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On the Exponential Map of the Lie Groups Locally Isomorphic to SU(p,q)

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Abstract. In this paper we classify all exponential Lie groups which are locally isomorphic to $\mathrm{SU}(p,q)$.

1. Introduction

A Lie group G is called *exponential* if its exponential function is surjective, and it is called *weakly exponential* if it has dense exponential image [1]. A Lie algebra $\mathfrak{g}$ is *exponential*, respectively, *weakly exponential* if there is an exponential, respectively, weakly exponential Lie group G with Lie algebra isomorphic to $\mathfrak{g}$, and *completely exponential*, respectively, *completely weakly exponential* if the simply connected Lie group G with Lie algebra $\mathfrak{g}$ is exponential, respectively, weakly exponential.

There is no practical criterion for exponentiality in the general case, though we have criteria for some classes of Lie groups.

Theorem 1.1. ([6]) *Let G be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$. The following conditions are equivalent:*

(1) *G is exponential;*

(2) *For each nilpotent $X \in \mathfrak{g}$, the centralizer $Z(X,G)$ is weakly exponential.*

Thus, the exponentiality question of semisimple Lie groups is reduced to the weak exponentiality question of some set of their subgroups. For weak exponentiality there are the following theorems.

Theorem 1.2. (A.Borel, published in [3]) *A connected Lie group is weakly exponential if and only if all Cartan subgroups are connected.*

Theorem 1.3. ([3]) *All connected solvable Lie groups are weakly exponential. The underlying real Lie group of any complex connecteld Lie group is weakly exponential.*

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Theorem 1.4. ([3]) *Let N be a connected normal Lie subgroup of a Lie group G. Then the following conditions are equivalent:*

(1) *G is weakly exponential;*

(2) *N and G/N are weakly exponential.*

Hence, the determination of weakly exponential Lie groups is reduced to the case of semisimple Lie groups. In [4], NEEB gives a list of all weakly exponential and completely weakly exponential simple real Lie algebras. In particular, he proves the following statement:

Theorem 1.5. ([4]) *The algebra $\mathfrak{su}(p,q)$ is weakly exponential for all p and q, and is completely weakly exponential if $p > q$.*

In [2], ĐOKOVIĆ and NGUYÊÑ give a list of all weakly exponential and exponential simple linear real Lie groups. In particular, they prove

Theorem 1.6. ([2]) *The group $G = \mathrm{SU}(p,q)/\mathbb{Z}_r$, $p > q$, is exponential iff every odd prime divisor of $|Z(G)| = \frac{p+q}{r}$ is greater than $\frac{p+q}{p-q}$. The group $\mathrm{SU}(p,p)/\mathbb{Z}_r$ is exponential iff $r = 2p$.*

In [7], WÜSTNER considers the question of exponentiality of simply connected simple real Lie groups. In particular, he proves that the universal covering group of $\mathrm{SU}(p,1)$ is exponential iff $p \geqslant 3$. In this paper we give a criterion for a Lie group that is locally isomorphic to $\mathrm{SU}(p,q)$ to be exponential (Theorem 3.5). For example, for the covering groups of $\mathrm{SU}(p,q)$ we prove the following

Theorem 1.7. 1) *Let numbers p,q be such that the group $\mathrm{SU}(p,q)$ is exponential, and let G be an s-fold covering group of $\mathrm{SU}(p,q)$. Then the following conditions are equivalent:*

(i) *G is exponential;*

(ii) $\mathrm{GCD}(s,p,q) = 1$ *and* $\mathrm{GCD}(s, q - j(p-q)) = 1$ *for* $j = 0, 1, \ldots, [\frac{q}{p-q}]$.

2) *The universal covering of $\mathrm{SU}(p,q)$ is not exponential if $p \geq q > 1$.*

2. Nilpotent elements and their centralizers

Let G be a Lie group locally isomorphic to $\mathrm{SU}(p,q)$, $p \geqslant q \geqslant 1$. Set

$$\mathrm{GCD}(p,q) = d,\ p = p'd,\ q = q'd,\ n = p+q,\ n' = p' + q'.$$

For an arbitrary Lie group H we denote by $\widetilde{H}$ the simply connected Lie group locally isomorphic to H. Also we denote the commutator subgroup of H by (H,H).

For each nilpotent $X \in \mathfrak{su}(p,q)$ there exists a linear representation $R : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{su}(p,q)$ such that $X = R(e)$, where $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Therefore, the nilpotent elements in the algebra $\mathfrak{su}(p,q)$ are parametrized by the pairs (R,η), where R is a $(p+q)$-dimension representation of the algebra $\mathfrak{sl}(2,\mathbb{R})$ and η is an R-invariant Hermitian form of signature (p,q) (if $p = q$ then the class of nilpotent elements corresponding to (R,η) is equal to the class of nilpotent elements corresponding to $(R,-\eta)$).

Consider an irreducible linear representation $R : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(V)$, $\dim V = n$. There exists a nondegenerate R-invariant Hermitian form η_n in V, which is unique up to multiplication by a nonzero real number. In a basis of eigenvectors of some semisimple element in $\mathfrak{sl}(2,\mathbb{R})$ it is represented by matrix:

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & & 0 & 1 & 0 \\ \vdots & & & 0 & \vdots \\ 0 & & & & \vdots \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix},$$

the signature of the form being $([\frac{n-1}{2}]+1, [\frac{n}{2}])$.

Let $R = \sum_{i=1}^{m} k_i R_i$, where $R_i : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(V_i)$ are non-isomorphic irreducible representations, $\dim V_i = n_i$. We may assume that $V = \bigoplus_{i=1}^{m} V_i \otimes \mathbb{C}^{k_i}$, where $\mathfrak{sl}(2,\mathbb{R})$ acts on $\mathbb{C}^{k_i}$ trivially. Any R-invariant Hermitian form on the space V is represented as $\eta = \bigoplus_{i=1}^{m} \eta_{n_i} \otimes f_i$, where η_{n_i} is an R_i-invariant Hermitian form, f_i is a Hermitian form on $\mathbb{C}^{k_i}$. Let f_i be of signature (k_i^+, k_i^-), $i = 1, \dots, m$.

A *signed Young diagram* is the Young diagram in which every box is labelled with plus or minus so that signs alternate along the rows. We identify two signed Young diagrams iff they can be obtained from each other by permuting rows of equal length. Assume p and q be the number of pluses and minuses in the signed Young diagram J. Then the pair (p,q) is called *the signature* of J.

Let us consider the signed Young diagram J which consists of $\sum_{i=1}^{m} k_i$ rows, with k_i rows of length n_i, of which k_i^+ rows begin with plus and k_i^- rows begin with minus. This diagram coresponds to the pair (R,η). The signature of the form η is equal to the signature of Young diagram J. Therefore, the classes of nilpotent elements in the algebra $\mathfrak{su}(p,q)$ are parametrized by the signed Young diagrams of signature (p,q) (if $p = q$ we also can exchange all signs and their opposites).

Let X be a nilpotent element in $\mathfrak{su}(p,q)$ and J be a corresponding signed Young diagram. Consider the centralizer $Z(X,G)$ of X in a Lie group G locally isomorphic to $\mathrm{SU}(p,q)$. By Theorem 1.4, the weak exponentiality of the centralizer $Z(X,G)$ is equivalent to the weak exponentiality of its maximal reductive subgroup $S(X,G)$ which is equal to the centralizer of the subalgebra $R(\mathfrak{sl}(2,\mathbb{R})) \subset \mathfrak{su}(p,q)$.

We denote the space of linear operators on the space of dimention n by $\mathrm{L}(n,\mathbb{C})$. Let us consider the centralizer $S = Z(\mathfrak{sl}(2,\mathbb{R}), \mathrm{L}(n,\mathbb{C}))$. By Schur's lemma, it consists of the elements $\bigoplus_{i=1}^{m} E_{n_i} \otimes A_i$, where E_n is the identity operator, $A_i \in \mathrm{L}(k_i,\mathbb{C})$. Thus,

$$S(X, \mathrm{SU}(p,q)) = \mathrm{SU}(p,q) \cap S = \\ = \{(A_1,\dots,A_m) \in \mathrm{U}(k_1^+,k_1^-) \times \dots \times \mathrm{U}(k_m^+,k_m^-) : \textstyle\prod_{i=1}^{m} (\det A_i)^{n_i} = 1\},$$

$$\mathrm{Lie}(S(X,G)) = \mathfrak{s}(X) = \mathfrak{su}(p,q) \cap S = \\ = \{(A_1,\dots,A_m) \in \mathfrak{u}(k_1^+,k_1^-) \oplus \dots \oplus \mathfrak{u}(k_m^+,k_m^-) : \textstyle\sum_{i=1}^{m} n_i \mathrm{tr} A_i = 0\}$$

The following statement is proven by ĐOKOVIĆ and NGUYÊÑ in [2]. We give a new proof.

Proposition 2.1. *A Lie group locally isomorphic to* $\mathrm{SU}(p,p)$ *is exponential iff it is isomorphic to* $\mathrm{PSU}(p,p)$.

Proof. Consider the signed Young diagram J that consists of one row of length $2p$. Let X be a nilpotent element corresponding to J. Then $\mathfrak{s}(X)=\{0\}$ and for a Lie group G locally isomorphic to $\mathrm{SU}(p,p)$ the group $S(X,G)$ is a finite subgroup containing $Z(G)$. Thus, the group $S(X,G)$ is connected only if $Z(G)=\{e\}$, i.e. if G is isomorphic to the corresponding adjoint group. And by Theorem 1.6, the group $\mathrm{PSU}(p,p)$ is exponential for all p. ■

We assume that $p>q$ for the rest of the paper.

A Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is called *compactly embedded* if $\overline{\exp \operatorname{ad}\mathfrak{k}}$ is compact in $\mathrm{Aut}(\mathfrak{g})$.

Lemma 2.2. ([7]) *Let* H *be a connected Lie group. If* H' *is a covering group of* H *and* φ *is the corresponding covering map, then* H' *is connected iff* $\mathrm{Ker}\varphi \subseteq \exp_{H'}\mathfrak{k}$, *where* $\mathfrak{k}$ *is a maximal abelian compactly embedded subalgebra in* $\mathfrak{h}$.

This lemma holds for any maximal abelian compactly embedded subalgebras, because all of them are conjugated.

Let us prove the following simple lemma:

Lemma 2.3. *The group* $\mathrm{U}(p,q)$ *is weakly exponential for any* p *and* q.

Proof. The center $Z(\mathrm{U}(p,q))$ is connected and hence is weakly exponential. The group $\mathrm{U}(p,q)/Z(\mathrm{U}(p,q)) \cong \mathrm{PSU}(p,q)$ is weakly exponential by Theorem 1.5. Thus, by Theorem 1.4, the group $\mathrm{U}(p,q)$ is weakly exponential. ■

Let us notice the simple corollary from this lemma: the group $\prod_{i=1}^{m}\mathrm{U}_i(p_i,q_i)$ and its quotients by connected central subgroups are weakly exponential.

Theorem 2.4. *Let* G *be a Lie group locally isomorphic to* $\mathrm{SU}(p,q)$ *and for each nilpotent element* $X\in\mathfrak{su}(p,q)$ *the group* $S(X,G)$ *is connected. Then* G *is exponential.*

Proof. We denote the identity component of $S(X,\mathrm{SU}(p,q))$ by $S(X)$. First assume that $k_i^+\neq k_i^-$ for any $i=1,\dots,m$. Then the universal covering of $S(X)$ is isomorphic to the group $\mathbb{R}^{m-1}\times\mathrm{SU}(k_1^+,k_1^-)\times\ldots\times\mathrm{SU}(k_m^+,k_m^-)$, which is weakly exponential by Theorem 1.5. Therefore, the group $S(X,G)$ is weakly exponential if it is connected.

Now assume that $k_1^+=k_1^-,\dots,k_s^+=k_s^-$ and $k_i^+\neq k_i^-$ for $s<i\le m$. One can notice that in this case $m>1$. For each $i=1,\dots,s$ in the Young diagram J_X corresponding to X there are k_i rows of the same length, half of them begins with plus, another half begins with minus. Consider the diagram which is obtained from J_X by joining all such rows in one (we can do this because the number minuses in such rows equals the number of pluses). We denote the corresponding nilpotent element by Y. The group $S(Y,G)$ is connected by the condition of the theorem. Let us consider the cover of the group $S(Y,\mathrm{PSU}(p,q))$

by the group $S(Y,G)$. By Lemma 2.2 the kernel of the covering map is contained in $\exp\mathfrak{h}$, where $\mathfrak{h}$ is the compactly embedded subalgebra consisting of all diagonal matrices in $\mathfrak{s}(Y)$. Notice that this subalgebra consists of matrices which are scalar on the subspace corresponding to the first $\sum_{i=1}^{s} k_i$ rows of the Young diagram corresponding to X. This subalgebra is contained in a maximal abelian compactly embedded subalgebra of $\mathfrak{s}(X)$. Let us consider the subalgebra $\mathfrak{s}_1(X)$ of elements from $\mathfrak{s}(X)$ which are scalar on the subspace corresponding to the sum of the first s unitary subalgebras. The subalgebra $\mathfrak{h}$ is contained in it. Consider the corresponding subgroup S_1' of $\mathrm{PSU}(p,q)$. It is connected; thus the subgroup S_1 of G which is the inverse image of S_1', is connected. Its universal covering group is isomorphic to the direct product of some components, isomorphic to $\mathbb{R}$, and some components, isomorphic to $\widetilde{\mathrm{SU}}(k_i^+,k_i^-)$, $k_i^+\neq k_i^-$. Hence, this group is weakly exponential. The quotient of $S(X,G)$ by this group is isomorphic to the quotient of $\mathrm{U}(k_1^+,k_1^-)\times\ldots\times\mathrm{U}(k_s^+,k_s^-)$ by the subgroup of scalar matrices. This group is weakly exponential hence, by Theorem 1.4, the group $S(X,G)$ is weakly exponential. Therefore, G is exponential. ∎

3. Criterion of connectivity of $\mathbf{S}(X,G)$

We denote by $\varphi:\widetilde{\mathrm{SU}}(p,q)\to\mathrm{SU}(p,q)$ the covering map. The center of $\mathrm{SU}(p,q)$ is isomorphic to $\mathbb{Z}_n=\langle y\rangle$, where $y=\exp\frac{2\pi i}{n}E$. The center of $\widetilde{\mathrm{SU}}(p,q)$ is isomorphic to $\mathbb{Z}_d\times\mathbb{Z}$ ([5]), moreover we may assume that $\mathrm{Ker}\varphi=\langle(1,n')\rangle$. Let $\nu: Z(\widetilde{\mathrm{SU}}(p,q))\to\mathbb{Z}$ be the projection.

Let us consider the representation space of $\mathrm{SU}(p,q)$ as $V=V_+\oplus V_-$, where V_+ (respectively V_-) is the maximal subspace of V, on which the Hermitian form is positively (negatively) definite. Assume $K=\{(A,B)\in\mathrm{U}(p)\times\mathrm{U}(q): \det A\times\det B=1\}\subset\mathrm{SU}(p,q)$. The group K is a maximal compact subgroup of $\mathrm{SU}(p,q)$ and it is isomorphic to the almost direct product of $\mathrm{SU}(p)\times\mathrm{SU}(q)$ and circumference. It is well-known that the fundamental group of any Lie group is isomorphic to the fundamental group of its maximal compact subgroups. Thus, $\pi_1(\mathrm{SU}(p,q))\cong\pi_1(K)$. The commutator subgroup $(K,K)\cong\mathrm{SU}(p)\times\mathrm{SU}(q)$ is simply connected and the quotient $K/(K,K)$ is isomorphic to the circumference. Therefore $\pi_1(K)\cong\pi_1(K/(K,K))\cong\mathbb{Z}$.

Lemma 3.1. *Let* $\gamma(t)=\exp_{\mathrm{SU}(p,q)}(2\pi i\ \xi(t))$, *where* $\xi(t)=$ $=t\ \mathrm{diag}(\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q)$, $0\le t\le 1$, $\alpha_i,\beta_i\in\mathbb{Z}$. *Then* $\gamma\subset\mathrm{SU}(p,q)$ *is homotopic to the loop* ${\gamma_0}^r$, *where* γ_0 *is the generator of* $\pi_1(\mathrm{SU}(p,q))$ *and* $r=\sum_{i=1}^{p}\alpha_i=-\sum_{j=1}^{q}\beta_j$.

Proof. If $\sum_{i=1}^{p}\alpha_i=0$ then $\xi(t)\in\mathrm{Lie}((K,K))$ for any $t\in[0,1]$. The group (K,K) is simply connected, hence the loop γ is trivial.

Now let $\sum_{i=1}^{p}\alpha_i\neq 0$. The isomorphism $\pi_1(K)\cong\pi_1(K/(K,K))$ is generated by the projection of K on $K/(K,K)$. Thus, r is equal to the number of intersections of γ and (K,K). For any point $\gamma(t)$ of this intersection $t\sum_{i=1}^{p}\alpha_i$ is an integer. Hence, $r=\sum_{i=1}^{p}\alpha_i=-\sum_{j=1}^{q}\beta_j$.

∎

Lemma 3.2. *Let* $\xi = \operatorname{diag}(\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q) \in \mathfrak{su}(p,q)$, $\alpha_i, \beta_i \in \mathbb{Q}$ *be such that the curve* $\gamma(t) = \exp_{\widetilde{\mathrm{SU}}(p,q)} 2\pi i t\xi$, $0 \leqslant t \leqslant 1$ *connects the identity element of the group with an element* $z \in Z(\widetilde{\mathrm{SU}}(p,q))$. *Then* $\nu(z) = n' \sum_{i=1}^{p} \alpha_i$.

Proof. Notice that if $\varphi(z) = y^a$ then $\alpha_i = \frac{a}{n} + z_i$, $\beta_j = \frac{a}{n} + z'_j$, $z_i, z'_j \in \mathbb{Z}$ for any $i = 1, \dots, p$, $j = 1, \dots, q$ and thus, $\exp_{\widetilde{\mathrm{SU}}(p,q)}(2\pi i n\xi) = e$. Consider a loop $\gamma_1 = \varphi(\gamma^n(t)) = \exp_{\mathrm{SU}(p,q)}(2\pi i t n\xi)$. By Lemma 3.1, we have $\varphi(\gamma_1) = \gamma_0^r$, where $r = n\sum_{i=1}^{p} \alpha_i = -n\sum_{i=1}^{q} \beta_i$. Thus, $\gamma^n(1) = \tilde{z}^r$, where $\tilde{z}$ is the generator of $\mathrm{Ker}\varphi$, $\nu(\tilde{z}) = n'$. Therefore $\nu(z) = \nu(\gamma(1)) = \frac{1}{n}\nu(\gamma^n(1)) = \frac{1}{n} r\nu(\tilde{z}) = n'\sum_{i=1}^{p} \alpha_i = -n'\sum_{i=1}^{q} \beta_i$. ■

This lemma implies that if $z \in Z(\widetilde{\mathrm{SU}}(p,q))$ is such that $\varphi(z) = y^a$ and $\nu(z) = b$ then $b + q'a \equiv 0 \ (\mathrm{mod}\, n')$. Indeed, z is equal to

$$\exp_{\widetilde{\mathrm{SU}}(p,q)}(2\pi i(\operatorname{diag}(z_1, \dots, z_p, z'_1, \dots, z'_q) + \frac{a}{n}E)),$$

where z_i, $z'_j \in \mathbb{Z}$. By Lemma 3.2, $\nu(z) = -n'\sum_{i=1}^{q} \beta_i = -\frac{n'qa}{n} - n'\sum_{i=1}^{q} z'_i = -q'a - n'\sum_{i=1}^{q} z'_i$. Hence, $b + aq' = n'x$, where $x = \sum_{i=1}^{q} z'_i \in \mathbb{Z}$. Moreover, for any a, b that satisfy this condition there exists a $z \in Z(\widetilde{\mathrm{SU}}(p,q))$ such that $\nu(z) = b$, $\varphi(z) = y^a$.

Set $D(X,G) = S(X,G)^0 \cap Z(G)$ for a Lie group G locally isomorphic to $\mathrm{SU}(p,q)$ and a nilpotent element $X \in \mathfrak{su}(p,q)$. Let $\psi : \widetilde{\mathrm{SU}}(p,q) \to G$ be the covering map, $\mathrm{Ker}\psi = D$. Then $D(X,G) = \psi(D(X, \widetilde{\mathrm{SU}}(p,q)))$. The center $Z(G)$ is contained in $S(X,G)$, and the connectivity of $S(X,G)$ implies that $Z(G) = D(X,G)$. The latter equation is equivalent to the condition $D \cdot D(X, \widetilde{\mathrm{SU}}(p,q)) = Z(\widetilde{\mathrm{SU}}(p,q))$. This condition is sufficient for connectivity because the group $\mathrm{PSU}(p,q)$ is exponential and hence the group $S(X, \mathrm{PSU}(p,q))$ is connected.

A row of a signed Young diagram is called *good* if it is of odd length and begins with plus. If a row is *bad*, i.e. is of even length or begins with minus, then the number of pluses in it is less than or equal to the number of minuses. Thus, if the signature of a Young diagram is (p,q), $p > q$, then it contains at least one good row. A Young diagram J and a corresponding nilpotent element X are called *good* if all rows of J are good. Notice that a good Young diagram consists of $(p-q)$ rows. Moreover, if a nilpotent element X is good then all f_i are positively definite, and thus, the algebra $\mathfrak{s}(X)$, which is isomorphic to the quotient of $\bigoplus_{i=1}^{m} \mathfrak{u}(f_i)$ by the subalgebra of the scalar matrices, is compact and the identity component $S(X,G)^0$ is compact for any G.

Let us consider a bad signed Young diagram J and its longest bad row. We can obtain a new diagram J' by joining this row with the longest good one. Let X (respectively X') be the nilpotent element corresponding to the diagram J (respectively J'). Then $S(X',G)$ is contained in $S(X,G)$ as the set of operators, which are scalar on the subspaces corresponding to the joined rows.

One can see that after several such operations each signed Young diagram becomes good. Moreover, the intersection of the reductive part of the corresponding centralizer with the center of the group after each operation is contained in

$D(X,G)$. Thus, the condition of connectivity for all centralizers of nilpotent elements of $\mathfrak{su}(p,q)$ is equivalent to this condition for all centralizers of good nilpotents of $\mathfrak{su}(p,q)$. We assume that the nilpotent X is good for the rest of the paper.

For each $j=0,\ldots,[\frac{q}{p-q}]$ we consider the element
$z_j=\exp_{\widetilde{\mathrm{SU}}(p,q)}2\pi i(-\frac{2j+1}{n}E+\mathrm{diag}(j+1,0,\ldots,0,j))\in Z(\widetilde{\mathrm{SU}}(p,q))$.
By Lemma 3.2, $\nu(z_j)=q'-j(p'-q')$, $\varphi(z_j)=y^{-2j-1}$.

Lemma 3.3. *Let $X\in\mathfrak{su}(p,q)$ be a nilpotent element such that the length of the shortest row in the corresponding Young diagram J is equal to $2j+1$. Then $z_j\in D(X,\widetilde{\mathrm{SU}}(p,q))$.*

Proof. Notice that $j\leqslant\frac{q}{p-q}$ because the number of minuses in each row is greater or equals j. We choose the shortest row in J and consider the element $\xi=\frac{-1-2j}{n}E+\xi'=\mathrm{diag}(\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q)\in\mathfrak{s}(X)$, where ξ' is the diagonal matrix, which acts identically on the subspace V_i corresponding to this row and trivially on the orthogonal supplement. The dimension of the space $V_+\cap V_i$ is equal to $j+1$, therefore $n'\sum_{i=1}^{p}\alpha_i=n'(\frac{-1-2j}{n}p+j+1)=q'-j(p'-q')$. So, $\nu(\exp_{\widetilde{\mathrm{SU}}(p,q)}2\pi\xi)=\nu(z_j)$ and $\phi(z_j)=\phi(\exp_{\widetilde{\mathrm{SU}}(p,q)}2\pi\xi)$, hence $z_j=\exp_{\widetilde{\mathrm{SU}}(p,q)}2\pi\xi$. It follows, that the lemma is true. ∎

Now let us consider the Young diagram J_j consisting of $p-q-1$ rows of length $2j+1$ and one row of length $2n_j+1$, where $n_j=q-j(p-q)+j$. We denote by X_j the corresponding nilpotent element. We will prove that the group $D(X_j,\widetilde{\mathrm{SU}}(p,q))$ is generated by z_j.

Let $p'-q'=1$ and $j=q'$. Then the diagram J_j consists of $p-q$ rows of equal length and $\mathfrak{s}(X_j)\cong\mathfrak{su}(p-q)$, the group $S(X_j,\mathrm{SU}(p,q))$ is simply connected. Hence $S(X_j,\widetilde{\mathrm{SU}}(p,q))^0\cap\mathrm{Ker}\varphi=\{e\}$ (otherwise the image of the continuous curve connecting e with $z\in\mathrm{Ker}\varphi$ would be a nontrivial loop in $S(X_j,\mathrm{SU}(p,q))$). For any $z\in Z(\widetilde{\mathrm{SU}}(p,q))$ there is a power s such that $z^s\in\mathrm{Ker}\varphi$, therefore $D(X_j,\widetilde{\mathrm{SU}}(p,q))\subseteq\mathbb{Z}_d$. The group $\mathbb{Z}_d\subseteq Z(\mathrm{SU}(p,q))$ is generated by the element $y^{-1-2j}=\varphi(z_j)$. Hence, $D(X_j,\widetilde{\mathrm{SU}}(p,q))$ coincides with $\mathbb{Z}_d\subseteq Z(\widetilde{\mathrm{SU}}(p,q))$ and is generated by z_j.

Now assume either $p'-q'\neq 1$ or $p'-q'=1$ but $j\neq q'$. Then $n_j>j$ and

$$\mathfrak{s}(X_j)\cong\{(\lambda,A)\in\mathbb{R}\times\mathfrak{u}(p-q-1):(2n_j+1)i\lambda+(2j+1)\mathrm{tr}A=0\},$$

$$S(X_j,\mathrm{SU}(p,q))\cong\{(\mu,A)\in\mathbb{T}\times\mathrm{U}(p-q-1):\mu^{2n_j+1}\mathrm{det}^{2j+1}A=1\}.$$

We denote $u=\mathrm{GCD}(2n_j+1,2j+1)$ and consider the identity component

$$S(X_j,\mathrm{SU}(p,q))^0\cong\{(\mu,A)\in\mathbb{T}\times\mathrm{U}(p-q-1):\mu^{\frac{2n_j+1}{u}}\mathrm{det}^{\frac{2j+1}{u}}A=1\}=S.$$

The commutator subgroup $S'=\{(1,A):\mathrm{det}A=1\}\cong\mathrm{SU}(p-q-1)$ is simply connected and the quotient S/S' is isomorphic to the circumference. We denote by χ the embedding of S in the group $\mathrm{SU}(p,q)$: $\chi(\mu,A)=E_{2n_j+1}\otimes\mu\oplus E_{2j+1}\otimes A$. Notice that $\chi(\mu,A)|_{V^-}=E_{n_j}\otimes\mu\oplus E_j\otimes A$, hence $\det(\chi(\mu,A))|_{V^-}=$

$\mu^{n_j}\det^j A$. The image $\chi(S)$ is contained in the maximal compact subgroup K of $\mathrm{SU}(p,q)$, so we have an homomorphism of quotients $\theta : S/S' \to K/K'$ and hence the embedding of fundamental groups $\theta' : \pi_1(S) \to \pi_1(K)$. The index of the image of this embedding in the group $\pi_1(K)$ is equal to the number of elements in the kernel of the map θ. The latter is equal to the number of contiguous classes of S by S' contained in K', and it is equal to $|\chi^{-1}(\chi(S) \cap K')/S'|$. Then

$$\chi^{-1}(\chi(S) \cap K') = \{(\mu, A) \in S : \mu^{n_j}\det^j A = 1,\ \mu^{\frac{2n_j+1}{u}}\det^{\frac{2j+1}{u}} A = 1\} = \\ = \{(\mu, A) \in S : \mu\det A = 1,\ \mu^{\frac{n_j-j}{u}} = 1\}.$$

Thus, $[\pi_1(K) : \theta'(\pi_1(S))] = \frac{n_j-j}{u}$.

Lemma 3.4. *If $p' - q' \neq 1$ or $j \neq q'$, then $D(X_j, \widetilde{\mathrm{SU}}(p,q)) \cap \mathbb{Z}_d = \{e\}$.*

Proof. Assume that there is $Y = \frac{k}{n}E + \chi(\mathrm{diag}(\lambda_1, \dots \lambda_{p-q}))$, $\lambda_i \in \mathbb{Z}$, $0 \le k < n$, such that $\exp_{\widetilde{\mathrm{SU}}(p,q)} 2\pi i t Y = z \in \mathbb{Z}_d$. By Lemma 3.2, this is equivalent to the system:

$$\begin{cases} kp' + n'(n_j+1)\lambda_1 + n'(j+1)\sum_{i=2}^{p-q}\lambda_i &= 0, \\ kq' + n'n_j\lambda_1 + n'j\sum_{i=2}^{p-q}\lambda_i &= 0. \end{cases}$$

By excluding all variables but the first one we get

$$n\lambda_1 = -k$$

Since $k < n$, this system has no solution in integer numbers. ■

The simple corollary of this lemma is that $D(X_j, \widetilde{\mathrm{SU}}(p,q))$ has only one generator z. Since $[Z(\widetilde{\mathrm{SU}}(p,q)) : D(X, \widetilde{\mathrm{SU}}(p,q))] =$
$= [Z(\mathrm{SU}(p,q)) : D(X, \mathrm{SU}(p,q))] \cdot [\mathrm{Ker}\varphi : \mathrm{Ker}\varphi \cap D(X, \widetilde{\mathrm{SU}}(p,q))] =$
$= [Z(\mathrm{SU}(p,q)) : D(X, \mathrm{SU}(p,q))] \cdot [\pi_1(\mathrm{SU}(p,q)) : \theta'(\pi_1(S^0))] = u\frac{n_j-j}{u} = q - j(p-q)$ we have $\nu(z) = (q - j(p-q))/d = q' - j(p'-q') = \nu(z_j)$. By Lemma 3.3 $z_j \in D(X_j, \widetilde{\mathrm{SU}}(p,q))$, therefore $z_j z^{-1} \in D(X_j, \widetilde{\mathrm{SU}}(p,q))$, but $\nu(z_j z^{-1}) = \nu(z_j) - \nu(z) = 0$. Hence, by Lemma 3.4 $z = z_j$.

Thus, the exponentiality of $G \cong \widetilde{\mathrm{SU}}(p,q)/D$ implies that $D \cdot \langle z_j \rangle = Z(\widetilde{\mathrm{SU}}(p,q))$, $j = 0, \dots, [\frac{q}{p-q}]$. Moreover, this condition is sufficient because for each nilpotent element X there exists one of the elements z_j in the group $D(X, \widetilde{\mathrm{SU}}(p,q))$.

Before we can prove our main theorem, let us consider nontrivial subgroups in $\mathbb{Z}_d \times \mathbb{Z}$. For any subgroup D we can choose two generators x_1 and x_2, $x_1 \neq x_2$, such that $x_1 \in \mathbb{Z}_d$. In particular, if $D \cap \mathbb{Z}_d = e$ we will assume $x_1 = (0,0)$, if $D \subseteq \mathbb{Z}_d$ we will assume $x_2 = (0,0)$.

Theorem 3.5. 1) *A Lie group G locally isomorphic to $\mathrm{SU}(p,p)$ is exponential iff $G = \mathrm{PSU}(p,p)$.*

2) *Let $p \neq q$ and $D = \langle x_1, x_2 \rangle$ be a nontrivial central subgroup of $\widetilde{\mathrm{SU}}(p,q)$, $\varphi(x_1) = y^{an'}$, $a|d$, $\nu(x_2) = b$, $\varphi(x_2) = y^c$, $0 \leqslant c < an'$, $b + cq' = ln'$. The group $G = \widetilde{\mathrm{SU}}(p,q)/D$ is exponential iff for all $j = 0, \dots, [\frac{q}{p-q}]$ the following conditions are fulfilled:*

(i) $\mathrm{GCD}(b, q' - j(p'-q')) = 1$;
(ii) $\mathrm{GCD}(a, l(2j+1) - cj) = 1$.

Proof. The first part was proven in Proposition 2.1.

Assume $p \neq q$. The condition $D \cdot \langle z_j \rangle = Z(\widetilde{\mathrm{SU}}(p,q))$ is equivalent to two following conditions:

(i') The projection of $D\langle z_j \rangle$ on $\mathbb{Z}$ covers all elements of $\mathbb{Z}$;

(ii') $D\langle z_j \rangle$ contains $\mathbb{Z}_d$.

The projection of $D\langle z_j \rangle$ on $\mathbb{Z}$ is generated by $\mathrm{GCD}(b, \nu(z_j)) = \mathrm{GCD}(b, q' - jn')$, so the conditions (i) and (i') are equivalent. The condition (i') implies that the intersection $D \cdot \langle z_j \rangle \cap \mathbb{Z}_d$ is generated by the elements x_1 and $x_2' = x_2^{\nu(z_j)} z_j^{-\nu(x_2)} = x_2^{q'-j(p'-q')} z_j^{-b}$. It contains $\mathbb{Z}_d$ iff $\mathbb{Z}_d \subset Z(\mathrm{SU}(p,q))$ is generated by the elements $\varphi(x_1) = y^{an'}$ and $\varphi(x_2') = y^{c(q'-j(p'-q'))+b(2j+1)}$. The latter is equivalent to the following: $\mathrm{GCD}(an', c(q' - j(p' - q')) + b(2j + 1)) \equiv n' (\mathrm{mod} n)$. The theorem is proven. ∎

Remark 3.6. Let us prove that for a Lie group isomorphic to $\mathrm{SU}(p,q)/\mathbb{Z}_r$ the above criterion is equivalent to the result of Djoković and Nguyêñ (Theorem 1.6). Set $e = \mathrm{GCD}(r, d)$, $r_1 = \frac{r}{e}$, $d_1 = \frac{d}{e}$. Then under the conditions of Theorem 3.5 $a = d_1$, $b = \frac{n'}{r_1}$, and c, l are given by the equation $b + cq' = ln'$, $c < n$, $\frac{n}{r}$ divides c ($\frac{n}{r} = \frac{n'}{r_1} d_1$ divides c, $\frac{n'}{r_1}$ divides ln', hence $\frac{n'}{r_1}$ divides b. We choose b as a minimal number with such property, i.e. $b = \frac{n'}{r_1}$. Therefore, the index of $\varphi(\langle x_2 \rangle)$ is equal to $\frac{n}{bd} = r_1$, hence $\langle x_1 \rangle$ is the subgroup of $\mathbb{Z}_d$ of index e. Thus, $a = \frac{d}{e} = d_1$).

Let the condition (i) of Theorem 3.5 fail to be true for some $j < \frac{q}{p-q}$ and k be a prime divisor of $\mathrm{GCD}(b, q' - j(p' - q'))$. Since $\mathrm{GCD}(n', p' - q') = 1$ and $n' = br_1$, the equation $n' = (2j + 1)(p' - q') + 2(q' - j(p' - q'))$ implies that k divides $2j + 1$, hence k is an odd divisor of n which is lower than $\frac{n}{p-q}$. Since $\frac{n}{r} = bd_1$ and k divides b, k divides $\frac{n}{r}$, therefore the condition of Theorem 1.6 fails to be true.

Let the condition (ii) fail to be true for some $j < \frac{q}{p-q}$ and k be a prime divisor of $\mathrm{GCD}(d_1, l(2j+1) - cj)$. We denote with s_1 and s_2 the maximal powers of k which divides b and d_1 respectively, $s_2 > 0$. Since $\mathrm{GCD}(d_1, r_1) = 1$ and $b = \frac{n'}{r_1}$, the maximal power of k which divides n' is equal to s_1. Since $\frac{n}{r} = \frac{n'}{r_1} d_1$ divides c, $k^{s_1+s_2}$ divides c and the equation $b = ln' - cq'$ implies that k does not divide l. But k divides $l(2j + 1) - cj$, hence k is an odd divisor of $\frac{n}{r}$, which is lower than $\frac{n}{p-q}$. Thus, the condition of Theorem 1.6 fails to be true.

Now let the condition of Theorem 1.6 fail to be true, k be an odd prime divisor of $\frac{n}{r} = \frac{n'}{r_1} d_1$, $k < \frac{n}{p-q}$. Consider the condition of Theorem 3.5 for $j = \frac{k-1}{2}$. If k does not divide d_1 then k divides $b = \frac{n'}{r_1} = \frac{1}{d_1}\frac{n}{r}$. The equation $n' = k(p' - q') + 2(q' - j(p' - q'))$ implies that k divides $q' - j(p' - q')$, therefore the condition (i) of Theorem 3.5 fails to b true. Now let k divide d_1. Since $\frac{n}{r}$ divides c, k is a divisor of $\mathrm{GCD}(d_1, l(2j + 1) - cj)$, hence the condition (ii) of Theorem 3.5 fails to be true.

Proof of Theorem 1.7. a) If a Lie group G is an s-fold covering group of $\mathrm{SU}(p,q)$ then under the conditions of Theorem 3.5 the group D is generated by $\tilde{z}^s$, where $\tilde{z}$ is the generator of $\mathrm{Ker}\varphi$, thus $a = d$, $c = 0$, $b = sn'$. Therefore, the group G is exponential iff the conditions $\mathrm{GCD}(sn', q' - j(p' - q')) = 1$, $\mathrm{GCD}(s(2j+1), d) = 1$ are true for all $j = 0, \ldots, [\frac{q}{p-q}]$. Since $q' - j(p' - q') = (2j + 1)q' - jn'$, the first condition is equivalent to the following two conditions: $\mathrm{GCD}(n', 2j+1) = 1$ and $\mathrm{GCD}(s, q' - j(p' - q')) = 1$. The second condition is equivalent to the following two conditions: $\mathrm{GCD}(s, d) = \mathrm{GCD}(s, p, q) = 1$ and $\mathrm{GCD}(2j+1, d) = 1$. All of these conditions are equivalent to the following three conditions: $\mathrm{GCD}(n, 2j+1) = 1$, $\mathrm{GCD}(s, q - j(p-q)) = 1$ and $\mathrm{GCD}(s, p, q) = 1$. By Theorem 1.6, the group $\mathrm{SU}(p, q)$ is exponential iff every odd prime divisor of $p + q$ is greater than $\frac{p+q}{p-q}$. This condition is equivalent to the following: $\mathrm{GCD}(p+q, 2j+1) = 1$ for all $j = 0, \dots, [\frac{q}{p-q}]$. It follows that the theorem is true.

b) By Lemma 3.4, for the exponentiality of the group $\widetilde{\mathrm{SU}}(p, q)$ we need the condition $\mathrm{GCD}(p, q) = 1$. Consider the element X_0. We have $D(X_0, \widetilde{\mathrm{SU}}(p, q)) = \langle z_0 \rangle$, where $\nu(z_0) = q' = q$. Therefore $\nu(D(X_0, \widetilde{\mathrm{SU}}(p, q))) = \langle \nu(z_0) \rangle \neq \mathbb{Z}$, and $D(X_0, \widetilde{\mathrm{SU}}(p, q)) \neq Z(\widetilde{\mathrm{SU}}(p, q))$. Hence, the group $S(X_0, \widetilde{\mathrm{SU}}(p, q))$ is not connected, so it is not weakly exponential.

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