalgebra of $\mathbf{g}(n)_{\mathbf{C}}$. Let $\epsilon_l \in \mathbf{h}^*$ be such that for $H = \sum_{j=1}^k z_j h_j$, $\epsilon_j(H) = z_j$. Then, if $\Sigma(ad)$ denotes the root system of $\mathbf{g}(n)_{\mathbf{C}}$, we have $\Sigma(ad) = \{\pm\epsilon_j \pm \epsilon_l : 1 \leq j < l \leq k\}$.

If $n = 2k+1$, set $\tilde{\mathbf{n}} = \mathbf{n} + \langle f_j \cdot e_{2k+1} : 1 \leq j \leq k \rangle$ and
$\overline{\tilde{\mathbf{n}}} = \overline{\mathbf{n}} + \langle g_j \cdot e_{2k+1} : 1 \leq j \leq k \rangle$.

In this case, $\tilde{\mathbf{n}}$ and $\overline{\tilde{\mathbf{n}}}$ are maximal nilpotent Lie algebras normalized by $\mathbf{h}$, $\mathbf{g}(n)_{\mathbf{C}} = \tilde{\mathbf{n}} + \mathbf{h} + \overline{\tilde{\mathbf{n}}}$ and

$$\Sigma(ad) = \{\pm\epsilon_j \pm \epsilon_l : 1 \leq j < l \leq k\} \cup \{\pm\epsilon_j : 1 \leq j \leq k\}.$$

We now examine the representation γ of $\mathbf{g}(n)_{\mathbf{C}}$. If $H = \sum_{j=1}^{k}, z_j h_j$ and $I \subset S$,

$$\gamma(H) f_I = 1/2(\sum_{j\in I} z_j - \sum_{j\in S\sim I} z_j) f_I.$$

Hence, the weights of γ are $\Sigma(\gamma) = \{1/2(\pm\epsilon_1 \pm \cdots \pm \epsilon_k)\}$. If $n = 2k$, $\gamma = \gamma_e \oplus \gamma_o$ where γ_e and γ_o are the respective representations of $\mathbf{g}(n)_{\mathbf{C}}$ on $\bigwedge^e W$ and $\bigwedge^o W$. Then $\Sigma(\gamma_e)$ consists of those weights in $\Sigma(\gamma)$ that have an even number of plus signs in their expansions, and $\Sigma(\gamma_o)$ consists of the weights of $\Sigma(\gamma)$ that have an odd number of plus signs in their expansions.

Proposition 4.1. *As a $\mathbf{g}(2k+1)_{\mathbf{C}}$-module, $End \bigwedge^* W \cong \oplus_{l=0}^{k} \bigwedge^l \mathbf{C}^{2k+1}$.*

Proof. Consider the linear map $\Gamma : \bigwedge^l \mathbf{C}^n \to End \bigwedge^* W$ such that

$$\Gamma(\psi_1 \wedge \cdots \wedge \psi_l)\omega = 1/l! \sum_{\sigma\in P_l} (sgn\sigma)\gamma(\psi_{\sigma(1)}) \cdots \gamma(\psi_{\sigma(l)})\omega.$$

A simple calculation shows that Γ is an intertwining operator for any l. Using the fact that as $\mathbf{g}(n)_{\mathbf{C}}$-modules, $\bigwedge^l \mathbf{C}^n \cong \bigwedge^{n-l} \mathbf{C}^n$, and noting that

$$\dim(\mathbf{C} \oplus \cdots \oplus \bigwedge^{k} \mathbf{C}^{2k+1}) = 2^{2k} = End \bigwedge^{*} W,$$

we have our result. ∎

Since $\mathbf{C}^{2k+1} = \mathbf{C}^{2k} \oplus \mathbf{C}e_{2k+1}$, we see that as $\mathbf{g}(2k)_{\mathbf{C}}$-modules,

$$End \bigwedge^{*} W \cong \mathbf{C} \oplus \oplus_{l=1}^{k} (\bigwedge^{l} \mathbf{C}^{2k} \oplus \bigwedge^{l-1} \mathbf{C}^{2k}).$$

Thus we obtain the following.

Proposition 4.2. *If $n = 2k$, as a $\mathbf{g}(n)_{\mathbf{C}}$-module,*
$End \bigwedge^* W \cong 2(\oplus_{l=0}^{k} (\bigwedge^l \mathbf{C}^{2k})) \oplus \bigwedge^k \mathbf{C}^{2k}$.

Recall that the $\mathbf{g}(2k)_{\mathbf{C}}$-module, $\bigwedge^l \mathbf{C}^{2k}$, is irreducible for $l \neq k$, and $\bigwedge^k \mathbf{C}^{2k} = U_! + U_2$ where U_1 is the irreducible module with highest weight vector $f_1 \wedge \cdots \wedge f_k$, and U_2 is the irreducible module with highest weight vector $f_1 \wedge \cdots \wedge f_{k-1} \wedge g_k$.

Proposition 4.3. *The following are $\mathbf{g}(2k)_{\mathbf{C}}$-module isomorphisms.*

(α) *If $k = 2l+1$,*
$End \bigwedge^e W \cong End \bigwedge^o W \cong \bigoplus_{j=0}^{l} \bigwedge^{2j} \mathbf{C}^{2k}$,
$Hom(\bigwedge^e W, \bigwedge^o W) \cong \bigoplus_{j=0}^{l-1} \bigwedge^{2j+1} \mathbf{C}^{2k} \oplus U_1$,
and
$Hom(\bigwedge^o W, \bigwedge^e W) \cong \bigoplus_{j=0}^{l-1} \bigwedge^{2j+1} \mathbf{C}^{2k} \oplus U_2$.

(β) *If $k = 2l$,*
$End \bigwedge^e W \cong \bigoplus_{j=0}^{k-1} \bigwedge^{2j} \mathbf{C}^{2k} \oplus U_1$, $End \bigwedge^o W \cong \bigoplus_{j=0}^{k-1} \bigwedge^{2j} \mathbf{C}^{2k} \oplus U_2$,
and
$Hom(\bigwedge^o W, \bigwedge^e W) \cong Hom(\bigwedge^e W, \bigwedge^o W) \cong \bigoplus_{j=0}^{l-1} \bigwedge^{2j+1} \mathbf{C}^{2k}$.

Proof. Note that $\Gamma(f_1 \wedge \cdots \wedge f_k) = 0$ on $\bigwedge^o W$ and $\Gamma(f_1 \wedge \cdots \wedge f_{k-1} \wedge g_k) = 0$ on $\bigwedge^e W$. If k is even, these operators preserve the parity of a form and if k is odd they reverse the parity. Note also that $\Gamma(f_1 \wedge \cdots f_r)$ preserves parity if r is even and reverses parity if r is odd. Our proposition now follows from proposition 4.2. ■

We have already seen that the space of $\mathbf{g}(2k+1)_{\mathbf{C}}$-invariants in $\bigwedge^* W \otimes \bigwedge^* W$ is one dimensional. Suppose $\omega = \sum_{I \subset S} a(I, S \sim I) f_I \otimes f_{S \sim I}$ is an invariant. Then $\gamma(f_p{\cdot}f_q)(\omega) = 0$, and if $J \cap \{p,q\} = \emptyset$ and $K = J \cup \{p,q\}$, we have

$$a(J, S \sim J) f_p \wedge f_q \wedge f_J \otimes f_{S \sim J} + a(K, S \sim K) f_K \otimes f_p \wedge f_q \wedge f_{S \sim K} = 0.$$

Without loss of generality we may assume $p < q$, $|\{1, \ldots, p-1\} \cap J| = l$ and $|\{1, \ldots q-1\} \cap J| = m$. Then $f_p \wedge f_q \wedge f_J = (-1)^{l+m} f_K$ and

$$f_p \wedge f_q \wedge f_{S \sim K} = (-1)^{p+q+l+m+1} f_{S \sim J}.$$

Hence we have $a(K, S \sim K) = (-1)^{p+q} a(J, S \sim J)$. Similarly,

$$\gamma(f_q{\cdot}e_{2k+1})(\omega) = 0.$$

So, if $L = J \cup \{q\}$, we have $a(L, S \sim L) = (-1)^{q+|S|} a(J, S \sim J)$. Finally, setting $a(\emptyset, S) = 1$ and $\sigma(I) = \sum_{j \in I} j$, we have $a(I, S \sim I) = (-1)^{\sigma(I) + k|I|}$ where $k = |S|$. Now $\sigma(I) \cong \sigma(S \sim I) \bmod 2$ if and only if $k \cong 0 \bmod 4$ or $k \cong 3 \bmod 4$, and $\sigma(I) \cong -\sigma(S \sim I) \bmod 2$ if and only if $k \cong 1 \bmod 4$ or $k \cong 2 \bmod 4$. Thus we have $a(I, S \sim I) = a(S \sim I, I)$ if $k \cong 0 \bmod 4$ or $k \cong 1 \bmod 4$, and $a(I, S \sim I) = -a(S \sim I, I)$ if $k \cong 2 \bmod 4$ or $k \cong 3 \bmod 4$. Summarizing our results, we have the following.

Proposition 4.4. *For any $k > 0$ there is a unique, up to scalar multiple, $\mathbf{g}(2k+1)_{\mathbf{C}}$-invariant $\omega \in \bigwedge^* W \otimes \bigwedge^* W$ with the following properties.*

1. *ω is nondegenerate.*
2. *If k is odd, $\bigwedge^e W$ and $\bigwedge^o W$ are maximally isotropic spaces w.r.t. ω.*
3. *If k is even, $\bigwedge^e W$ and $\bigwedge^o W$ are mutually orthogonal.*
4. *If $k \cong 0 \bmod 4$ or $k \cong 1 \bmod 4$, ω is a symmetric form.*
5. *If $k \cong 2 \bmod 4$ or $k \cong 3 \bmod 4$, ω is a skew-symmetric form.*
6. *The form ω is the sum of two $\mathbf{g}(2k)_{\mathbf{C}}$-invariant forms.*

Corollary 4.5.

1. *If $k \cong 0 \bmod 4$ or $1 \bmod 4$, $\mathbf{g}(2k+1)_{\mathbf{C}} \subset \bigwedge^2(\bigwedge^* W)$.*
2. *If $k \cong 2 \bmod 4$ or $3 \bmod 4$, $\mathbf{g}(2k+1)_{\mathbf{C}} \subset S^2(\bigwedge^* W)$.*
3. *If $k \cong 0 \bmod 4$, $\mathbf{g}(2k)_{\mathbf{C}}$ may be considered as a subset of either $\bigwedge^2(\bigwedge^e W)$ or of $\bigwedge^2(\bigwedge^o W)$.*
4. *If $k > 2$ and $k \cong 2 \bmod 4$, $\mathbf{g}(2k)_{\mathbf{C}}$ may be considered as a subset of either $S^2(\bigwedge^e W)$ or of $S^2(\bigwedge^o W)$.*
5. *$\mathbf{g}(4)_{\mathbf{C}}$ is the direct sum of two ideals with one contained in $S^2(\bigwedge^e W)$ and the other contained in $S^2(\bigwedge^o W)$*

We conclude this section with another result about the group D_n.

Proposition 4.6. *If V_1, V_2, and V_3 are irreducible D_n-modules,*
$\dim \mathrm{Hom}_{D_n}(V_3, V_1 \otimes V_2) \leq 1$.

Proof. We prove this for n odd. If either V_1 or V_2 is one dimensional, the result is obvious. If $V_1 \cong V_2 \cong \bigwedge^* W$, the space $V_1 \otimes V_2$ is the direct sum of inequivalent one dimensional representations.

The proof for n even is similar. ∎

5. General Results

Let $\mathbf{k}$ be a complex reductive Lie algebra with Lie bracket $[,]_0$. Suppose $\rho : \mathbf{k} \to End\mathbf{p}$ is an irreducible representation and $\Phi : \bigwedge^2 \mathbf{p} \to \mathbf{k}$ is a $\mathbf{k}$ intertwining operator. Set $\mathbf{g} = \mathbf{k} + \mathbf{p}$ and define $[,] : \mathbf{g} \times \mathbf{g} \to \mathbf{g}$ by setting

$$[x + u, y + v] = [x, y]_0 + \rho(x)v - \rho(y)u + \Phi(u \wedge v)$$

for $x, y \in \mathbf{k}$ and $u, v \in \mathbf{p}$. Clearly, $[,]$ is skew symmetric and bilinear. For $\alpha, \beta, \gamma \in \mathbf{g}$ set

$$J(\alpha, \beta, \gamma) = [[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta].$$

Now $\mathbf{g}$ is a Lie algebra if and only if $J = 0$.

(i) If $\alpha, \beta, \gamma \in \mathbf{k}$, then $J(\alpha, \beta, \gamma) = 0$.

(ii) If $|\{\alpha, \beta, \gamma\} \cap \mathbf{k}| = 2$, $J(\alpha, \beta, \gamma) = 0$ as ρ is a representation of $\mathbf{k}$.

(iii) If $|\{\alpha, \beta, \gamma\} \cap \mathbf{k}| = 1$, $J(\alpha, \beta, \gamma) = 0$ as Φ intertwines $\mathbf{k}$.

Hence we see that $\mathbf{g}$ is a Lie algebra if and only if $J(\alpha, \beta, \gamma) = 0$ for any $\alpha, \beta, \gamma \in \mathbf{p}$.

For $X \in \mathbf{k}$ and $\alpha, \beta, \gamma \in \mathbf{p}$ we have the identity

$$\rho(X)(J(\alpha, \beta, \gamma)) = J(\rho(X)\alpha, \beta, \gamma) + J(\alpha, \rho(X)\beta, \gamma) + J(\alpha, \beta, \rho(X)\gamma).$$

Suppose $\mathbf{k} = \mathbf{n}^* + \mathbf{h} + \mathbf{n}$ with $\mathbf{h}$ a Cartan subalgebra, $\mathbf{h} + \mathbf{n}$ a Borel subalgebra, and $\mathbf{n}^* + \mathbf{h}$ the opposite Borel.

Lemma 5.1. *$J = 0$ if and only if $J(\alpha_0, \beta, \gamma) = 0$ where $0 \neq \alpha_0 \in \mathbf{p}^{\mathbf{n}}$ and β and γ are arbitrary elements of $\mathbf{p}$.*

Proof. If $J(\alpha_o, \beta, \gamma) = 0$ for any $\beta, \gamma \in \mathbf{p}$, it follows from the equation above that $J(\rho(X_1)\alpha_0, \beta, \gamma) = 0$ for any $X_1 \in \mathbf{n}^*$ and any $\beta, \gamma \in \mathbf{p}$. By induction on r, it follows that $J(\rho(X_1) \cdots \rho(X_r)\alpha_0, \beta, \gamma) = 0$ for any $X_1, \ldots, X_r \in \mathbf{n}^*$ and any $\beta, \gamma \in \mathbf{p}$. Since $\mathbf{p}$ is an irreducible $\mathbf{k}$-module, it follows that $J = 0$. The opposite implication is trivial. ∎

Fix $0 \neq \beta_0 \in \mathbf{p}^{\mathbf{n}^*}$

Theorem 5.2. *$J = 0$ if and only if $J(\alpha_0, \beta_0, \gamma) = 0$ for any $\gamma \in \mathbf{p}$.*

Proof. Suppose $J(\alpha_0, \beta_0.\gamma) = 0$ for any $\gamma \in \mathbf{p}$. For $X \in \mathbf{n}$ we have $J(\alpha_0, \rho(X)\beta_0, \gamma) = -J(\alpha_0, \beta_0, \rho(X)\gamma) = 0$, an d so, for $X_1, \ldots, X_r \in \mathbf{n}$ we have

$$J(\alpha_0, \rho(X_1) \cdots \rho(X_r)\beta_0, \gamma) = (-1)^r J(\alpha_0, \beta_0, \rho(X_r) \cdots \rho(X_1)\gamma) = 0.$$

Since $\rho(\mathbf{U}(\mathbf{n}))\beta_0 = \mathbf{p}$, it follows from lemma 1 that $J = 0$.

The opposite implication is obvious. ∎

Note that we may replace Φ by $c\Phi$ for any $c \neq 0$.

6. The Complex Lie Algebra $\mathbf{e}_{8,\mathbf{C}}$

In this section we will construct the complex Lie algebra $\mathbf{e}_{8,\mathbf{C}}$. Although it is apparent from Cartan's list of symmetric spaces (Helgason [6]) that such a construction exists, we will need the explicit construction given here in subsequent sections. Suppose $n = 2k$ and let $\mathbf{g}(n)_{\mathbf{C}}$ and $\bigwedge^* W$ be as in section 3. In order to construct a Lie algebra on $\mathbf{g}(n)_{\mathbf{C}} + \bigwedge^e W$ as in section 4 with $\mathbf{k} = \mathbf{g}(n)_{\mathbf{C}}$ and $\mathbf{p} = \bigwedge^e W$, since $\mathbf{k}$ is simple whenever $k > 2$, we must have $\mathbf{k} \subset \bigwedge^2 \bigwedge^e W$. Hence, $k = 4l$ for some integer l.

Since γ is an irreducible faithful representation of $\mathbf{k}$ not equivalent to ad, $\mathbf{k}+\mathbf{p}$ must be a simple Lie algebra if it is a Lie algebra provided $\Phi : \bigwedge^2 \mathbf{p} \to \mathbf{k}$ is nontrivial. Moreover, if $\mathbf{k}+\mathbf{p}$ is a Lie algebra, a Cartan subalgebra of $\mathbf{k}$ is also a Cartan subalgebra of $\mathbf{k}+\mathbf{p}$, since $0 \notin \Sigma(\gamma)$. So, if $\mathbf{k}+\mathbf{p} = \mathbf{l}_k$ is a Lie algebra, it is simple of rank k and dimension $k(2k-1) + 2^{k-1}$. A simple examination of dimensions and ranks shows that $\mathbf{l}_k$ can be a simple Lie algebra only for $k = 4$ or 8. *Apriori* we may have either $\mathbf{l}_4 = \mathbf{B}_4$ or $\mathbf{l}_4 = \mathbf{C}_4$, or $\mathbf{l}_8 = \mathbf{e}_8$. By construction, we see that $\mathbf{l}_4 \cong \mathbf{B}_4$. We devote the rest of this section to an examination of the case $k = 8$.

For the rest of this section we take $S = \{1, \ldots, 8\}$. For $I \subset S$ with $I = \{i_1, \ldots, i_p\}$ and $i_1 < \cdots < i_p$ set $f_I = f_{i_1} \wedge \cdots \wedge f_{i_p}$, $\tilde{f}_I = f_{i_1} \cdots f_{i_p}$, and $\tilde{g}_I = g_{i_1} \cdots g_{i_p}$. Set $I' = S \sim I$.

Suppose $\mathbf{l}_8$ has the structure of a Lie algebra. Let $I, J \subset S$ with $|I|$ and $|J|$ even and $[f_I, f_J] \neq 0$. Setting

$$\mu = 1/2(\sum_{j \in I \cup J} \epsilon_j - \sum_{j \in I' \cup J'} \epsilon_j)$$

we have $\mu \in \Sigma(ad)$ where $\Sigma(ad)$ is the root system of $\mathbf{k}$. Moreover, if $\mu \in \Sigma(ad)$, $\mu = \pm\epsilon_j \pm \epsilon_r$ for $j \neq r$. An elementary calculation yields the following.

Lemma 6.1. *Suppose $I, J \subset S$ with I and J even. If $[f_I, f_J] \neq 0$, one of the following holds.*

1. *$|I \cap J| = 2$ and $|I| + |J| = 10$, or equivalently, $|I' \cap J'| = 0$ and $|I'| + |J'| = 6$.*
2. *$|I \cap J| = 1$ and $|I| + |J| = 8$, or equivalently, $|I' \cap J'| = 1$ and $|I'| + |J'| = 8$.*
3. *$I' = J$.*
4. *$|I \cap J| = 0$ and $|I| + |J| = 6$, or equivalently, $|I' \cap J'| = 2$ and $|I'| + |J'| = 10$.*

Lemma 6.2. *Suppose $I, J \subset S$ with I and J even and $[f_I, f_J] \neq 0$.*

1. *If $|I \cap J| = 2$ and $|I| + |J| = 10$, $[f_I, f_J] = c\tilde{f}_{I \cap J}$.*
2. *If $|I \cap J| = 1$ and $|I| + |J| = 8$, $[f_I, f_J] = c f_{I \cap J} \cdot g_{I' \cap J'}$.*
3. *If $J = I'$, $[f_I, f_J] \in \mathbf{h}$.*

4. *If* $|I \cap J| = 0$ *and* $|I| + |J| = 6$, $[f_I, f_J] = c\tilde{g}_{I' \cap J'}$.

The proof of this lemma follows immediately from an examination of the root spaces of $\mathbf{k}$.

Lemma 6.3. *Suppose* $I, J, K \subset S$ *with* $|I|$, $|J|$, *and* $|K|$ *all even. If* $[f_I, f_J] \neq 0$ *and* $[f_I, f_K] \neq 0$, *then* $[f_J, f_K] = 0$.

Proof. If $|I| = 0$, lemma 6.1 guarantees that $|J| + |K| \geq 12$, and so by lemma 6.1, $[f_J, f_K] = 0$.

If $|I| = 8$, lemma 6.1 states that $|J| + |K| \leq 4$, and so again by lemma 6.1, $[f_J, f_K] = 0$.

If $|I| = 2$, lemma 6.1 states that $4 \leq |J|, |K| \leq 8$. From lemma 6.1 we need to consider only the cases $(|J|, |K|) = (4, 4)$, $(|J|, |K|) = (4, 6)$, and $(|J|, |K|) = (6, 4)$. If $|J| = |K| = 4$, $I \cap J = I \cap K = \emptyset$ and so $|J \cap K| \geq 2$ and hence the bracket is 0. If $|J| = 4$ and $|K| = 6$, we have $I \cap J = \emptyset$ and $|I \cap K| \leq 1$. Then $|J \cap K| \geq 3$ and the bracket again must be 0. The case $(|J|, |K|) = (6, 4)$ is clear.

If $|I| = 4$, lemma 6.1 states that $2 \leq |J|, |K| \leq 6$. Since $[f_J, f_K] = 0$ if $|J| + |K| \leq 4$ or $|J| + |K| \geq 12$ we need to consider the cases where $(|J|, |K|)$ is one of the pairs $(2, 4)$, $(2, 6)$, $(4, 2)$, $(4, 4)$, $(4, 6)$, $(6, 2)$ or $(6, 4)$. For $(|J|, |K|) = (2, 4)$, $|I \cap J| = 0$ and $|I \cap K| \leq 1$, and hence $|J \cap K| \geq 1$. Thus the bracket is 0in this case and also if $(|J|, |K|) = (4, 2)$. If $(|J|, |K|) = (2, 6)$, we now have $|J \cap K| = 2$, and hence the bracket is 0 in this case and also if $(|J|, |K|) = (6, 2)$. For $(|J|, |K|) = (4, 4)$, we have $|I \cap J| \leq 1$ and $|I \cap K| \leq 1$. This guarantees that $|J \cap K| \geq 2$ and hence the bracket is 0. If $(|J|, |K|) = (4, 6)$ or $(6, 4)$, we have $|J \cap K| \geq 3$ and again the bracket is 0.

If $|I| = 6$, $0 \leq |J|, |K| \leq 4$. Since the bracket is 0 if $|J| + |K| \leq 4$, we need now only consider the cases where $(|J|, |K|) = (2, 4)$, $(4, 2)$ or $(4, 4)$. If $(|J|, |K|) = (2, 4)$, $|I \cap J| \leq 1$ and $|I \cap K| = 2$. Thus $|J \cap K| \geq 1$ and the bracket is 0 in this case and in the case where $(|J|, |K|) = (4, 2)$. Finally, if $|J| = |K| = 4$, we have $|I \cap J| = |I \cap K| = 2$ and thus $|J \cap K| \geq 2$. This guarantees that the bracket is 0 in the case. ■

We now proceed to define $[,] : \mathbf{p} \times \mathbf{p} \to \mathbf{k}$. From the simplicity of $\mathbf{k}$ we know that $[f_1 \wedge f_2, f_S]$ is a non zero multiple of $f_1 \cdot f_2$. Setting $[f_1 \wedge f_2, f_S] = f_1 \cdot f_2$, the intertwining condition now forces the remaining brackets.

Lemma 6.4. *If* $|I| = 2$, $[f_I, f_S] = \tilde{f}_I$.

Proof. If $I = \{1, 2\}$ we already have the result. Suppose $I = \{1, k\}$ with $k > 2$. Then we must have

$$-2\tilde{f}_I = [f_k \cdot g_2, f_1 \cdot f_2]_0 = [\gamma(f_k \cdot g_2)(f_1 \wedge f_2), f_S] + [f_1 \wedge f_2, \gamma(f_k \cdot g_2) f_S] =$$

$$[\gamma(f_k \cdot g_2) f_1 \wedge f_2, f_S] + 0 = -2[f_I, f_S].$$

Thus we have our result in this case. The proofs for $I = \{2, k\}$ $(2 < k)$ and $I = \{k, l\}$ $(2 < k < l)$ are similar. ■

Recall that $\sigma(I) = \sum_{j \in I} j$.

Lemma 6.5. $[f_I, f_{I'}] = (-1)^{\sigma(I)}(1/2)(\sum_{j\in I} h_j - \sum_{j\in I'} h_j)$.

Proof. Clearly, $[f_\emptyset, f_S] = \sum_{j=1}^{8} a_j h_j$ for some $a_1, \dots, a_8$. As

$$[f_j \cdot f_k, [f_\emptyset, f_S]] = [f_j \wedge f_k, f_S] = -(a_j + a_k) f_j \cdot f_k = f_j \cdot f_k,$$

$a_j + a_k = -1$ for all j, k. Hence $a_j = -1/2$ for all j, and $[f_\emptyset, f_S] = -1/2(\sum_{j=1}^{8} h_j)$.

Now

$$[g_j \cdot g_k, [f_j \wedge f_k, f_S]] = [-4f_\emptyset, f_S] + 4(-1)^{j+k}[f_j \wedge f_k, f_{S\sim\{j,k\}}] =$$

$$[g_j \cdot g_k, f_j \cdot f_k] = -2(f_j \cdot g_j - g_k \cdot f_k) = 4(h_j + h_k).$$

Hence

$$[f_j \wedge f_k, f_{S\sim\{j,k\}}] = (-1)^{j+k}(h_j + h_k - (1/2)(\sum_{l=1}^{8} h_l)),$$

and the result holds in this case. Moreover, since $(-1)^{\sigma(I)} = (-1)^{\sigma(I')}$, the result holds if $|I|$ is $0, 2, 6$ or 8, It remains only to consider the case where $|I| = 4$.

Suppose $I = \{a, b, c, d\}$ where $a < b < c < d$. Since

$$[f_{\{c,d\}}, f_{\{c,d\}'}] = (-1)^{c+d}(1/2)(h_c + h_d - \sum_{j\in\{c,d\}'} h_j),$$

we have after applying $f_a \cdot f_b$ that $[f_I, f_{\{c,d\}'}] = (-1)^{c+d} f_a \cdot f_b$. Thus, we have

$$[g_a \cdot g_b, [f_I, f_{\{c,d\}'}]] = -4[f_{\{c,d\}}, f_{\{c,d\}'}] + 4(-1)^{a+b}[f_I, f_{I'}] = (-1)^{c+d} 4(h_a + h_b).$$

After rearranging terms, our result also holds in this case. ∎

The remaining brackets will be obtained as follows. Suppose $u, v \in \mathbf{p}$ with $[u, v]$ known and $X \in \mathbf{k}$ such that $\gamma(X)u = 0$. Then $[u, \gamma(X)v] = [X, [u, v]]$.

Lemma 6.6. *Suppose* $|I| = 2$, $|J| = 6$, *and* $|I \cap J| = 1$, *then* $[f_I, f_J] = \pm(1/2) f_{I\cap J} \cdot g_{I'\cap J'}$.

Proof. Suppose $\{a\} = I \cap J$ and $\{b\} = I' \cap J'$. Then $\gamma(f_a \cdot g_b) f_{I'} = \pm 2 f_J$. Thus $[f_I, f_J] = \pm 1/2[f_a \cdot g_b, [f_I, f_{I'}]]$ and our result now follows from lemma 6.5. ∎

Lemma 6.7. *Suppose* $|I| = 2$, $|J| = 4$ *and* $I \cap J = \emptyset$. *Then* $[f_I, f_J] = \pm(1/4)\tilde{g}_{(I\cup J)'}$.

Proof. Suppose $\{a, b\} = (I \cup J)'$. The proof follows from the identity

$$[f_I, \gamma(g_a \cdot g_b) f_{I'}] = [g_a \cdot g_b, [f_I, f_{I'}]].$$

∎

We summarize the remaining possible brackets in the following lemma; the proofs are similar to those above.

Lemma 6.8. *1. If $|J| = 6$, $[f_\emptyset, f_J] = (-1)^{\sigma(J')+1}(1/4)g_{J'}$.*

2. If $|I| = 4$, $|J| = 6$ and $|I \cap J| = 2$, then $[f_I, f_J] = \pm f_{I\cap J}$.

3. If $|I| = 4$, $|J| = 4$ and $|I \cap J| = 1$, then $[f_I, f_J] = \pm(1/2)f_a \cdot g_b$ where $\{a\} = I \cap J$ and $\{b\} = I' \cap J'$.

The remaining brackets are obtain from the skew symmetry of $[,]$.

Theorem 6.9. *The bracket operation turns $\mathbf{k} + \mathbf{p}$ into a simple Lie algebra.*

Proof. Since $[,] : \bigwedge^2 \mathbf{p} \to \mathbf{k}$ is a $\mathbf{k}$ intertwining operator, we need only check that the Jacobi identity $J(u, v, w) = 0$ holds for $u, v, w \in \mathbf{p}$. From theorem 5.2 we need only check that $J(f_S, f_\emptyset, f_I) = 0$ for any $I \subset S$ with $|I|$ even. For $|I| = 0$ or 8 the result is obvious.

For $|I| = 2$,

$$J(f_S, f_\emptyset, f_I) = [[f_S, f_\emptyset], f_I] + [[f_I, f_S], f_\emptyset] =$$

$$\gamma(1/2\sum_{j=1}^{8} h_j)f_I + f_I = 1/4(-2+4)f_I + f_I = 0.$$

For $|I| = 4$,

$$J(f_S, f_\emptyset, f_I) = [[f_S, f_\emptyset], f_I] = \gamma(1/2\sum_{j=1}^{8} h_j)f_I = 0.$$

For $|I| = 6$,

$$J(f_S, f_\emptyset, f_I) = [[f_S, f_\emptyset], f_I] + [[f_\emptyset, f_I] f_S] = f_I + [[f_\emptyset, f_I], f_S].$$

If $I' = \{a, b\}$ with $a < b$, $[f_\emptyset, f_I] = (-1)^{a+b+1}1/4 g_a \cdot g_b$. Hence

$$J(f_S, f_\emptyset, f_I) = f_I + (-1)^{a+b+1}(1/4)\gamma(g_a \cdot g_b)f_S = f_I - f_I = 0. \qquad \blacksquare$$

Thus we see that $\mathbf{g} = \mathbf{k} + \mathbf{p}$ has the structure of a Lie algebra. Since it is simple of rank 8 and dimension 248, it must be $\mathbf{e}_{8,\mathbf{C}}$.

7. Real Forms $\mathbf{e}_{8(8)}, \mathbf{e}_{7(7)}$, $\mathbf{e}_{6(6)}$, and $\mathbf{f}_{4(4)}$

Throughout this section $\mathbf{e}_{j(j)}$ will denote a real split Lie algebra of $\mathbf{e}_{j,\mathbf{C}}$. Fix $\mathbf{k}_\mathbf{R} = \mathbf{g}(16) = \mathbf{spin}(16)$. For $I \subset S$ set

$$X(I) = 2^{|I'|/2} f_I + (-1)^{\sigma(I)} 2^{|I|/2} f_{I'}$$

and

$$Y(I) = i(2^{|I'|/2} f_I - (-1)^{\sigma(I)} 2^{|I|/2} f_{I'}).$$

Let $\mathbf{p}_\mathbf{R} = \langle X(I), Y(I) : I \subset S, |I| \text{ even } \rangle$. For convenience set $\epsilon(I) = (-1)^{\sigma(I)}$. Note that $X(I) = \epsilon(I)X(I')$ and $Y(I) = -\epsilon(I)Y(I')$.

Suppose $I \subset S$, $l \notin I$ and $J = I \cup \{l\}$. If $|I \cap \{1, \ldots, l-1\}| = p$, we have the following equations:

$\gamma(e_{2l-1})X(I) = (-1)^p X(J)$; $\gamma(e_{2l-1})Y(I) = (-1)^p Y(J)$;
$\gamma(e_{2l})X(I) = (-1)^p Y(J)$; and, $\gamma(e_{2l})Y(I) = (-1)^{p+1} X(J)$.

Since $\mathbf{k}_\mathbf{R} = \langle e_j \cdot e_l : j \neq l \rangle$, we have that $[\mathbf{k}_\mathbf{R}, \mathbf{p}_\mathbf{R}] \subset \mathbf{p}_\mathbf{R}$.

Theorem 7.1. $\mathbf{g_R} = \mathbf{k_R} + \mathbf{p_R}$ *is a Lie algebra.*

Proof. Since $[\mathbf{k_R}, \mathbf{p_R}] \subset \mathbf{p_R}$, it remains only to prove $[\mathbf{p_R}, \mathbf{p_R}] \subset \mathbf{k_R}$. Since $\mathbf{p_R}$ is an irreducible $\mathbf{k_R}$-module, it suffices to prove $[\mathbf{p_R}, u] \subset \mathbf{k_R}$ for a $0 \neq u \in \mathbf{p_R}$. Take $u = X(\emptyset)$. Now $[X(I), X(\emptyset)] = 0$ unless $|I| = 2$ or $|I| = 6$. Since $X(I') = \epsilon(I)X(I)$ we need only consider the case $|I| = 2$. In this case,

$$[X(I), X(\emptyset)] = 2^{|I'|/2}[f_I, f_S] + \epsilon(I)2^{|I|/2+4}[f_{I'}, f_\emptyset] = 2^3\tilde{f}_I + \epsilon(I)2^5[f_{I'}, f_\emptyset].$$

Applying 1 of lemma 9 we see

$$[X(I), X(\emptyset)] = 2^3\tilde{f}_I + \epsilon(I)2^5(1/4)\epsilon(I)\tilde{g}_I = 8(\tilde{f}_I + \tilde{g}_I) \in \mathbf{k_R}.$$

Similarly, for $|I| = 2$, we see that $[Y(I), X(\emptyset)] = i8(\tilde{f}_I - \tilde{g}_I) \in \mathbf{k_R}$. Finally,

$$[Y(\emptyset), X(\emptyset)] = 2^5 i[f_\emptyset, f_S] = -2^4 i\sum_{j=1}^{8} h_j \in \mathbf{k_R}.$$

Hence, $\mathbf{g_R}$ is a Lie algebra. ■

Since $\mathbf{g_R} \otimes \mathbf{C} = \mathbf{g}$, we obtain the following.

Corollary 7.2. $\mathbf{g_R}$ *is a real form of* $\mathbf{e}_{8,\mathbf{C}}$.

We now construct a maximal abelian subalgebra $\mathbf{a}$ of $\mathbf{p_R}$. If $I, J \subset S$ with $|I| = |J| = 4$ and $|I \cap J| = 2$ observe that $[X(I), X(J)] = 0$. Let

$$C = \{\emptyset, \{1,2,3,4\}, \{1, 2.5.6\}, \{1,2,7,8\},$$
$$\{1,3,5,7\}, \{1,3,6,8\}, \{1,4,5,8\}, \{1,4,6,7\}\}.$$

If $\mathbf{a} = \langle X(I) : I \in C\rangle$, $\mathbf{a}$ is a maximal abelian subalgebra of $\mathbf{p_R}$. As $\dim \mathbf{a} = 8$, the following is obvious.

Theorem 7.3. *The Lie algebra* $\mathbf{g_R}$ *is a real split form of* $\mathbf{e}_{8,\mathbf{C}}$. *Thus* $\mathbf{g_R} = \mathbf{e}_{8(8)}$.

Suppose $Tr : \mathbf{h} \to \mathbf{C}$ is the map $Tr(\sum_{j=1}^{8} z_j h_j) = \sum_{j=1}^{8} z_j$ and $\mathbf{h}_0 = \{H : Tr(H) = 0\}$. Setting $\mathbf{k}_0 = \mathbf{h}_0 + \langle f_j \cdot g_k : j \neq k\rangle$, we see that $\mathbf{k}_0 \cong \mathbf{sl}(8, \mathbf{C})$. Now $[\mathbf{k}_0, \bigwedge^l W] = \bigwedge^l W$ and if $\mathbf{p}_0 = \bigwedge^4 W$ it follows from lemmas 7 and 10 that $[\mathbf{p}_0, \mathbf{p}_0] \subset \mathbf{k}_0$. Hence $\mathbf{g}_0 = \mathbf{k}_0 + \mathbf{p}_0$ is a Lie algebra. As $\mathbf{p}_0$ is an irreducible $\mathbf{k}_0$-module and $\mathbf{k}_0$ is simple, $\mathbf{g}_0$ is a simple Lie algebra. Moreover, since $\operatorname{rank}(\mathbf{g}_0) = 7$ and $\dim \mathbf{g}_0 = 133$, $\mathbf{g}_0 = \mathbf{e}_{7,\mathbf{C}}$. Now $\mathbf{g}_{0,\mathbf{R}} = \mathbf{g}_0 \cap \mathbf{g_R}$ is a real form of $\mathbf{g}_0$, and $\mathbf{g}_{0,\mathbf{R}} = \mathbf{k}_{0,\mathbf{R}} + \mathbf{p}_{0,\mathbf{R}}$ where $\mathbf{k}_{0,\mathbf{R}} = \mathbf{k}_0 \cap \mathbf{k_R}$ and $\mathbf{p}_{0,\mathbf{R}} = \mathbf{p}_0 \cap \mathbf{p_R}$. If $C_0 = C \sim \emptyset$, $\mathbf{a}_0 = \langle X_I : I \in C_0\rangle$ is a maximal abelian subalgebra of $\mathbf{p}_{0,\mathbf{R}}$ of dimension 7. Hence we have

Theorem 7.4. *The Lie algebra* $\mathbf{g}_{0,\mathbf{R}}$ *is the split Lie algebra* $\mathbf{e}_{7(7)}$.

We now construct the lie algebra $\mathbf{e}_{6(6)}$.

Let

$$\Omega = f_1 \wedge f_2 + f_3 \wedge f_4 + f_5 \wedge f_6 + f_7 \wedge f_8$$

and

$$\Omega^* = g_1 \wedge g_2 + g_3 \wedge g_4 + g_5 \wedge g_6 + g_7 \wedge g_8.$$

Set $\mathbf{k}_{00} = \{X \in \mathbf{k}_0 : X(\Omega) = 0\}$ and $\mathbf{p}_{00} = \{u \in \mathbf{p}_0 : \iota(\Omega^*)u = 0\}$. Since Ω is a non degenerate 2-form, we obtain the following.

Proposition 7.5. $\mathbf{k}_{00}$ *is the Lie algebra* $\mathbf{sp}(4, \mathbf{C})$.

It is an elementary exercise to see that

$$\mathbf{k}_{00} = \langle h_{2j-1} - h_{2j} : j \leq 4 \rangle +$$

$$\langle f_{2j-1} \cdot g_{2j}, f_{2j} \cdot g_{2j-1} : j \leq 4 \rangle + \langle f_{2j-1} \cdot g_{2k-1} - f_{2k} \cdot g_{2j} : j \neq k \rangle +$$

$$\langle f_{2j-1} \cdot g_{2k} + f_{2k-1} \cdot g_{2j}, f_{2j} \cdot g_{2k-1} + f_{2k} \cdot g_{2j-1} : j < k \rangle.$$

Moreover, restricting ourselves only to $I \subset S$ with $|I| = 4$, we have

$$\mathbf{p}_{00} = \langle f_I : |I \cap \{2j-1, 2j\}| = 1, 1 \leq j \leq 4 \rangle +$$

$$\langle f_I - f_J : |I \cap J| = 2, |I \cap J \cap \{2j-1, 2j\}| \leq 1, 1 \leq j \leq 4,\ \{2k-1, 2k\} \subset I,$$

$$\{2k-1, 2k\} \cap J = \emptyset, \{2l-1, 2l\} \subset J, \{2l-1, 2l\} \cap I = \emptyset \text{ for some } k, l \leq 4 \rangle +$$

$$\langle aX_J + bX_K + cX_L : J = \{1,2,3,4\}, K = \{1,2,5,6\}, L = \{1,2,7,8\}, a+b+c = 0 \rangle.$$

Now an elementary calculation yields

Proposition 7.6. $\mathbf{g}_{00} = \mathbf{k}_{00} + \mathbf{p}_{00}$ *is a Lie algebra.*

Now $\mathbf{g}_{00}$ is easily seen to be simple of dimension 78 with Cartan subalgebra

$$\langle h_{2j-1} - h_{2j} : j \leq 4 \rangle +$$

$$\langle aX_J + bX_K + cX_L : a+b+c = o, J = \{1,2,3,4\}, K = \{1,2,5,6\}, L = \{1,2,7,8\} \rangle.$$

As rank $\mathbf{g}_{00} = 6$, we see that $\mathbf{g}_{00} = \mathbf{e}_{6,\mathbf{C}}$. If we set $\mathbf{g}_{00,\mathbf{R}} = \mathbf{g}_{00} \cap \mathbf{g}_{\mathbf{R}}$, we have $\mathbf{g}_{00,\mathbf{R}}$ is a real form of $\mathbf{e}_{6,\mathbf{C}}$. Since

$$\mathbf{a}_{00} = \langle X_I : I = \{1,3,5,7\}, \{1,3,6,8\}, \{1,4,5,8\}, \{1,4,6,7\} \rangle +$$

$$\langle aX_J + bX_K + cX_L : a, b, c \in \mathbf{R}, a+b+c = 0,$$

$$J = \{1,2,3,4\}, K = \{1,2,5,6\}, L = \{1,2,7,8\} \rangle$$

is a maximal abelian subalgebra of $\mathbf{p}_{00,\mathbf{R}}$, and is of dimension 6, $\mathbf{g}_{00,\mathbf{R}}$ is a split real form of $\mathbf{e}_{8,\mathbf{C}}$. In other words

Theorem 7.7. $\mathbf{g}_{00,\mathbf{R}} = \mathbf{e}_{6(6)}$.

Suppose $\Omega^0 = f_3 \wedge f_4 + f_5 \wedge f_6 + f_7 \wedge f_8$ and $\Omega^{0*} = g_3 \wedge g_4 + g_5 \wedge g_6 + g_7 \wedge g_8$. Set $\mathbf{k}_{000} = \{X \in \mathbf{k}_{00} : X\Omega^0 = \Omega^0\}$ and $\mathbf{p}_{000} = \{u \in \mathbf{p}_{00} : \iota(\Omega^{0*})u = 0\}$.

Proposition 7.8. $\mathbf{k}_{000} \cong \mathbf{sl}(2, \mathbf{C}) \oplus \mathbf{sp}(3, \mathbf{C})$

Proof. A direct calculation we have:

$$\mathbf{k}_{000} = \langle h_1 - h_2, f_1 \cdot g_2, f_2 \cdot g_1 \rangle \oplus$$
$$(\langle h_{2j-1} - h_{2j} : 2 \le j \le 4\rangle + \langle f_{2j-1} \cdot g_{2j}, f_{2j} \cdot g_{2j-1} : 2 \le j \le 4\rangle +$$
$$\langle f_{2j-1} \cdot f_{2k-1} - f_{2k} \cdot g_{2j} : j, k > 1, j \ne k\rangle +$$
$$\langle f_{2j-1} \cdot g_{2k} + f_{2k-1} \cdot g_{2j}, f_{2j} \cdot g_{2k-1} + f_{2k} \cdot g_{2j-1} : 1 < j < k\rangle).$$

Note that the first direct summand is $\mathbf{sl}(2, \mathbf{C})$ and the second is $\mathbf{sp}(3, \mathbf{C})$. ■

Similarly, we have

$$\mathbf{p}_{000} = \langle f_I : |I \cap \{2j-1, 2j\}| = 1, 1 \le j \le 4\rangle +$$
$$\langle f_I - f_J : |I \cap J| = 2, |I \cap \{1,2\}| = 1, I \cap \{1,2\} = J \cap \{1,2\}, \{2j-1, 2j\} \subset I,$$
$$\{2j-1, 2j\} \cap J = \emptyset, \{2k-1, 2k\} \subset J, \{2k-1, 2k\} \cap I = \emptyset \text{ for some j, k} >$$

Now $\mathbf{g}_{000} = \mathbf{k}_{000} + \mathbf{p}_{000}$ is a simple Lie algebra of rank 4 and dimension 52. Setting $\mathbf{g}_{000,\mathbf{R}} = \mathbf{g}_{000} \cap \mathbf{g}_{00,\mathbf{R}}$ we have

$$\mathbf{a}_{000} = \langle X_I : I = \{1,3,5,7\}, \{1,3,6,8\}, \{1,4,5,8\}, \{1,4,6,7\}\rangle$$

is a maximal abelian subalgebra of, $\mathbf{p}_{000,\mathbf{R}}$. Thus we have

Theorem 7.9. $\mathbf{g}_{000} = \mathbf{f}_{4,\mathbf{C}}$, *and* $\mathbf{g}_{000,\mathbf{R}} = \mathbf{f}_{4(4)}$.

8. The Groups $\tilde{E}_{6(6)}, \tilde{E}_{7(7)}$, $\tilde{E}_{8(8)}$, and $\tilde{F}_{4(4)}$

From Bourbaki [4], we know that $Z(E_{6,\mathbf{C}}) = \mathbf{Z}_3$, $Z(E_{7,\mathbf{C}}) = \mathbf{Z}_2$, and $Z(E_{8,\mathbf{C}}) = \{I\}$. Since $Z(\tilde{E}_{8(8)}) = \{1, e_1 \cdots e_{16}\} \cong \mathbf{Z}_2$, $\tilde{E}_{8(8)}$ is not a linear group. However, $\tilde{E}_{8(8)}/Z(\tilde{E}_{8(8)}) = E_{8(8)}$ is a linear group and hence so are all of its subgroups. Also, note from [4] that $Z(F_{4,\mathbf{C}}) = \{1\}$.

Proposition 8.1. *The analytic subgroup of* $\tilde{E}_{8(8)}$ *with Lie algebra* $\mathbf{e}_{7(7)}$ *is* $\tilde{E}_{7(7)}$.

Proof. Let G_0 be the analytic subgroup of $\tilde{E}_{8(8)}$ with Lie algebra $\mathbf{e}_{7(7)}$. Now the maximal compact subgroup of $\tilde{E}_{7(7)}$ is $SU(8)$, and the center of $\tilde{E}_{7(7)}$ is $\mathbf{Z}_4$. If $u_j = (1 + e_{2j-1} \cdot e_{2j})/\sqrt{2}$ and $z = u_1 \cdots u_8$, we have $z \cdot f_j \cdot z^{-1} = i f_j$ and $z \cdot g_j \cdot z^{-1} = -i g_j$ and so z centralizes $\mathbf{e}_{7(7)}$. As $z^2 = e_1 \cdots e_{16}$, it suffices to show $z \in G_0$. Since $\exp(\pi i/2) h_j = u_j$, $u_j^4 = 1$, and $\mathbf{h}_0 = \{H \in \mathbf{h} : Tr(H) = 0\}$ is a Cartan sualgebra of $\mathbf{e}_{7(7)}$, it follows that $u_1^{-1} u_j \in G_0$ for any $j \le 8$. Hence $(u_1^{-1} \cdot u_2) \cdots (u_1^{-1} u_8) = z \in G_0$ and so $G_0 = \tilde{E}_{7(7)}$. ■

Note that since $Z(\tilde{E}_{7(7)}) = \mathbf{Z}_4$ and $Z(E_{7,\mathbf{C}}) = \mathbf{Z}_2$, $\tilde{E}_{7(7)}$ is not a linear group.

Proposition 8.2. *The analytic subgroup of* $\tilde{E}_{7(7)}$ *with Lie algebra* $\mathbf{e}_{6(6)}$ *is* $\tilde{E}_{6(6)}$.

Proof. Let G_{00} be the analytic subgroup of $\tilde{E}_{7(7)}$ with Lie algebra $\mathbf{e}_{6(6)}$. Since the maximal compact subgroup of $\tilde{E}_{6(6)}$ is $Sp(4)$, $Z(\tilde{E}_{6(6)}) = \mathbf{Z}_2$. It thus suffices to show that $z^2 \in G_{00}$. Since $\langle h_{2j-1} - h_{2j} : j \leq 4\rangle \subset \mathbf{k}_{00,\mathbf{R}}$,
$\exp(\pi i)\sum_{j=1}^{4}(h_{2j-1} - h_{2j}) = z^2 \in G_{00}$ and hence $G_{00} = \tilde{E}_{6(6)}$. ■

Again since $Z(E_{6,\mathbf{C}}) = \mathbf{Z}_3$, $\tilde{E}_{6(6)}$ is not a linear group.

Remark. We have $Z(\tilde{E}_{8(8)}) \subset \tilde{E}_{6(6)} \subset \tilde{E}_{7(7)} \subset \tilde{E}_{8(8)}$ with no $\tilde{E}_{j(j)}$ linear. However, all $\tilde{E}_{j(j)}/Z(\tilde{E}_{8(8)})$ are linear. As $Z(\tilde{F}_{4(4)}) \cong \mathbf{Z}_2$, $\tilde{F}_{4(4)}$ is not a linear group. Since $Z(\tilde{F}_{4(4)}) = Z(\tilde{E}_{8(8)})$, $\tilde{F}_{4(4)}/Z(\tilde{E}_{8(8)}) = F_{4(4)}$ is a linear group.

Since $\sum_{j=1}^{4} i(h_{2j-1} - h_{2j}) \in \mathbf{f}_{4(4)}$ and $\exp\pi\sum_{j=1}^{4} i(h_{2j-1} - h_{2j}) = e_1\cdots e_{16}$, the analytic subgroup of $\tilde{E}_{6(6)}$ having Lie algebra $\mathbf{f}_{4(4)}$, is $\tilde{F}_{4(4)}$. We state the following.

Proposition 8.3. *The analytic subgroup of $\tilde{E}_{6(6)}$ having Lie algebra $\mathbf{f}_{4(4)}$ is $\tilde{F}_{4(4)}$.*

9. The Group $\tilde{M}_j$ for the Groups $\tilde{E}_{j(j)}$ and $\tilde{F}_{4(4)}$

Recall $\mathbf{a}$, $\mathbf{a}_0$, $\mathbf{a}_{00}$, and $\mathbf{a}_{000}$ from section 7. For convenience set $\mathbf{a}_8 = \mathbf{a}$, $\mathbf{a}_7 = \mathbf{a}_0$, $\mathbf{a}_6 = \mathbf{a}_{00}$ and $\mathbf{a}_4 = \mathbf{a}_{000}$. Let $\tilde{K}_j$ be the maximal compact subgroup of $\tilde{E}_{j(j)}$ for $j = 6, 7$ or 8 or of $\tilde{F}_{4(4)}$ for $j = 4$; set $\tilde{M}_j = \tilde{K}_j \cap Z(\mathbf{a}_j)$. We now give an explicit description of the groups $\tilde{M}_j$. Recall from [10] that $\tilde{M}_j/Z(\tilde{E}_{8(8)}) \cong (\mathbf{Z}_2)^j$ and hence $|\tilde{M}_j| = 2^{j+1}$.

Consider the element $* = e_1{\cdot}e_3{\cdot}\cdot{\cdot}e_{15}$ of $\tilde{K}_8 = Spin(16)$ and note that $*^2 = 1$. A simple calculation yields the following.

Proposition 9.1. *If $I \subset S$, $\gamma(*)f_I = (-1)^{\sigma(I)+|I|}2^{(|I|-|I'|)/2}f_{I'}$*

Corollary 9.2. $* \in \tilde{M}_8$.

Recall that

$$C = \{\varnothing, \{1,2,3,4\}, \{1,2.5.6\}, \{1,2,7,8\},$$

$$\{1,3,5,7\}, \{1,3,6,8\}, \{1,4,5,8\}, \{1,4,6,7\}\}$$

and $\mathbf{a} = \langle X_I : I \in C\rangle$. Note that for $I \in C$, $\sigma(I) = 1$ and so $X_I = 4(f_I + f_{I'})$. We first give an explicit description of $\tilde{M}_7$. If P_8 denotes the permutation group on 8 elements $P_8 \subset U(8)$ and the elements of even order are in $SU(8)$.Suppose

$P = \{(12)(34)(56)(78), (13)(24)(57)(68), (14)(23)(58)(67), (15)(26)(37)(48), (16)(25)(38)(47), (17)(28)(35)(46), (18)(27)(36)(45), I\} \subset P_8$.

It is easy to see that P is an abelian group of order 8 with every element other than the identity of order 2.

Now let Q be the group generated by the following elements:

$\omega_1 = iI$, $\omega_2 = diag(-1,-1,1,1,1,1,-1,-1)$,
$\omega_3 = diag(-1,-1,1,1,-1,-1,1,1)$, and $\omega_4 = diag(-1,1,-1,1,-1,1,-1,1)$.

Proposition 9.3. $\tilde{M}_7 = PQ$ *and* $[\tilde{M}_7, \tilde{M}_7] = Z(\tilde{E}_{8(8)})$. *It has precisely two inequivalent irreducible representations of dimension 8, and all other irreducible representations are one dimensional.*

Proof. If $\omega_1^{k_1}\omega_2^{k_2}\omega_3^{k_3}\omega_4^{k_4} = 1$, the (4,4)-entry is 1 and so it follows that $\omega_1^{k_1} = 1$. Now, since the (3,3)-entry is 1, $\omega_4^{k_4} = 1$. Finally, since the (8,8)-entry is 1, $\omega_2^{k_2} = 1$ and hence $\omega_3^{k_3} = 1$. Thus Q is a an abelian group of order 32. Now $\omega_1^2 = e_1 \cdots e_{16} = -I$, and observe that for $p \in P$ and $q \in Q$, $pq = \pm qp$. Now $P \subset \tilde{M}_7$ and $Q \subset \tilde{M}_7$ and hence $PQ = \{pq : p \in P, q \in Q\}$ is a subgroup of $\tilde{M}_7$. Since $|PQ| = |\tilde{M}_7| = 2^8$, $PQ = |\tilde{M}_7|$. Now $\tilde{M}_7$ is a subgroup of $SU(8)$ and both $W \cong \mathbf{C}^8$ and $\bigwedge^7 W \cong (\mathbf{C}^8)^*$ are irreducible inequivalent $\tilde{M}_7$-modules. As $|\chi(\tilde{M}_7)| = 2^7$ and $|\tilde{M}_7| = 8^2 + 8^2 + 2^7 = 2^8$, these are the only irreducible modules of $\tilde{M}_7$ of dimension > 1. ∎

We now describe $\tilde{M}_8$.

Proposition 9.4. $\tilde{M}_8 = \tilde{M}_7 \cup *\tilde{M}_7$ *and* $[\tilde{M}_8, \tilde{M}_8] = Z(\tilde{E}_{8(8)})$. *It has –up to equivalence– precisely one irreducible representation of dimension 16, and all other irreducible representations are one dimensional.*

Proof. Now $\tilde{M}_7 \subset \tilde{M}_8$ and $* \in \tilde{M}_8$. Since $|\tilde{M}_8| = 2^9$, we have $\tilde{M}_8 = \tilde{M}_7 \cup *\tilde{M}_7$. Moreover, since $\tilde{M}_8 / Z(\tilde{E}_{8(8)}) \cong (\mathbf{Z}_2)^8$ and $\tilde{M}_8$ is not abelian, we have $[\tilde{M}_8, \tilde{M}_8] = \{1, e_1 \cdots e_{16}\}$. Since $* : \bigwedge^r W \to \bigwedge^{8-r} W$ and W and $\bigwedge^7 W$ are inequivalent irreducible $\tilde{M}_7$-modules, $W + \bigwedge^7 W$ is an irreducible 16-dimensional $\tilde{M}_8$-module. As $|\chi(\tilde{M}_8)| = 2^8$ and $|\tilde{M}_8| = 2^9 = 2^8 + (16)^2$ we see that this –up to equivalence– is the only irreducible $\tilde{M}_8$-module of dimension > 1. Observe that for $p \in P$ with $p(1) = j$ that

$$\gamma(*p)(f_1) = \gamma(*)(f_j) = (-1)^{j+1} 2^{-3} f_{\{j\}'}$$

and

$$\gamma(p*)(f_1) = 2^{-3}\gamma(p) f_{\{1\}'} = 2^{-3}(-1)^{j+1} f_{\{j\}'}.$$

Hence $[*, p] = 1$ for any $p \in P$. Similarly, we obtain $[*, \omega_j] = 1$ for $2 \le j \le 4$, and $[*, \omega_1] = -I = e_1 \cdots e_{16}$. The remaining commutation relations in $\tilde{M}_8$ are easy to compute. ∎

Proposition 9.5. $\tilde{M}_6 = \{m \in \tilde{M}_7 : \wedge^2\gamma(m)\Omega = \Omega\}$, *and – up to equivalence – $\tilde{M}_6$ has exactly one irreducible representation of dimension > 1.*

Proof. Let $F = \{m \in \tilde{M}_7 : \wedge^2\gamma(m)\Omega = \Omega\}$. Clearly, F is a subgroup of $\tilde{M}_6$. Now

$$P' = F \cap P = \{(13)(24)(57)(68), (15)(26)(37)(48), (17)(28)(35)(46), 1\}$$

a subgroup of order 4, and $Q' = F \cap Q$ a subgroup of Q generated by $\omega_1^2, \omega_2, \omega_3$ and $\omega_1\omega_4$ and the order of Q' is 16. So $P'Q'$ is a subgroup of F of order 2^6.

Setting $\tau = (12)(34)(56)(78)$, we see that $\tau\omega_1 \in F$. Thus $P'Q' \cup \tau\omega_1 P'Q'$ is a subgroup of F of order 2^7. Hence we must have $P'Q' \cup \tau\omega_1 P'Q' = F = \tilde{M}_6$. Moreover, up to equivalence, W is the only irreducible $\tilde{M}_6$-module of dimension > 1. ∎

We now construct the group $\tilde{M}_4$. Let $L = \{m \in \tilde{M}_6 : \wedge^2\gamma(m)\Omega^* = \Omega^*\}$. Clearly, $L \subset M_4$ and so to prove $L = \tilde{M}_4$, it suffices to prove $|L| = 2^5$. Now $\tau\omega_1, \tau\omega_4, \omega_2, \omega_3 \in L$ with $(\tau\omega_1)^2 = (\tau\omega_4)^2 = -1$ and $\omega_2^2 = \omega_3^2 = 1$. It is easy to see that the group $\langle \tau\omega_1, \omega_2, \omega_3\rangle$ is a subgroup of L of order 16. Since $\tau\omega_4$ is not in this group we have that $|L| = 32$.

Proposition 9.6. *$\tilde{M}_4 = L$, $[\tilde{M}_4, \tilde{M}_4] = \{\pm I\}$ and $\tilde{M}_4$ has – up to equivalence – 4 irreducible inequivalent representations of degree 2. The remaining representations are one dimensional.*

Proof. The fact that $\tilde{M}_4 = L$ and $[\tilde{M}_4, \tilde{M}_4] = \{\pm 1\}$ are clear. It remains only to exhibit 4 inequivalent irreducible modules of dimension 2. The spaces $\langle f_1, f_2\rangle$, $\langle f_3, f_4\rangle, \langle f_5, f_6\rangle$, and $\langle f_7, f_8\rangle$ are all $\tilde{M}_4$-modules. Note that $Z(\tilde{M}_4) = \{\pm 1, \pm\omega_2, \pm\omega_3, \pm\omega_2\omega_3\}$ acts on each of these spaces with a different character. Therefore these modules are inequivalent $\tilde{M}_4$-modules. ∎

Remark. It is easy to see that $\tilde{M}_j \neq D_{j+1}$ for $j = 4, 6, 7$, or 8.

10. The Construction of $\mathbf{g}_{2(2)}$ and $\tilde{G}_{2(2)}$

Let

$$\mathbf{k}' = \bigoplus_{j=1}^{4} \langle f_{2j-1} \cdot g_{2j}, f_{2j} \cdot g_{2j-1}, h_{2j-1} - h_{2j}\rangle$$

and

$$\mathbf{p}' = \langle f_I : |I| = 4, |I \cap \{2j-1, 2j\}| = 1, 1 \leq j \leq 4\rangle.$$

Now $\mathbf{k}' \subset \mathbf{k}_{000}$ and $\mathbf{p}' \subset \mathbf{p}_{000}$. It is easy to see that $\mathbf{g}' = \mathbf{k}' + \mathbf{p}'$ is a simple Lie subalgebra of $\mathbf{f}_{4,\mathbf{C}}$ and, in fact, $\mathbf{g}' \cong \mathbf{s}o(8, \mathbf{C})$. Consider the linear map $\sigma : \mathbf{g}' \to \mathbf{g}'$ defined as follows:

$f_1 \cdot g_2 \to f_1 \cdot g_2, f_2 \cdot g_1 \to f_2 \cdot g_1$, $h_1 - h_2 \to h_1 - h_2$,
$f_3 \cdot g_4 \to f_5 \cdot g_6 \to f_7 \cdot g_8 \to f_3 \cdot g_4$, $f_4 \cdot g_3 \to f_6 \cdot g_5 \to f_8 \cdot g_7 \to f_4 \cdot g_3$,
$h_3 - h_4 \to h_5 - h_6 \to h_7 - h_8 \to h_3 - h_4$,
and for $1 \leq j \leq 2$
$f_{\{j,3,5,7\}} \to f_{\{j,3,5,7\}}$, $f_{\{j,4,6,8\}} \to f_{\{j,4,6,8\}}$,
$f_{\{j,4,5,7\}} \to -f_{\{j,3,6,7\}} \to f_{\{j,3,5,8\}} \to f_{\{j,4,5,7\}}$,
$f_{\{j,3,6,8\}} \to -f_{\{j,4,5,8\}} \to f_{\{j,4,6,7\}} \to f_{\{j,3,6,8\}}$.

Proposition 10.1. *The map $\sigma : \mathbf{g}' \to \mathbf{g}'$ is a Lie algebra automorphism.*

Proof. Clearly, $\sigma : \mathbf{k}' \to \mathbf{k}'$ is a Lie algebra homomorphism, and it is also easy to see that $\sigma : \mathbf{p}' \to \mathbf{p}'$ intertwines the action of $\mathbf{k}'$. It suffices to show that $\sigma([f_I, f_J]) = [\sigma(f_I), \sigma(f_J)]$ for any $f_I, f_J \in \mathbf{p}'$. We recall the following identities:

$[f_{\{1,3,5,7\}}, f_{\{2,4,6,8\}}] = 1/2 \sum_{j=1}^{4}(h_{2j-1} - h_{2j})$,

$$[f_{\{1,3,5,7\}}, f_{\{1,4,6,8\}}] = 1/2 f_1 \cdot g_2\,,$$
$$[f_{\{1,3,5,7\}}, f_{\{2,3,6,8\}}] = -1/2 f_3 \cdot g_4\,,$$
$$[f_{\{1,3,5,7\}}, f_{\{2,4,5,8\}}] = 1/2 f_5 \cdot g_6\,,$$
$$[f_{\{1,3,5,7\}}, f_{\{2,4,5,7\}}] = -1/2 f_7 \cdot g_8\,.$$

It follows that $[\sigma(f_{\{1,3,5,7\}}), \sigma(f_J)] = \sigma([f_{\{1,3,5,7\}}, f_J])$ for any $f_J \in \mathbf{p}'$. Thus, if we fix $u = f_{\{1,3,5,7\}}$, we have $[u, \sigma(v)] = [\sigma(u), \sigma(v)] = \sigma([u,v])$ for any $v \in \mathbf{p}'$. From the Jacobi identity and the facts that σ is a Lie algebra automorphism of $\mathbf{k}'$ and a $\mathbf{k}'$ intertwining operator on $\mathbf{p}'$ we have $[\sigma([x,u]), \sigma(v)] = \sigma([[x,u],v])$ for any $x \in \mathbf{k}'$ and any $v \in \mathbf{p}'$, and the proof follows. ■

Our above discussion yields the following result.

Proposition 10.2. $\mathbf{g}'^{\sigma} = \mathbf{k}'^{\sigma} + \mathbf{p}'^{\sigma}$ *is a Lie algebra.*

Now $\mathbf{k}'^{\sigma} = \langle f_1 \cdot g_2, f_2 \cdot g_1, h_1 - h_2 \rangle \oplus \langle \zeta, \eta, \omega \rangle$ where

$$\zeta = \sum_{j=2}^{4} f_{2j-1} \cdot g_{2j}, \quad \eta \sum_{j=2} \cdot f_{2j} \cdot g_{2j-1}, \text{ and } \omega = \sum_{j=2}^{4} (h_{2j-1} - h_{2j}).$$

The following elements are a basis for $\mathbf{p}'^{\sigma}$.
$f_{\{1,3,5,7\}}, f_{\{1,4,5,7\}} - f_{\{1,3,6,7\}} + f_{\{1,3,5,8\}},\ f_{\{1,3,6,8\}} - f_{\{1,4,5,8\}} + f_{\{1,4,6,7\}}, f_{\{1,4,6,8\}},$
$f_{\{2,3,5,7\}}, f_{\{2,4,5,7\}} - f_{\{2,3,6,7\}} + f_{\{2,3,5,8\}},\ f_{\{2,3,6,8\}} - f_{\{2,4,5,8\}} + f_{\{2,4,6,7\}}, f_{\{2,4,6,8\}}.$

So $\mathbf{g}'^{\sigma}$ is a Lie algebra of rank 2 and dimension 14. Since it is clearly simple we have a direct proof of the folowing.

Theorem 10.3. $\mathbf{g}'^{\sigma} = \mathbf{g}_{2,\mathbf{C}}$.

Let L be the analytic subgroup of $\tilde{F}_{4(4)}$ having Lie algebra $\mathbf{g}_{2(2)} = \mathbf{g}_{2.\mathbf{C}} \cap \mathbf{f}_{4(4)}$. Since $Z(G_{2,\mathbf{C}}) = \{1\}$, $Z(G_{2(2)}) \cong \mathbf{Z}_2$, and $e_1 \cdots e_{16} \in Z(L)$, we see that $L = \tilde{G}_{2(2)}$ and $Z(\tilde{G}_{2(2)}) = Z(\tilde{E}_{8(8)})$.

If $\mathbf{g}_{2(2)} = \mathbf{k}_2 + \mathbf{p}_2$ is the corresponding Cartan decomposition of $\mathbf{g}_{2(2)}$,

$$\mathbf{a}_2 = \langle X_{\{1,3,5,7\}}, X_{\{1,3,6,8\}} - X_{\{1,4,5,8\}} + X_{\{1,4,6,7\}} \rangle$$

is a maximal abelian subalgebra of $\mathbf{p}_2$. Let $\tilde{K}_2$ be the maximal compact subgroup of $\tilde{G}_{2(2)}$, and set $\tilde{M}_2 = Z(\mathbf{a}_2) \cap \tilde{K}_2$. Recall that $\tilde{M}_4 = \langle \tau\omega_1, \tau\omega_4, \omega_2, \omega_3 \rangle$. If G' is the analytic subgroup of $\tilde{F}_{4(4)}$ having Lie algebra $\mathbf{g}' \cap \mathbf{f}_{4(4)}$, we have that $\tilde{M}_4 \subset G'$, and so $\tilde{M}_4^{\sigma}$ is a subgroup of $\tilde{M}_2$. It is easy to see that $\sigma(\omega_2) = \omega_3$, $\sigma(\omega_3) = \omega_2\omega_3$, $\sigma(\tau\omega_1) = \tau\omega_1$, and $\sigma(\tau\omega_4) = \tau\omega_4$. Hence $\tilde{M}_4^{\sigma} = \langle \tau\omega_1, \tau\omega_4 \rangle$.

Proposition 10.4. $\tilde{M}_2 = \tilde{M}_4^{\sigma}$ *is a nonabelian group with one – up to equivalence – irreducible representation of degree 2.*

Proof. Since $\tilde{M}_4^{\sigma}$ is of order 8, we have our equality. Since $\tau\omega_1\tau\omega_4 = -\tau\omega_4\tau\omega_1$, $\tilde{M}_2$ is not abelian. Moreover, as $\tilde{M}_2/Z(\tilde{G}_{2(2)}) \cong \mathbf{Z}_2 \times \mathbf{Z}_2$, our result follows. ■

Remark. It is easy to see that $\tilde{M}_2 = D_3$.

This concludes our examination of the universal covering groups of the split simple exceptional groups.

11. $\mathbf{e}_{8(-24)}$ and $\tilde{E}_{8(-24)}$

Recall $\mathbf{e}_{8,\mathbf{C}} = \mathbf{s}o(16,\mathbf{C}) + \bigwedge^e W$, $\mathbf{e}_{7,\mathbf{C}} = \mathbf{s}l(8,\mathbf{C}) + \bigwedge^4 W$, and $\mathbf{sl}(2,\mathbf{C}) = \langle \sum_{j=1}^{8} h_j, f_\varnothing, f_S\rangle$. Set
$\widehat{\mathbf{k}}_\mathbf{C} = \mathbf{e}_{7,\mathbf{C}} \oplus \mathbf{s}l(2,\mathbf{C})$, and $\widehat{\mathbf{p}}_\mathbf{C} = \mathbf{F} + \mathbf{G} + \bigwedge^2 W + \bigwedge^6 W$, where
$\mathbf{F} = \langle f_j \cdot f_k : j < k\rangle$, $\mathbf{G} = \langle g_j \cdot g_k : j < k\rangle$.
We now have the following commutation relations.

$$[\mathbf{F},\mathbf{F}] = 0,\ [\mathbf{F}, \textstyle\bigwedge^j W] \subset \bigwedge^{j+2} W,\ [\mathbf{F},\mathbf{G}] \subset \mathbf{s}l(8,\mathbf{C}),\ [\mathbf{G},\mathbf{G}] = 0,$$

$$[\mathbf{G}, \bigwedge^j W] \subset \bigwedge^{j-2} W, \quad [\bigwedge^2 W, \bigwedge^2 W] = 0, \quad [\bigwedge^6 W, \bigwedge^6 W] = 0,$$

and

$$[\bigwedge^2 W, \bigwedge^6 W] \subset \mathbf{s}l(2,\mathbf{C}) \oplus \mathbf{s}l(8,\mathbf{C}).$$

Thus $[\widehat{\mathbf{p}}_\mathbf{C}, \widehat{\mathbf{p}}_\mathbf{C}] \subset \widehat{\mathbf{k}}_\mathbf{C}$, and $[\widehat{\mathbf{k}}_\mathbf{C}, \widehat{\mathbf{p}}_\mathbf{C}] \subset \widehat{\mathbf{p}}_\mathbf{C}$. Hence $(\widehat{\mathbf{k}}_\mathbf{C}, \widehat{\mathbf{p}}_\mathbf{C})$ is a symmetric pair for $\mathbf{e}_{8,\mathbf{C}}$. Finally, if $\widehat{\mathbf{k}} = \widehat{\mathbf{k}}_\mathbf{C} \cap (\mathbf{k}_\mathbf{R} + i\mathbf{p}_\mathbf{R})$ and $\widehat{\mathbf{p}} = \widehat{\mathbf{p}}_\mathbf{C} \cap (i\mathbf{k}_\mathbf{R} + \mathbf{p}_\mathbf{R})$, the Lie algebra $\widehat{\mathbf{g}} = \widehat{\mathbf{k}} + \widehat{\mathbf{p}}$ is a real form of $\mathbf{e}_{8,\mathbf{C}}$. If $\widehat{G}$ is the simply connected analytic group with Lie algebra $\widehat{\mathbf{g}}$, $\widehat{G}$ has maximal compact subgroup $\widehat{K} = E_{7(-133)} \times SU(2)$. Thus $\mathbf{e}_{8(-24)} = \widehat{\mathbf{g}}$ and $\widehat{G} = \tilde{E}_{8(-24)}$. The subgroup of $SU(2)$ having Lie algebra $\langle ih_0\rangle$ where $h_0 = \sum_{j=1}^{8} h_j$ is

$$\langle \exp tih_0 : t \in \mathbf{R}\rangle / \langle \omega_1^2\rangle (\omega_1 = \exp \pi i h_0),$$

and the subgroup of $E_{7(-133)}$ having Lie algebra $\mathbf{su}(8)$ is $SU(8)/\{\pm I\}$. Now $Z(SU(2)) \cong \mathbf{Z}_2$ and $Z(E_{7(-133)}) \cong \mathbf{Z}_2$ (see [4]). A simple calculation thus shows that

$$Z(SU(2)) = \langle \omega_1\rangle / \langle \omega_1^2\rangle = \{\pm I_2\}, \quad Z(E_{7(-133)}) = \langle iI_8\rangle / \{\pm I_8\},$$

and

$$Z(\tilde{E}_{8(-24)}) = \langle (iI_8, \omega_1)\rangle / \langle (-I_8, \omega_1^2)\rangle \cong \mathbf{Z}_2.$$

Thus $\tilde{E}_{8(-24)}$ is not a linear group.

If $\widehat{\mathbf{a}} = \langle X_{\{1,2\}}, X_{\{3,4\}}, X_{\{5,6\}}, X_{\{7,8\}}\rangle$, $\widehat{\mathbf{a}}$ is a maximal abelian subalgebra of $\widehat{\mathbf{p}}$. Then $\widehat{\mathbf{k}} \cap \mathbf{z}(\widehat{\mathbf{a}}) = \mathbf{k}' + i\mathbf{p}' \cong \mathbf{so}(8)$ where $\mathbf{k}'$ and $\mathbf{p}'$ are as defined in section 10. Set $\widehat{M} = \widehat{K} \cap Z(\widehat{\mathbf{a}})$ and let $\widehat{M_0}$ be the connected component of the identity. It is easy to see that the subgroup $\widehat{M_0}$ of $\widehat{K}$ or of $\widehat{K}/Z(\widehat{G})$ having Lie algebra $\widehat{\mathbf{k}} \cap \mathbf{z}(\widehat{\mathbf{a}})$ is $Spin(8)$. Hence from [10] we see that if $\widehat{M} = Z(\widehat{\mathbf{a}}) \cap \widehat{K}$, $|\widehat{M}/\widehat{M_0}| = 8$. Clearly, $Z(\tilde{E}_{8(-24)}) \subset \widehat{M}$ and $Z(\tilde{E}_{8(-24)}) \cap \widehat{M_0} = \{1\}$. To analyze the group $\widehat{M}$ further we recall some results from [7] and [10].

Suppose Σ_r is the set of restricted roots and let Φ_r be the set of simple restricted roots. Then from [7] we have that $\widehat{M}/Z(\widehat{G}) = Z_1 \cdot \widehat{M_0}$ where

$$Z_1 = \langle \exp 2\pi i H_\alpha/|\alpha|^2 : \alpha \in \Sigma_r\rangle = \langle \exp 2\pi i H_\alpha/|\alpha|^2 : \alpha \in \Phi_r\rangle.$$

Moreover, since the Satake diagram of $\mathbf{e}_{8(-24)}$ has four white dots and only two white dots are not adjacent to any black dots, we have from [10] that $Z_1 = \langle \exp 2\pi i H_\alpha/|\alpha|^2 : \alpha \in \Phi_r$ and α is long $\rangle \cong \mathbf{Z}_2 \times \mathbf{Z}_2$.

For $j \leq 4$ let $\epsilon_j \in \widehat{\mathbf{a}}^*$ be the map

$$\epsilon_j(a_1 X_{\{1,2\}} + a_2 X_{\{3,4\}} + a_3 X_{\{5,6\}} + a_4 X_{\{7,8\}}) = a_j.$$

An elementary but tedious calculation shows that

$$\Sigma_r = \{\pm\ 4\epsilon_j \pm\ 4\epsilon_k : j < k \leq 4\} \cup \{\pm 8\epsilon_j : j \leq 4\} \cup \{\pm 4\epsilon_1 \pm 4\epsilon_2 \pm 4\epsilon_3 \pm 4\epsilon_4\}$$

and $\Phi_r = \{4\epsilon_1 - 4\epsilon_2 - 4\epsilon_3 - 4\epsilon_4, 8\epsilon_4, 4\epsilon_3 - 4\epsilon_4, 4\epsilon_2 - 4\epsilon_3\}$.

The 24 long restricted roots are all of multiplicity 1, and the 24 short restricted roots are all of multiplicity 8. Setting $\alpha_1 = 4\epsilon_1 - 4\epsilon_2 - 4\epsilon_3 - 4\epsilon_4$ another routine calculation yields $H_{\alpha_1}/|\alpha_1|^2 = 1/16(X_{\{1,2\}} - X_{\{3,4\}} - X_{\{5,6\}} - X_{\{7,8\}})$ and $H_{4\epsilon_4}/|4\epsilon_4|^2 = 1/8 X_{\{7,8\}}$. If x is in the preimage of $\exp(1/8)\pi i(X_{\{1,2\}} - X_{\{3,4\}} - X_{\{5,6\}} - X_{\{7,8\}})$ in $\widehat{M}$ and y is in the preimage of $\exp(1/4)\pi i X_{\{7,8\}}$, we have that $Ad(x)$ and $Ad(y)$ are both non trivial on $\mathbf{sl}(2,\mathbf{C})$. Hence $x^2 = y^2$ is the nontrivial element of $Z(\tilde{E}_{8(-24)})$, and $x^4 = y^4 = 1$. Since $Ad(x)16 f_\emptyset = -f_S$, $Ad(x) f_S = -16 f_\emptyset$, $Ad(y) f_\emptyset = -f_\emptyset$ and $Ad(y) f_S = -f_S$, we have $xyx^{-1} = y^{-1}$. If $\widehat{Z}_1$ is the preimage of Z_1 in $\tilde{E}_{8(-24)}$ $\widehat{Z}_1 = \langle x, y\rangle$ is a nonabelian group of order 8 centralized by $\widehat{M_0}$, and so we have the following proposition.

Proposition 11.1. $\widehat{M} = \widehat{Z}_1 \cdot \widehat{M_0}$ *and* $\widehat{Z}_1 \cap \widehat{M_0} = \{1\}$.

Corollary 11.2. $\widehat{M} = \widehat{Z}_1 \times \widehat{M_0}$.

Remark. Note that $\widehat{Z}_1 \cong D_3$ is the quaternionic group.

12. The real form $\mathbf{e}_{7(-5)}$ and $\tilde{E}_{7(-5)}$

Recall $\mathbf{e}_{7,\mathbf{C}} = \mathbf{sl}(8,\mathbf{C}) + \bigwedge^4 W$. Fix $W_1 = \langle f_1, f_2, f_3, f_4, f_5, f_6\rangle$, $W_2 = \langle f_7, f_8\rangle$, and let $\mathbf{gl}(6,\mathbf{C})$ be the Lie subalgebra of $\mathbf{sl}(8,\mathbf{C})$ generated by all elements of the form $f_j \cdot g_k$ where $j \neq k \leq 6$, and all sums of the form $\sum_{j=1}^8 a_j h_j$ where $a_7 = a_8$ and $\sum_{j=1}^8 a_j = 0$. Now

$$\bigwedge^4 W = \bigwedge^4 W_1 \oplus (\bigwedge^3 W_1 \otimes W_2) \oplus (\bigwedge^2 W_1 \otimes \bigwedge^2 W_2).$$

If we now set

$$\mathbf{k}_{0,\mathbf{C}} = \mathbf{g}l(6,\mathbf{C}) + \bigwedge^4 W_1 + \bigwedge^2 W_1 \otimes \bigwedge^2 W_2,$$

we have that $\mathbf{k}_{0,\mathbf{C}}$ is a simple Lie algebra of dimension 66 and rank 6. Hence $\mathbf{k}_{0,\mathbf{C}} = \mathbf{so}(12,\mathbf{C})$. Set also

$$\mathbf{sl}(2,\mathbf{C}) = \langle f_7 \cdot g_8, f_8 \cdot g_7, h_7 - h_8\rangle, \mathbf{k}_{\mathbf{C}} = \mathbf{k}_{0,\mathbf{C}} \oplus \mathbf{sl}(2,\mathbf{C}),$$

and

$$\mathbf{p}_{\mathbf{C}} = \langle f_j \cdot g_k : j \leq 6 < k \quad \text{or} \quad j > 6 \geq k > + \bigwedge^3 W_1 \otimes W_2.$$

Then $(\mathbf{k}_{\mathbf{C}}, \mathbf{p}_{\mathbf{C}})$ is a symmetric decomposition of $\mathbf{e}_{7,\mathbf{C}}$, and if $\mathbf{g} = \mathbf{k} + \mathbf{p}$ is the corresponding real form, $\mathbf{g} = \mathbf{e}_{7(-5)}$. Let $\tilde{E}_{7(-5)}$ be the corresponding real Lie group with maximal compact subgroup $K = Spin(12) \times SU(2)$. Now $Z(K) = \mathbf{Z}_2^3$ and by examining the representation of K on $\mathbf{p}$, we see $Z(E_{7(-5)} = \mathbf{Z}_2^2$. Thus $E_{7(-5)}$ is not a linear group.

Setting

$$H_1 = X_{\{1,3,5,7\}}, H_2 = X_{\{1,3,6,8\}}, H_3 = X_{\{1,4,5,8\}}, H_4 = X_{\{1,4,6,7\}},$$

$\mathbf{a} = \langle H_1, H_2, H_3, H_4 \rangle$ is easily seen to be a maximal abelian subalgebra of $\mathbf{p}$. For $j \leq 4$ let $\epsilon_j \in \mathbf{a}^*$ be the map

$$\epsilon_j(a_1H_1 + a_2H_2 + a_3H_3 + a_4H_4) = a_j$$

Then, if Σ_r is the set of restricted roots and Φ_r is a set of simple restricted roots, Σ_r and Φ_r are exactly as in section 11; this time the short roots are all of multiplicity 4.

If $I \subset S$, set $h_I = \sum_{j \in I} h_j - \sum_{j \in I'} h_j$. Then, if $|I| = 4$, $Tr(h_I) = 0$. We now see that

$$\mathbf{m_C} = \langle f_{\{1,2,3,4\}}, f_{\{5,6,7,8\}}, h_{\{1,2,3,4\}} \rangle \oplus$$
$$\langle f_{\{1,2,5,6\}}, f_{\{3,4,7,8\}}, h_{\{1,2,5,6\}} \rangle \oplus \langle f_{\{1,2,7,8\}}, f_{\{3,4,5,6\}}, h_{\{1,2,7,8\}} \rangle.$$

Let $\widehat{M} = K \cap Z(\mathbf{a})$ and let $\widehat{M_0}$ be the connected component of the identity of $\widehat{M}$. Setting $x_1 = \exp(\pi i/2)h_{\{1,2,3,4\}}$, $x_2 = \exp(\pi i/2)h_{\{1,2,5,6\}}$, and $x_3 = \exp(\pi i/2)h_{\{1,2,7,8\}}$,we see that

$$\widehat{M_0} = (SU(2) \times SU(2) \times SU(2))/L$$

where L is a finite subgroup of

$$Z(SU(2) \times SU(2) \times SU(2)) = \langle x_1, x_2, x_3 \rangle.$$

A simple calculation on $\bigwedge^4 W$ shows that L is trivial. Furthermore, since the representation of $\widehat{M_0}$ on any short restricted root space is the tensor pruduct of two irreducible 2-dimensional repesentations of two of the $SU(2)$ factors of $\widehat{M_0}$, $(-I_2, -I_2, -I_2) \in Z(E_{7(-5)})$. Hence $\widehat{M_0} \cap Z(E_{7(-5)}) = \mathbf{Z}_2$. Setting $\alpha_1 = 4(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)$, we have

$$H_{\alpha_1}/|\alpha_1|^2 = 1/16(H_1 - H_2 - H_3 - H_4) \quad \text{and} \quad H_{8\epsilon_4}/|8\epsilon_4|^2 = (1/8)H_4.$$

If x is a preimage of $\exp(\pi i/8)(H_1 - H_2 - H_3 - H_4)$ in K and y is a preimage of $\exp(\pi i/4)(H_4)$ in K, we have that $\widehat{M} = \widehat{Z_1} \cdot \widehat{M_0}$ where $\widehat{Z_1} = \langle x, y \rangle$ is a group of order 8. Now $Ad(x)(f_7 \cdot g_8) = -f_8 \cdot g_7$, $Ad(x)(f_8 \cdot g_7) = -f_7 \cdot g_8$, $Ad(y)(f_7 \cdot g_8) = -f_7 \cdot g_8$, and $Ad(y)(f_8 \cdot g_7) = -f_8 \cdot g_7$. Hence we have $x^2 = y^2$ $x^4 = y^4 = 1$ and $xyx^{-1} = y^{-1}$.Finally, $\widehat{M_0}$ centralizes $\widehat{Z_1}$, and we have the following.

Proposition 12.1. $\widehat{M} = \widehat{Z_1} \cdot \widehat{M_0}$ *and* $\widehat{M_0} \cap \widehat{Z_1} = \{1\}$.

Corollary 12.2. $\widehat{M} = \widehat{Z_1} \times \widehat{M_0}$.

Remark. As in section 11, $\widehat{Z_1} \cong D_3$ is the quaternionic group.

13. $\mathbf{e}_{6(2)}$ and $\tilde{E}_{6(2)}$

Set

$$\mathbf{sl}(2,\mathbf{C}) = \langle f_7 \cdot g_8, f_8 \cdot g_7, h_7 - h_8 \rangle \text{ and } \quad \mathbf{sl}(6,\mathbf{C}) = \langle f_i \cdot g_j : i \neq j \leq 6 \rangle + \mathbf{h}_0$$

where $\mathbf{h}_0 = \{\sum_{j=1}^6 a_j h_j : \sum_{j=1}^6 a_j = 0\}$. If $W_1 = \langle f_j : 1 \leq j \leq 6 \rangle$ and $W_2 = \langle f_7, f_8 \rangle$, $\mathbf{p_C} = \bigwedge^3 W_1 \otimes W_2$ is an irreducible $\mathbf{k_C} = \mathbf{sl}(6,\mathbf{C}) \oplus \mathbf{sl}(2,\mathbf{C})$-module. As before $(\mathbf{k_C}, \mathbf{p_C})$ is a symmetric pair for $\mathbf{e}_{6,\mathbf{C}}$. Now $\mathbf{e}_{6(2)}$ is the corresponding real form of $\mathbf{e}_{6,\mathbf{C}}$. Let $\tilde{E}_{6(2)}$ be the real connected Lie group with Lie algebra $\mathbf{e}_{6(2)}$ and maximal compact subgroup $K = SU(6) \times SU(2)$.

Now $\mathbf{p}$ has a maximal abelian subalgebra $\mathbf{a} = \langle H_j : 1 \leq j \leq 4 \rangle$ is the same $\mathbf{a}$ as in section 12. Then $\mathbf{m}_0 = \langle h_1 + h_2 - h_3 - h_4, h_1 + h_2 - h_5 - h_6 \rangle$, and $\widehat{M}_0 = S^1 \times S^1$. Moreover, if $\widehat{M} = K \cap Z(\mathbf{a})$ and $\widehat{Z}_1$ is the quaternionic group in section 12 we have the following result.

Proposition 13.1. $\widehat{M} = \widehat{Z}_1 \cap \widehat{M}_0$ *and* $\widehat{M}_0 \cap \widehat{Z}_1 = \{1\}$.

Corollary 13.2. $\widehat{M} = \widehat{Z}_1 \times \widehat{M}_0$.

Remarks.

1. Since $Z(\tilde{E}_{6(2)}) = \langle (\omega I_6, \epsilon I_2) : \omega^3 \epsilon = 1, \omega^6 = \epsilon^2 = 1 \rangle \cong \mathbf{Z}_6$, $\tilde{E}_{6(2)}$ is not a linear group.
2. A simple calculation shows that $\widehat{M}_0 \cap Z(E_{6(2)}) = \mathbf{Z}_3$
3. The short restricted roots are all of multiplicity 2.

14. The Hermitian real form $\mathbf{e}_{7(-25)}$ and the group $\tilde{E}_{7(-25)}$

Recall from sections 12 and 13 that $\mathbf{e}_{7,\mathbf{C}} = \mathbf{sl}(8,\mathbf{C}) + \bigwedge^4 W$ and

$$\mathbf{e}_{6,\mathbf{C}} = \mathbf{sl}(6,\mathbf{C}) + \mathbf{sl}(2,\mathbf{C}) + \bigwedge^3 W_1 \otimes W_2.$$

If $W_1^* = \langle g_1, g_2, g_3, g_4, g_5, g_6 \rangle$, $W_2^* = \langle g_7, g_8 \rangle$ and $h_0 = \sum_{j=1}^6 h_j - 3(h_7 + h_8)$,

$$\mathbf{e}_{7,\mathbf{C}} = \mathbf{e}_{6,\mathbf{C}} + \langle h_0 \rangle + (W_1 \cdot W_2^* + \bigwedge^4 W_1) + (W_1^* \cdot W_2 + \bigwedge^2 W_1 \otimes \bigwedge^2 W_2)$$

where all the sums are direct as vector spaces. If

$$\mathbf{k_C} = \mathbf{e}_{6,\mathbf{C}} + \mathbf{C}h_0, \mathbf{p}_+ = (W_1 \cdot W_2^* + \bigwedge^4 W_1), \mathbf{p}_- = (W_1^* \cdot W_2 + \bigwedge^2 W_1 \otimes \bigwedge^2 W_2),$$

$ad(h_0)$ is 0 on $\mathbf{k_C}$, 4 on $\mathbf{p}_+$ and -4 on $\mathbf{p}_-$. It follows immediately that $\mathbf{k_C}$ is a Lie algebra, $[\mathbf{p}_\pm, \mathbf{p}_\pm] = 0$ and $[\mathbf{p}_+, \mathbf{p}_-] \subset \mathbf{k_C}$. If $\mathbf{p_C} = \mathbf{p}_+ + \mathbf{p}_-$, $(\mathbf{k_C}, \mathbf{p_C})$ is a symmetric pair for $\mathbf{e}_{7,\mathbf{C}}$. If $\mathbf{g} = \mathbf{k} + \mathbf{p}$ is the corresponding real form, $\mathbf{g} = \mathbf{e}_{7p(-25)}$.

Now $\tilde{E}_{7(-25)}$ is the simply connected analytic group with Lie algebra $\mathbf{e}_{7(-25)}$. Let $\Phi : \tilde{E}_{7(-25)} \longrightarrow E_{7(-25)}$ be the covering homomorphism. The maximal compact subgroup of $E_{7(-25)}$ is $K = E_{6(-78)} \cdot S^1$,and the pullback of K is $\tilde{K} = \Phi^{-1}(K) = E_{6(-78)} \times \mathbf{R}$. For $(x, \tau) \in \tilde{K}$, $\Phi(x, \tau) = x \cdot \exp(i\tau h_0)$.

Using the fact that $\mathbf{Z}_3 \cong Z(E_{6(-78)}) \subset SU(6) \cdot SU(2)$, we see that $Z(\tilde{E}_{7(-25)}) \subset SU(6) \cdot SU(2) \times \mathbf{R}$, and a direct calculation yields

$$Z(\tilde{E}_{7(-25)}) = \{(\omega I_6, \epsilon I_2, \tau) : \omega^3 \epsilon = 1, \omega^6 = 1, \omega^4 \exp 4i\tau = 1\}/\{\pm(I_6, I_2, 0)\}.$$

If ω_0 is a primitive sixth root of unity, we have that $Z(E_{7(-25)})$ is generated by $(\omega_0 I_6, -I_2, \pi/6)$ and so $Z(\tilde{E}_{7(-25)}) \cong \mathbf{Z}$. Now $\mathbf{a} = \langle X_{\{1,2,3,4\}}, X_{\{1,2,5,6\}}, X_{\{1,2,7,8\}} \rangle$ is a maximal abelian subalgebra of $\mathbf{p}$, and from [10] we have that $\widehat{M} = \tilde{K} \cap Z(\mathbf{a})$ is generated by $\widehat{M}_0 = Spin(8)$ and the pullback of $Ad(\exp(\pi i/4)X_{\{1,2,7,8\}})$. Since

$$Ad(\exp(\pi i/4)X_{\{1,2,7,8\}}) = Ad(diag(1, 1, -1, -1, -1, -1, 1, 1))$$

and $diag(1, 1, -1, -1, -1, -1, 1, 1) \in Spin(8)$, we have the following.

Proposition 14.1. $\widehat{M} = \widehat{M}_0 \cdot Z(\tilde{E}_{7(-25)}) \cong \widehat{M}_0 \times \mathbf{Z}$.

Remark. From [10] and [13] we have that $\Phi(\widehat{M}) \cong \widehat{M}_0 \times \mathbf{Z}_2$.

15. The Hermitian real form $\mathbf{e}_{6(-14)}$ and group $\tilde{E}_{6(-14)}$

Now

$$\mathbf{so}(10, \mathbf{C}) \cong \langle f_i \cdot f_j, g_i \cdot g_j, f_i \cdot g_j : i, j \leq 5, i \neq j \rangle + \langle h_j : j \leq 5 \rangle,$$

and if $h_0 = h_6 - h_7 - h_8$, $\mathbf{k_C} = \mathbf{so}(10, \mathbf{C}) \oplus \mathbf{C}h_0$ is a reductive Lie algebra. Setting $V = \langle f_1, f_2, f_3, f_4, f_5 \rangle$, $\mathbf{p}_+ = \bigwedge^o V \cdot f_6$, $\mathbf{p}_- = \bigwedge^e V \cdot f_7 \cdot f_8$, and $\mathbf{p_C} = \mathbf{p}_+ + \mathbf{p}_-$, we have that $\mathbf{k_C} + \mathbf{p_C} = \mathbf{e}_{6,\mathbf{C}}$ and $(\mathbf{k_C}, \mathbf{p_C})$ is a symmetric pair. The Lie algebra corresponding to the real form of this symmetric pair is $\mathbf{e}_{6(-14)}$. Now $K = Spin(10) \cdot S^1$ is the maximal compact subgroup of $E_{6(-14)}$ and $\tilde{K} = Spin(10) \times \mathbf{R}$ is the pullback of K in $\tilde{E}_{6(-14)}$. Note that $ad(h_0) = \pm 3/2$ on $\mathbf{p}_\pm$.

Now $\mathbf{a} = \langle X_{\{1,2,5,6\}}, X_{\{1,2,7,8\}} \rangle$ is a maximal abelian subalgebra of $\mathbf{p}$. If $\widehat{M} = Z(\mathbf{a}) \cap \tilde{K}$, we have from [10] and [12], that the image of $\widehat{M}$ in K is connected and $\mathbf{m} = \mathbf{su}(4) \oplus \langle H_0 \rangle$ where $H_0 = h_5 - 1/3h_0$. A simple calculation shows that $Z(\tilde{E}_{6(-14)}) \cong \mathbf{Z}$ and has a generator the pullback of

$$\exp \pi i(h_1 + h_2 + h_3 + h_4 + h_5 - 1/3h_0).$$

Since $h_1 + h_2 + h_3 + h_4 \in \mathbf{su}(4)$, we have the following.

Proposition 15.1. *The group* $\widehat{M} = \widehat{M}_0 \cong SU(4) \times \mathbf{R}$.

16. The groups $E_{6(-26)}$ and $F_{4(-20)}$

The groups $E_{6(-26)}$ and $F_{4(-20)}$ are both simply connected linear groups with trivial centers. From [10] we see that for $E_{6(-26)}$ the group $\widehat{M}$ is $Spin(8)$, and for $F_{4(-20)}$ the group $\widehat{M}$ is $Spin(7)$. This concludes our analysis of the exceptional groups.

17. The Classical Groups

Throughout this section we will use the notation (G, K) to denote a pair where G is a simply connected real simple Lie group and K is the pullback in G of a maximal compact subgroup of $Ad(G)$.

1. $(\tilde{SL}(n, \mathbf{R}), Spin(n))$: Now $\mathbf{p}$ is the space of real $n \times n$ symmetric matrices of trace 0. If $\mathbf{a}$ is the space of diagonal matrices in $\mathbf{p}$, $\mathbf{a}$ is a maximal abelian subalgebra of $\mathbf{p}$. We then have the following.

Proposition 17.1. $\widehat{M} = Z(\mathbf{a}) \cap Spin(n) = D_n$.

Remark. If P is a non minimal parabolic subgroup of $\tilde{SL}(n, \mathbf{R})$, an easy calculation shows that $\{\pm I\} = [D_n, D_n] \subset P_0$.

2. $(SU^*(2n), Sp(n))$: Since $SU^*(2n)$ is a linear group we have the following proposition.

Proposition 17.2. $\widehat{M} = (Sp(1))^n$.

3. $(\tilde{SU}(p, q), SU(p) \times SU(q) \times \mathbf{R}H)$ with $1 \le p < q$ and $H = \begin{pmatrix} qI_p & O \\ 0 & -pI_q \end{pmatrix}$, and the map of K onto $S(U(p) \times U(q))$ sends tH to $\exp(itH)$:
Then $\mathbf{a} = \langle E_{j,p+j} + E_{p+j,j} : 1 \le j \le p \rangle$ is a maximal abelian subalgebra of $\mathbf{p}$. Setting $\widehat{M} = K \cap Z(\mathbf{a})$ we now state.

Proposition 17.3. $\widehat{M} \cong (S^1)^{p-1} \times SU(q-p) \times \mathbf{R}$.

Proof. By a simple calculation, we have
$\widehat{M} = \langle (\alpha, \beta, \theta H) : \alpha = diag(a_1, \dots, a_p), \beta = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}$,
$\beta_1 = diag(b_1, \dots, b_p), \beta_2 \in U(q-p), a_1 \cdots a_p = 1, b_1 \cdots b_p \cdot det\beta_2 = 1$,

$$|a_1| = \ldots = |a_p| = 1, b_j = a_j e^{i(p+q)\theta)} \rangle \cong$$

$$\langle (b_1, \dots, b_p, \beta_2, \theta) : b_1, \dots, b_p \in \mathbf{C}, |b_1| = \ldots = |b_p| = 1, \beta_2 \in U(q-p),$$

$$b_1 \cdots b_p = e^{ip(p+q)\theta} = (det\beta_2)^{-1} \rangle \cong$$

$$\langle (b_1, \dots, b_p, \beta_2, \theta) : |b_1| = \ldots = |b_p| = 1, \beta_2 \in SU(q-p),$$
$$b_1 \cdots b_p = e^{ip(p+q)\theta} \rangle \cong$$

$$\langle (b_1, \dots, b_{p-1}, \beta_2, \theta) : |b_1| = \ldots = |b_p| = 1, \beta_2 \in SU(q-p), \theta \in \mathbf{R} \rangle \cong$$
$$(S^1)^{p-1} \times SU(q-p) \times \mathbf{R}.$$

■

4. $(\tilde{SU}(p, p), SU(p) \times SU(p) \times \mathbf{R}H), H = diag(I_p, -I_p)$: Taking $\mathbf{a}$ is as in 3, we obtain:

Proposition 17.4. $\widehat{M} \cong (S^1)^{p-1} \times \mathbf{Z}$

Proof. : A simple calculation yields

$$\begin{gathered}\widehat{M} = \langle (a_1, g \ldots, a_p, b_1, \ldots, b_p, \theta) : |a_1| = \ldots = |a_p| = 1, b_j = a_j e^{2i\theta}, \\ a_1 \cdots a_p = b_1 \cdots b_p = 1, e^{2pi\theta} = 1 \rangle \cong \\ \langle (b_1, \ldots, b_{p-1}, \theta) : |b_1| = \ldots = |b_p| = 1, \theta \in (\pi/p)\mathbf{Z} \rangle \cong \\ (S^1)^{p-1} \times \mathbf{Z}.\end{gathered}$$

■

Remarks. Note that $Z(G) \cong \langle (z, (k/p)\pi) : z^p = 1, k \in \mathbf{Z} \rangle$, and the kernel of the map of K onto $S(U(p) \times U(p))$ is isomorphic to

$$\langle (\exp(-2\pi i/p), 2\pi/p) \rangle \cong \mathbf{Z}.$$

Then the image of $\widehat{M}$ in $S(U(p) \times U(p))$ is $(S^1)^{p-1} \times \mathbf{Z}_2$, and the image of $\widehat{M}$ in AdG is $(S^1)^{p-1}$.

5. $(\tilde{Spin}_0(p,q), Spin(p) \times Spin(q))$, where $1 \le p \le q$, $p \ne 2$, and $q \ne 2$: Taking $\mathbf{a}$ as in 3, and keeping our notation we obtain the following.

Proposition 17.5. $\widehat{M} = D_p \times Spin(q-p)$

Note that D_1 and $Spin(1)$ are trivial.

a. If $p = 1$, $\tilde{Spin}_0(p,q)$ is a linear group and $\widehat{M} = Spin(q-1)$.

b. If $q = p$ or if $q = p+1$, $\widehat{M} = D_p$.

6. $(\tilde{Spin}_0(2,q), \mathbf{R} \times Spin(q))$ with $q \ne 2$: We have a map $\phi : \mathbf{R} \longrightarrow Spin(q+2)$ where $\phi(t) = \exp(te_1 \cdot e_2)$. If $\mathbf{a}$ is as in 3, then $\widehat{M} = \tilde{Z}_1 \cdot \widehat{M}_0$ where $\widehat{M}_0 = Spin(q-2)$ and

$$\tilde{Z}_1 = \{(k\pi/2, \pm e_3 \cdot e_4) : k \text{ odd}\} \cup \{(\mathrm{k}\pi/2, \pm 1) : \mathrm{k\ even}\}.$$

With addition in the first variable and multiplication in the second variable, $\tilde{Z}_1$ is easily seen to be a group. Since $(0,-1) \in \widehat{M}_0$, if $q > 3$,this product is not direct. If $\widehat{Z}_1 = \{(k\pi/2, (e_3 \cdot e_4)^k) :\ k \in \mathbf{Z}\}$, $\widehat{Z}_1$ is a group isomorphic to $\mathbf{Z}$. Note that $\tilde{Z}_1 \cong \widehat{Z}_1 \times \langle (0, \pm 1) \rangle \cong \mathbf{Z} \times \mathbf{Z}_2$.

Proposition 17.6. *If* $q > 3$, $\widehat{M} \cong \mathbf{Z} \times \widehat{M}_0$. *If* $q = 3$, $\widehat{M} \cong \mathbf{Z} \times \mathbf{Z}_2$

Remarks. (a) Note that $\tilde{Spin}_0(2,3) \cong \tilde{Sp}(2, \mathbf{R})$.

(b) The image of $\widehat{Z}_1$ in $Spin(q+2)$ is $\langle e_1 \cdot e_2 \cdot e_3 \cdot e_4 \rangle \cong \mathbf{Z}_2$.

(c) If $q > 3$, the image of $\widehat{M}$ in $Spin(q+2)$ is isomorphic to $\mathbf{Z}_2 \times Spin(q-2)$.

(d) If $q = 3$, the image of $\widehat{M}$ in $Spin(5)$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$.

(e) The image of $\widehat{M}$ in $Ad(\tilde{Spin}_0(2,q))$ is connected if and only if q is even.

(f) If $p = q = 2$, $\tilde{Spin}_0(2,2) \cong \tilde{SL}(2,\mathbf{R}) \times \tilde{SL}(2,\mathbf{R})$ is not a simple group.

7. $(\tilde{Sp}(n,\mathbf{R}), SU(n) \times \mathbf{R}H)$, where $H = \begin{pmatrix} 0 & I \\ -1 & 0 \end{pmatrix}$: Then $\mathbf{p} = \{\begin{pmatrix} A & B \\ B & -A \end{pmatrix} : A^T = A, B^T = B\}$, with

$$\mathbf{a} = \{diag(x_1, \ldots, x_n, -x_1, \ldots, -x_n) : x_1, \ldots, x_n \in \mathbf{R}\}$$

is a maximal abelian subalgebra of $\mathbf{p}$.

Proposition 17.7. $\widehat{M} \cong (\mathbf{Z}_2)^{n-1} \times \mathbf{Z}$.

Proof. Since the image of $\widehat{M}$ in $U(n)$ is

$$M = \langle diag(\epsilon_1, \ldots, \epsilon_n) : \epsilon_j = \pm 1 \rangle \cong (\mathbf{Z}_2)^n,$$

$$\widehat{M} \cong \langle (A, \theta) : A \in SU(n), \theta \in \mathbf{R}, A \exp(i\theta) \in M \rangle.$$

Hence $\widehat{M} \cong (M \cap SU(n)) \times \overline{M}$ where $M \cap SU(n) \cong \mathbf{Z}_2^{n-1}$ and

$$\overline{M} = \{(A, \theta) : A \in SU(n), \theta \in \mathbf{R}, A \exp(i\theta) = diag(\pm 1, 1, \ldots, 1)\} =$$

$$\{(diag((-1)^k, 1, \ldots, 1) \exp(-ik\pi/n), k\pi/n) : k \in \mathbf{Z}\} \cong \mathbf{Z}.$$

Thus the result holds. ■

8. $(\tilde{SO}^*(2n), SU(n) \times \mathbf{R}H)$, $H = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$: Now

$$\mathbf{p} = \{ i \begin{pmatrix} A & B \\ B & -A \end{pmatrix} : A, B \in M_n(\mathbf{R}), A^T = -A, B^T = -B\}.$$

If $\mathbf{a}$ is the set

$$\{ i \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \in \mathbf{p} : X = \sum_{k=1}^{[n/2]} x_k (E_{2k-1,2k} - E_{2k,2k-1}),\ x_k \in \mathbf{R}\},$$

$\mathbf{a}$ is a maximal abelian subalgebra of $\mathbf{p}$.
A simple calculation shows that $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \times \theta H \in \widehat{M}$ if and only if $Ze^{i\theta}X = X\overline{Z}e^{-i\theta}$ and $XZe^{i\theta} = \overline{Z}e^{-i\theta}X$ where $Z = A + iB$.

a. If $n = 2k+1$, $Z = diag\ (Z(1), \ldots, Z(k),\ e^{i\phi})$ where $Z(1),\ \ldots, Z(k) \in M_2(\mathbf{C})$,

$$Z(j) = \begin{pmatrix} z(j)_{11} & z(j)_{12} \\ -\overline{z(j)}_{12} e^{-2i\theta} & \overline{z(j)}_{11} e^{-2i\theta} \end{pmatrix},$$

and $\det Z(1) \cdots \det Z(k)\ e^{i\phi} = 1$. Setting $Z_o(j) = \begin{pmatrix} z(j)_{11} & z(j)_{12} \\ -\overline{z(j)}_{12} & \overline{z(j)}_{11} \end{pmatrix}$, we have

$$\det Z_o(1) = \cdots = \det Z_o(k) = 1$$

and $\exp(i\phi - 2ki\theta) = 1$. Hence we obtain the following.

Proposition 17.8. *If $n = 2k+1$, then* $\widehat{M} \cong SU(2)^k \times \mathbf{R}$.

b. If $n = 2k$, $Z = diag(Z(1), \ldots, Z(k))$ where $Z(1), \ldots, Z(k) \in M_2(\mathbf{C})$,

$$Z(j) = \begin{pmatrix} z(j)_{11} & z(j)_{12} \\ -\overline{z(j)}_{12} e^{-2i\theta} & \overline{z(j)}_{11} e^{-2i\theta} \end{pmatrix},$$

and $\det Z(1) \cdots \det Z(k) = 1$. Setting $Z_o(j) = \begin{pmatrix} z(j)_{11} & z(j)_{12} \\ -\overline{z(j)}_{12} & \overline{z(j)}_{11} \end{pmatrix}$, we have

$$\det Z_o(1) = \cdots = \det Z_o(k) = 1$$

and $\exp(-2ki\theta) = 1$.

Proposition 17.9. *If $n = 2k$, then $\widehat{M} \cong SU(2)^k \times \mathbf{Z}$.*

Remarks. For $n = 2k$ the image of $\widehat{M}$ in $Spin^*(4k)$ is $SU(2)^k \times \mathbf{Z}_2$. The image of $\widehat{M}$ in $SO^*(4k)$ is $SU(2)^k$, and the image of $\widehat{M}$ in $Ad(SO^*(4k))$ is $SU(2)^k/\{\pm I\}$.

9. $(Sp(p,q), Sp(p) \times Sp(q))$ with $p \le q$: Then $Sp(p,q)$ is a simply connected linear group and $\widehat{M} = Sp(1)^p \times Sp(q-p)$.

18. Summary

Suppose G is a connected real simple Lie group with Lie algebra $\mathbf{g}$ contained in a simply connected group $G_{\mathbf{C}}$ having Lie algebra $\mathbf{g}_{\mathbf{C}}$. Let $\tilde{G}$ be the universal cover of G. Fix K, a maximal compact subgroup of G and denote the pullback of K in $\tilde{G}$ by $\tilde{K}$. Fixing $\mathbf{a}$ as in the previous sections, set $\widehat{M} = Z(\mathbf{a}) \cap \tilde{K}$ and let M denote the image of $\widehat{M}$ in K.

1. Then $\widehat{M} = \widehat{Z}_1 \times \widehat{M}_0$ where $\widehat{Z}_1$ is a discrete group and $\widehat{M}_0$ is the identity component of $\widehat{M}$.

2. The group $\widehat{Z}_1$ is infinite if and only if G/K is a tube type domain.

3. Since, as topological spaces, $\tilde{K}/\widehat{M} = K/M$,

$$\pi_1(K/\ M) \cong \pi_0(\widehat{M}) \cong \widehat{Z}_1.$$

   Hence K/M is simply connected if and only if $\widehat{M}$ is connected. From the homotopy exact sequence, it follows that $\widehat{M}$ is connected if and only if M is connected.

4. Suppose l is the number of white dots of the Satake diagram of G that are not adjacent to a black dot nor connected to another white dot by an arrow.

   (a) The group $\widehat{M}$ is connected if and only if $l = 0$.

   (b) If $l = 1$, G/K is a tube type domain and $Z_1 = \mathbf{Z}$.

   (c) If $l > 1$ and G/K is a tube type domain, $G = \tilde{Sp}(l, \mathbf{R})$ and

$$\widehat{M} = Z_1 = \mathbf{Z}_2^{l-1} \times \mathbf{Z}.$$

   (d) If $l > 1$ and G/K is a not tube type domain, $\widehat{Z}_1$ is a non abelian group of order 2^{l+1}.

5. If G is a split group, $\widehat{M} = \widehat{Z}_1$.

References

[1] Adams, J. F., "Lectures on Exceptional Groups," Zafer Mahmud and Mamoru Mimira, Editors,, University of Chicago Press, Chicago, 1996.

[2] Araki, S., *On root systems and an infinitesimal classification of irreducible symmetric spaces*, J. Math. Osaka City Univ. **13** (1962), 1–34.

[3] Baez, J. C., *The Octonions*, Bulletin Amer. Math. Soc. **39** (2002), 145–205.

[4] Bourbaki, N., " Groupes et algèbres de Lie," Hermann, Paris, 1968.

[5] Cartan, E., "The Theory of Spinors," Dover Publications, Inc., New York, 1981.

[6] Helgason, S., "Differential geometry and symmetric spaces, " Academic Press, New York, 1962.

[7] —, *A duality for symmetric spaces with applications to group representations*," Advances in Math. **5** (1970), 1–154.

[8] Jacobson, N., "Lectures in Abstract Algebra, Volume II–Linear Algebra, " Van Nostrand, Princeton, 1953.

[9] —, "Lie Algebras," Interscience, New York, 1962.

[10] Johnson, K. D., *A strong generalization of Helgason's theorem*, Trans. Amer. Math. Soc. **304** (1987), 171–192.

[11] —, *A constructive approach to tensor product decompositions*, J. für die reine und angewandte Mathematik **388** (1988), 129–148.

[12] Satake, I., *On representations and compactifications of symmetric Riemann spaces*, Ann. of Math. **72** (1960), 555–580.

[13] Schlichtkrull, H., *One-dimensional K-types in finite dimensional representations of semisimple Lie groups: A generalization of Helgason's theorem*, Math. Scand. **54** (1984), 279–294.

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Received April 2, 2003
and in final form July 20, 2002