

Analysis on real affine G -varieties

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Abstract. We consider the action of a real linear algebraic group G on a smooth, real affine algebraic variety $M \subset \mathbb{R}^n$, and study the corresponding left regular representation of G on the Banach space $C_0(M)$ of continuous, complex valued functions on M vanishing at infinity. We show that the differential structure of this representation is already completely characterized by the action of the Lie algebra \mathfrak{g} of G on the dense subspace $\mathcal{P} = \mathbb{C}[M] \cdot e^{-r^2}$, where $\mathbb{C}[M]$ denotes the algebra of regular functions of M and r the distance function in \mathbb{R}^n . We prove that the elements of this subspace constitute analytic vectors of the considered representation, and by taking into account the algebraic structure of \mathcal{P} , we obtain G -invariant decompositions and discrete reducing series of $C_0(M)$. In case that G is reductive, K a maximal compact subgroup, \mathcal{P} turns out to be a (\mathfrak{g}, K) -module in the sense of Harish-Chandra and Lepowsky, and by taking suitable subquotients of \mathcal{P} , respectively $C_0(M)$, one gets admissible (\mathfrak{g}, K) -modules as well as K -finite Banach representations.

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1. Introduction

Consider the regular action of a real linear algebraic group $G \subset \mathrm{GL}(n, \mathbb{R})$ on a smooth, real affine algebraic variety $M \subset \mathbb{R}^n$, and the corresponding left regular representation π of G on the Banach space $C_0(M)$ of continuous, complex valued functions on M vanishing at infinity. While the harmonic analysis of real reductive groups is well established, the representation theory of real affine G -varieties, that is, the study of group actions on function spaces associated with such varieties, is much less developed, and it is to them the present article is devoted to. Fix a compact subgroup K of G . In general representation theory, a crucial role is played by the space of differentiable, K -finite vectors. If E denotes a locally convex, complete, Hausdorff, topological vector space, and σ a continuous representation

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on E , this space is defined as the algebraic sum

$$E_K = \sum_{\lambda \in \hat{K}} E_\infty \cap E(\lambda),$$

where E_∞ is the space of differentiable vectors in E for σ , \hat{K} the set of all equivalence classes of finite dimensional irreducible representations of K , and $E(\lambda)$ the isotypic K -submodule of E of type $\lambda \in \hat{K}$, see [9]. E_K is dense in E , and the study of σ can be reduced to a great extent to the study of the module E_K . In case of the Banach representation $(\pi, C_0(M))$, a more natural submodule associated with the algebraic structure of M arises. Let $\mathbb{C}[M]$ denote the ring of regular functions of M , r the distance function in \mathbb{R}^n , and consider the subspace

$$\mathcal{P} = \mathbb{C}[M] \cdot e^{-r^2} \subset C_0(M).$$

It was introduced by Agricola and Friedrich in [1]. They proved that it is dense in $C_0(M)$, which in turn implies the density of $\mathbb{C}[M]$ in the Hilbert space $L^2(M, e^{-r^2} d\mu)$, where $d\mu$ denotes the volume form of M . Note that this is a generalization of the well-known fact that, for $M = \mathbb{R}^n$, the Hermite polynomials furnish a complete orthonormal basis of $L^2(\mathbb{R}^n, e^{-r^2} d\mu)$. As a consequence, the decomposition of $\mathbb{C}[M]$ into G -isotypic components according to Frobenius can be used to obtain an analogous decomposition of $L^2(M, e^{-r^2} d\mu)$. However, the regular representation of G on the coordinate ring $\mathbb{C}[M]$ can not be extended to a continuous representation on $L^2(M, e^{-r^2} d\mu)$. Instead, in this paper we examine the differential structure of the Banach representation $(\pi, C_0(M))$, and derive G -invariant decompositions of $C_0(M)$, by using the density of \mathcal{P} , and a classical theorem of Harish-Chandra.

More precisely, let \mathfrak{g} denote the Lie algebra of G , and $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ the universal enveloping algebra of the complexification of \mathfrak{g} . Let $\mathfrak{Z}(M)$ be the center of the algebra of invariant differential operators on M . As a first result, we show that the differential structure of the representation $(\pi, C_0(M))$ is already determined by the action of \mathfrak{g} on \mathcal{P} . An analogous dense graph theorem, but with respect to the space of differentiable vectors of an arbitrary Banach representation, was already derived by Langlands in [6], while studying the holomorphic semigroup generated by certain elliptic differential operators associated with the given representation. In a similar way, the dense graph theorem proved in this paper might be helpful in the study of the spectral properties of certain invariant operators in $\mathfrak{Z}(M)$. We then prove that \mathcal{P} is contained in the space of analytic elements of $C_0(M)$. By a result of Harish-Chandra [4], this allows us to derive $\pi(G)$ -invariant decompositions of the Banach space $C_0(M)$ from algebraic decompositions of \mathcal{P} into $d\pi(\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}))$ -invariant subspaces. In particular, we obtain discrete reducing series in $C_0(M)$. In case that G is reductive, \mathcal{P} turns out to be a (\mathfrak{g}, K) -module in the sense of Harish-Chandra and Lepowsky, and one has $\mathcal{P} = \sum_{\delta \in \hat{K}} \mathcal{P} \cap C_0(M)(\lambda)$. By taking suitable subquotients of \mathcal{P} and $C_0(M)$, one gets admissible (\mathfrak{g}, K) -modules as well as K -finite Banach representations.

2. Some general remarks on Banach representations

Let us begin with some generalities concerning Banach representations. Thus, consider a weakly continuous representation π of a Lie group G on a Banach space

\mathcal{B} by linear bounded operators. According to a result of Yosida from the theory of (C_0) -semigroups [10], π is also continuous with respect to the strong topology on \mathcal{B} , so that both continuity concepts coincide. Such a representation is called a *Banach representation*. Similarly, the generators of the weakly, respectively strongly, continuous one-parameter groups of operators $h \mapsto \pi(e^{hX})$ coincide, where $h \in \mathbb{R}$, and X is an element of the Lie-algebra \mathfrak{g} of G . We will denote the corresponding generators by $d\pi(X)$, which are given explicitly by

$$d\pi(X)\varphi = \left. \frac{d}{dh} \pi(e^{hX})\varphi \right|_{h=0}$$

for those $\varphi \in \mathcal{B}$, for which the limit exists. These operators are closed and densely defined with respect to the weak and strong topology on \mathcal{B} . Let d_G be left invariant Haar-measure on G , $f \in L^1(G, d_G)$, and \mathcal{B}^* the dual of \mathcal{B} . Then, for each $\varphi \in \mathcal{B}$, $f(g)\mu(\pi(g)\varphi)$ is d_G -integrable for arbitrary $\mu \in \mathcal{B}^*$, and there exists a $\psi \in \mathcal{B}$ such that

$$\mu(\psi) = \int_G f(g)\mu(\pi(g)\varphi) d_G(g)$$

for all $\mu \in \mathcal{B}^*$, i. e. $f(g)\pi(g)\varphi$ is integrable in the sense of Pettis, see e. g. [5]. Here ψ is given as a weak limit, and one defines as this limit the integral

$$\int_G f(g)\pi(g)\varphi d_G(g) = \psi,$$

in this way getting a linear operator $\pi(f) : \mathcal{B} \rightarrow \mathcal{B}, \varphi \mapsto \int_G f(g)\pi(g)\varphi d_G(g)$. Note that $\|\pi(f)\| \leq \|f\|_{L^1}$. In case that f is L^1 -integrable and continuous, $f(g)\pi(g)\varphi$ is also integrable in the sense of Bochner, and ψ given directly by the corresponding strongly convergent integral. Let

$$\mathcal{B}_\infty = \left\{ \sum_{i=1}^l \pi(f_i)\varphi_i : f_i \in C_c^\infty(G), \varphi_i \in \mathcal{B}, l = 1, 2, 3, \dots \right\}$$

be the Gårding-subspace of \mathcal{B} with respect to π , and $\mathcal{B}_{\infty,s}$, respectively $\mathcal{B}_{\infty,w}$, the subspace of differentiable elements in \mathcal{B} with respect to the strong, respectively weak, topology. \mathcal{B}_∞ is norm-dense in \mathcal{B} and, according to Langlands [6], the generators $d\pi(X)$ are already completely determined by their action on the Gårding-subspace. Thus, if Γ_{X_1, \dots, X_k} denotes the graph of the generators $d\pi(X_i), i = 1, \dots, k$, and $\Gamma_{X_1, \dots, X_k|_{\mathcal{B}_\infty}}$ its restriction to \mathcal{B}_∞ , one has $\Gamma_{X_1, \dots, X_k} = \overline{\Gamma_{X_1, \dots, X_k|_{\mathcal{B}_\infty}}}$. As an immediate consequence, the differential structure of the representation π is completely characterized by the action of the operators $d\pi(X)$ on \mathcal{B}_∞ . In particular, this implies that the strongly, respectively weakly, differentiable elements in \mathcal{B} do coincide with those that are differentiable with respect to the one-parameter groups of operators $h \mapsto \pi(e^{hX})$, the underlying topology being the strong, respectively weak, topology. Since, by Yosida, strong and weak generators coincide, one finally has

$$\mathcal{B}_{\infty,s} = \bigcap_{i=1}^d \bigcap_{k \geq 1} \mathcal{D}(d\pi(a_i)^k) = \mathcal{B}_{\infty,w}, \tag{1}$$

where a_1, \dots, a_d denotes a basis of \mathfrak{g} , compare e. g. [7]. On the other hand, by Dixmier and Malliavin [3], \mathcal{B}_∞ coincides with $\mathcal{B}_{\infty,s}$, and therefore it follows that

$$\mathcal{B}_\infty = \mathcal{B}_{\infty,s} = \mathcal{B}_{\infty,w}.$$

\mathcal{B}_∞ is invariant under the G -action π and the \mathfrak{g} -action $d\pi$, by which one gets representations of G and \mathfrak{g} on \mathcal{B}_∞ . The fact that $\mathcal{B}_{\infty,s} = \mathcal{B}_{\infty,w}$ can also be deduced from the following general argument going back to Grothendieck: If M is a non-compact C^∞ -manifold and E a locally convex, complete, Hausdorff, topological vector space, then $f : M \rightarrow E$ is a C^∞ -mapping with respect to the locally convex topology if and only if, for all $\mu \in E^*$, the function $\mu(f(m))$ on M is infinitely often differentiable, see [9], page 484.

Assume now that G is a real linear algebraic group. As a smooth, real, affine algebraic variety, G is a real analytic manifold, and hence a real analytic Lie group. Therefore, the exponential map is, locally, a real analytic homeomorphism. Taking a sufficiently small neighbourhood of zero in \mathfrak{g} , and assuming a decomposition of \mathfrak{g} of the form $\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_l$, $(X_1, \dots, X_l) \mapsto g e^{X_1} \dots e^{X_l}$ becomes an analytic homeomorphism of the aforementioned neighbourhood onto an open neighbourhood of $g \in G$. With the identification $\mathfrak{g} \simeq \mathbb{R}^d$, and with respect to a basis a_1, \dots, a_d of \mathfrak{g} , the canonical coordinates of second type of a point $g \in G$ are then given by

$$\Phi_g : gU_e \ni g e^{t_1 a_1} \dots e^{t_d a_d} \mapsto (t_1, \dots, t_d) \in W_0, \tag{2}$$

where W_0 denotes a sufficiently small neighbourhood of 0 in \mathbb{R}^d , and $U_e = \exp(W_0)$. We will write for Φ_e simply Φ . Let $\varphi \in \mathcal{B}_\infty$, so that $g \mapsto \pi(g)\varphi$ becomes a C^∞ -map from G to \mathcal{B} with respect to the strong and weak topology of \mathcal{B} . This is equivalent to the fact that, for all $g \in G$, the map $(t_1, \dots, t_d) \mapsto \pi(\Phi_g^{-1}(t))\varphi$ is infinitely often strongly, respectively weakly, differentiable on W_0 . With regard to any of these topologies, we obtain for $t \in W_0$ the relations

$$\begin{aligned} d\pi(a_j)\pi(\Phi_g^{-1}(t))\varphi &= \lim_{h \rightarrow 0} h^{-1}[\pi(e^{ha_j}) - \mathbf{1}]\pi(\Phi_g^{-1}(t))\varphi \\ &= \frac{d}{dh} \left(\pi(\Phi_g^{-1}(s^j(h, t)))\varphi \right) \Big|_{h=0} = \sum_{k=1}^d \frac{\partial}{\partial t_k} \left(\pi(\Phi_g^{-1}(t))\varphi \right) \frac{d}{dh} s_k^j(0, t); \end{aligned} \tag{3}$$

here the $s_k^j(h, t)$ are real analytic functions such that

$$e^{ha_j} g e^{t_1 a_1} \dots e^{t_d a_d} = g e^{s_1^j(h, t) a_1} \dots e^{s_d^j(h, t) a_d},$$

since $e^{ha_j} g e^{t_1 a_1} \dots e^{t_d a_d} \in \Phi_g^{-1}(W_0)$ for small h . In a similar way, we have

$$\pi(\Phi_g^{-1}(t))d\pi(a_j)\varphi = \sum_{k=1}^d \frac{\partial}{\partial t_k} \left(\pi(\Phi_g^{-1}(t))\varphi \right) \frac{d}{dh} r_k^j(0, t), \tag{4}$$

with real analytic functions $r_k^j(h, t)$ satisfying the relations

$$g e^{t_1 a_1} \dots e^{t_d a_d} e^{ha_j} = g e^{r_1^j(h, t) a_1} \dots e^{r_d^j(h, t) a_d}.$$

Clearly, $s_k^j(0, t) = r_k^j(0, t) = t_k$, and $s_k^j(h, 0) = r_k^j(h, 0) = \delta_{kj}h$ for $g = e$. Finally, we also note that

$$\begin{aligned} \frac{\partial}{\partial t_j} \left(\pi(\Phi_g^{-1}(t))\varphi \right) &= \lim_{h \rightarrow 0} h^{-1} \pi(\Phi_g^{-1}(t)) [\pi(\Phi_e^{-1}(t^j(h, t))) - 1] \varphi \\ &= \sum_{k=1}^d \pi(\Phi_g^{-1}(t)) d\pi(a_k) \varphi \frac{d}{dh} t_k^j(0, t), \end{aligned} \tag{5}$$

where the $t_k^j(h, t)$ are real analytic functions in h and t , and satisfy the relations

$$e^{-t_d a_d} \dots e^{-t_{j+1} a_{j+1}} e^{h a_j} e^{t_{j+1} a_{j+1}} \dots e^{t_d a_d} = e^{t_1^j(h, t) a_1} \dots e^{t_d^j(h, t) a_d}.$$

One has $t_k^j(0, t) = 0$, $t_k^j(h, 0) = \delta_{kj}h$, so that, in particular,

$$\frac{\partial}{\partial t_k} \left(\pi(\Phi_g^{-1}(t))\varphi \right) \Big|_{t=0} = \pi(g) d\pi(a_k) \varphi.$$

3. The regular representation $(\pi, C_0(M))$ of a real affine G -variety

We come now to the proper subject of this paper. Let M be a smooth, real affine algebraic variety and G a real linear algebraic group, which acts regularly on M . In what follows, we will view G as a closed subgroup of $GL(n, \mathbb{R})$, and M as embedded in \mathbb{R}^n . Denote by $C_0(M)$ the vector space of all continuous, complex valued functions on M which vanish at infinity; provided with the supremum norm, $C_0(M)$ becomes a Banach space. According to the Riesz representation theorem for locally compact, Hausdorff, topological vector spaces, its dual is given by the Banach space of all regular, complex measures $\mu : \mathfrak{B} \rightarrow \mathbb{C}$ on M with norm $|\mu|(M)$. Here \mathfrak{B} denotes the σ -algebra of all Borel sets of M , $|\mu|$ the variation of μ , and one has $C_0(M) \subset \mathcal{L}^1(\mu)$. The G -action on M induces a representation π of G on $C_0(M)$ by bounded linear operators according to

$$\pi(g) : C_0(M) \rightarrow C_0(M), \quad (\pi(g)\varphi)(m) = \varphi(g^{-1}m),$$

where $g \in G$. Henceforth, this representation will be called the *left regular representation* of G on $C_0(M)$. It is continuous with respect to the weak topology on $C_0(M)$, which is characterized by the family of seminorms $|\varphi|_{\mu_1, \dots, \mu_l} = \sup_i |\mu_i(\varphi)|$, $\mu_i \in C_0(M)^*$, $l \geq 1$. Indeed, by the theorem of Lebesgue on bounded convergence, one immediately deduces

$$\lim_{g \rightarrow e} \mu(\pi(g)\varphi) = \lim_{g \rightarrow e} \int \varphi(g^{-1}m) d\mu(m) = \int \varphi(m) d\mu(m) = \mu(\varphi)$$

for all $\varphi \in C_0(M)$ and complex measures μ , as well as $\mu \circ \pi(g) \in C_0(M)^*$ for all $g \in G$. Hence, by the considerations of the previous section, π is a Banach representation of G . In the following, let $\mathbb{C}[M]$ be the ring of all functions that arise by restriction of polynomials in \mathbb{R}^n to M and denote by \mathcal{P} the subspace

$$\mathcal{P} = \mathbb{C}[M] \cdot e^{-r^2},$$

where $r^2(m) = m_1^2 + \dots + m_n^2 = m^2$ is the square of the distance of a point $m \in M$ to the origin in \mathbb{R}^n with respect to the coordinates m_1, \dots, m_n . According to Agricola and Friedrich [1], \mathcal{P} is norm-dense in $C_0(M)$. Although \mathcal{P} is not invariant under the G -action π , the next proposition shows that \mathcal{P} is a $d\pi(\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}))$ -invariant subspace of $C_0(M)_{\infty}$. As will be shown later, the elements in \mathcal{P} are even analytic.

Proposition 3.1. \mathcal{P} is a \mathfrak{g} -submodule of $C_0(M)_\infty$.

Proof. As already explained, we can view $C_0(M)$ as endowed with the weak topology. Let a_1, \dots, a_d be a basis of \mathfrak{g} , $p \in \mathbb{C}[M]$ a polynomial on M , and $\varphi = p \cdot e^{-r^2}$ an element of $\mathcal{P} \subset C_0(M)$. One computes

$$\left. \frac{d}{dh} \left(\varphi(e^{-ha_j} m) \right) \right|_{h=0} = \left\langle (\text{grad } \varphi)(m), \left. \frac{d}{dh} (e^{-ha_j} m) \right|_{h=0} \right\rangle, \quad (6)$$

as well as

$$(\text{grad } \varphi)(m) = ((\text{grad } p)(m) - 2p(m)m)e^{-m^2}.$$

By assumption, $G \subset \text{GL}(n, \mathbb{R})$ and $\mathfrak{g} \subset M_n(\mathbb{R})$ act on $M \subset \mathbb{R}^n$ by matrices, and we obtain

$$\left. \frac{d}{dh} \left(e^{-ha_j} m \right) \right|_{h=0} = \lim_{h \rightarrow 0} h^{-1} \sum_{k=1}^{\infty} \frac{(-ha_j)^k}{k!} m = -a_j m.$$

Hence, if \tilde{a}_j denotes the vector field $(\tilde{a}_j)_m = a_j m$ on M , one has $-\tilde{a}_j \varphi = d/dh \varphi(e^{-ha_j} \cdot)|_{h=0} \in \mathcal{P} \subset C_0(M)$. For arbitrary $\mu \in C_0(M)^*$, we therefore get, according to Lebesgue,

$$\lim_{h \rightarrow 0} h^{-1} \mu([\pi(e^{ha_j}) - \mathbf{1}]\varphi) = \int_M \lim_{h \rightarrow 0} h^{-1} [\varphi(e^{-ha_j} m) - \varphi(m)] d\mu(m) = -\mu(\tilde{a}_j \varphi).$$

Hence $\varphi \in \mathcal{D}(d\pi(a_j))$ for all $j = 1, \dots, d$, and $d\pi(a_j)\varphi = -\tilde{a}_j \varphi$. Since $\tilde{a}_j \varphi \in \mathcal{P}$, the assertion now follows with (1). \blacksquare

As a consequence, the relations (3)–(5) hold for $\varphi \in \mathcal{P}$, also. Let a_1, \dots, a_d be a basis of \mathfrak{g} as above. In the following we will denote the generators $d\pi(a_i)$ by A_i . We set

$$\Gamma_{a_1, \dots, a_d | \mathcal{P}} = \{(\varphi, A_1 \varphi, \dots, A_d \varphi) \in C_0(M) \times \dots \times C_0(M) : \varphi \in \mathcal{P}\}.$$

As already noted, \mathcal{P} is not $\pi(G)$ -invariant in general, so that, for $\varphi \in \mathcal{P}$, it is not true that $(\pi(g)\varphi, A_1 \pi(g)\varphi, \dots, A_d \pi(g)\varphi)$ is contained in $\Gamma_{a_1, \dots, a_d | \mathcal{P}}$. Nevertheless, it will be shown in the following that the last assertion is correct, if instead of $\Gamma_{a_1, \dots, a_d | \mathcal{P}}$ one considers its closure $\overline{\Gamma_{a_1, \dots, a_d | \mathcal{P}}}$ in $C_0(M) \times \dots \times C_0(M)$ with respect to the strong product topology, and g is taken in a sufficiently small neighbourhood of the unit. To start with, we will need the following lemma.

Lemma 3.2. Let $\varrho > 0$ and $\kappa \in \mathbb{R}^*$. Then, for $l \rightarrow \infty$,

$$e^{-\varrho x^2} \sum_{k=0}^{l-1} \frac{(\kappa x^2)^k}{k!} \longrightarrow e^{-\varrho x^2} e^{\kappa x^2}$$

converges uniformly on \mathbb{R}^n , provided $\varrho/|\kappa| \geq 2$.

Proof. By the comparison criterion for series, one has

$$\left| e^{\kappa x^2} - \sum_{k=0}^{l-1} \frac{(\kappa x^2)^k}{k!} \right| \leq \frac{(|\kappa|x^2)^l}{l!} e^{|\kappa|x^2},$$

thus obtaining

$$\sup_{x \in \mathbb{R}^n} \left| e^{-\varrho x^2} \left(e^{\kappa x^2} - \sum_{k=0}^{l-1} \frac{(\kappa x^2)^k}{k!} \right) \right| \leq \sup_{x \in \mathbb{R}^n} \frac{(|\kappa|x^2)^l}{l!} e^{x^2(|\kappa|-\varrho)}.$$

Assume $\varrho/|\kappa| > 1$, so that $\lambda = \varrho/|\kappa| - 1 > 0$. The supremum on the right hand side of the last inequality, denoted in the following by $\Sigma_{l,\lambda}$, can be computed as

$$\Sigma_{l,\lambda} = \frac{1}{l!} \sup_{y \geq 0} y^l e^{-\lambda y} = \frac{1}{l!} (l/\lambda)^l e^{-l}. \tag{7}$$

By the Stirling formula, $l!$ is asymptotically given by $\sqrt{l} l^l e^{-l}$, so that, in case $\lambda \geq 1$, one deduces $\Sigma_{l,\lambda} \rightarrow 0$ for $l \rightarrow \infty$, and hence the assertion. ■

Now, we are able to prove the announced result. For this sake, let us define the set

$$\Gamma_{a_1, \dots, a_d | \pi(U)\mathcal{P}} = \{(\varphi, A_1\varphi, \dots, A_d\varphi) \in C_0(M) \times \dots \times C_0(M) : \varphi \in \pi(g)\mathcal{P}, g \in U\},$$

where U denotes a neighbourhood of $e \in G$.

Proposition 3.3. *Let U be a sufficiently small neighbourhood of the unit in G , $\varphi \in \mathcal{P}$, $g \in U$. Then there exists a series $\varphi_k^g \in \mathcal{P}$ such that $\|\pi(g)\varphi - \varphi_k^g\| \rightarrow 0$, and $\|A_j\pi(g)\varphi - A_j\varphi_k^g\| \rightarrow 0$ for all $j = 1, \dots, d$. In other words,*

$$\Gamma_{a_1, \dots, a_d | \pi(U)\mathcal{P}} \subset \overline{\Gamma_{a_1, \dots, a_d | \mathcal{P}}}.$$

Proof. Let $\varphi = p \cdot e^{-r^2} \in \mathcal{P}$, and define on the subspace \mathcal{P} the linear operators

$$\pi_k(g)\varphi = \varrho(g)p \sum_{j=0}^k \frac{(r^2 - \varrho(g)r^2)^j}{j!} \cdot e^{-r^2}, \quad g \in G, \tag{8}$$

where $\varrho(g)p(m) = p(g^{-1}m)$ denotes the left regular representation of G on $\mathbb{C}[M]$. Note that \mathcal{P} is left invariant under the operators (8), which, in general, can not be extended to continuous operators on $C_0(M)$. We set $\varphi_k^g = \pi_k(g)\varphi$, and compute

$$\begin{aligned} \|\pi(g)\varphi - \varphi_{k-1}^g\| &= \sup_{m \in M} \left| (\varrho(g)p)(m) \left[e^{-(g^{-1}m)^2} - e^{-m^2} \sum_{j=0}^{k-1} \frac{1}{j!} (m^2 - (g^{-1}m)^2)^j \right] \right| \\ &\leq \sup_{m \in M} \left| (\varrho(g)p)(m) e^{-m^2/2} \right| \sup_{m \in M} \left| e^{-m^2/2} \left(e^{m^2 - (g^{-1}m)^2} - \sum_{j=0}^{k-1} \frac{1}{j!} (m^2 - (g^{-1}m)^2)^j \right) \right|. \end{aligned}$$

Writing g_{ij} for the matrix entries of an arbitrary element $g \in G$, one computes

$$\begin{aligned} (gm)^2 &= (g_{11}m_1 + \dots + g_{1n}m_n)^2 + \dots + (g_{n1}m_1 + \dots + g_{nn}m_n)^2 \\ &= \sum_{k,i=1}^n g_{ki}^2 m_i^2 + 2 \sum_{k=1}^n \sum_{i<j}^n g_{ki} g_{kj} m_i m_j \\ &\leq \sum_{i=1}^n g_{ii}^2 m_i^2 + \sum_{i \neq k}^n g_{ki}^2 m_i^2 + \sum_{k=1}^n \sum_{i<j}^n g_{ki} g_{kj} (m_i^2 + m_j^2) \\ &\leq \left(\max_i g_{ii}^2 + \max_{k \neq i} g_{ki}^2 + n \max_{k;i<j} |g_{ki} g_{kj}| \right) m^2; \end{aligned}$$

in particular,

$$|(gm)^2 - m^2| \leq \left(\max_i |g_{ii}^2 - 1| + \max_{k \neq i} g_{ki}^2 + n \max_{k;i<j} |g_{ki} g_{kj}| \right) m^2.$$

Setting $\kappa_{g^{-1}} = \max_i |g_{ii}^2 - 1| + \max_{k \neq i} g_{ki}^2 + n \max_{k;i<j} |g_{ki} g_{kj}|$, we therefore obtain the estimate

$$\left| e^{m^2 - (g^{-1}m)^2} - \sum_{j=0}^{k-1} \frac{1}{j!} (m^2 - (g^{-1}m)^2)^j \right| \leq \sum_{j \geq k} \frac{1}{j!} |m^2 - (g^{-1}m)^2|^j \leq \sum_{j \geq k} \frac{1}{j!} (\kappa_g m^2)^j, \tag{9}$$

so that, with $C = \sup_{m \in M} \left| (\varrho(g)p)(m) e^{-m^2/2} \right|$,

$$\|\pi(g)\varphi - \varphi_{k-1}^g\| \leq C \sup_{m \in M} \left| e^{-m^2/2} \left(e^{\kappa_g m^2} - \sum_{j=0}^{k-1} \frac{1}{j!} (\kappa_g m^2)^j \right) \right|.$$

Assume that U is a sufficiently small neighbourhood of the unit in G such that $\kappa_g \leq 1/4$ for all $g \in U$. Then, by Lemma 3.2, $\|\pi(g)\varphi - \varphi_{k-1}^g\|$ goes to zero as $k \rightarrow \infty$, for arbitrary $g \in U$. It remains to show that, for $j = 1, \dots, d$, $\|A_j \pi(g)\varphi - A_j \varphi_{k-1}^g\|$ goes to zero as k goes to infinity. Now, since $A_j \varphi = -\tilde{a}_j \varphi = d/dh \varphi(e^{-ha_j} \cdot)|_{h=0}$, one has

$$\begin{aligned} \|A_j \pi(g)\varphi - A_j \varphi_{k-1}^g\| &= \sup_M \left| \frac{d}{dh} \pi(g)\varphi(e^{-ha_j} m) \Big|_{h=0} - \frac{d}{dh} \pi_{k-1}(g)\varphi(e^{-ha_j} m) \Big|_{h=0} \right| \\ &\leq \sup_{m \in M} \left| \frac{d}{dh} \pi(g)\varphi(e^{-ha_j} m) \Big|_{h=0} e^{(g^{-1}m)^2 - m^2} \left(e^{m^2 - (g^{-1}m)^2} - \sum_{j=0}^{k-1} \frac{1}{j!} (m^2 - (g^{-1}m)^2)^j \right) \right| \\ &\quad + \sup_{m \in M} \left| (\pi(g)\varphi)(m) \frac{d}{dh} \left(e^{-q(h)} \sum_{j=0}^{k-1} \frac{1}{j!} (q(h))^j \right) \Big|_{h=0} \right|, \end{aligned}$$

where we set $q(h) = (e^{-ha_j} m)^2 - (g^{-1}e^{-ha_j} m)^2$. The first summand on the right hand side converges to zero for $k \rightarrow \infty$ since, as a consequence of (9), and

$$\frac{d}{dh} \pi(g)\varphi(e^{-ha_j} m) \Big|_{h=0} = - \langle (\text{grad } p)(g^{-1}m) - 2p(g^{-1}m)g^{-1}m, g^{-1}a_j m \rangle e^{-(g^{-1}m)^2},$$

it can be estimated by

$$C' \sup_M \left| e^{-m^2/2} \left(e^{\kappa_g m^2} - \sum_{j=0}^{k-1} \frac{1}{j!} (\kappa_g m^2)^j \right) \right|$$

with $C' = \sup_{m \in M} \left| \langle (\text{grad } p)(g^{-1}m) - 2p(g^{-1}m)g^{-1}m, g^{-1}a_j m \rangle e^{-m^2/2} \right|$, and a repeated application of Lemma 3.2 yields the assertion. In order to estimate the second summand, we note that

$$\begin{aligned} \frac{d}{dh} \left(e^{-q(h)} \sum_{j=0}^{k-1} \frac{1}{j!} (q(h))^j \right) \Big|_{h=0} &= \frac{d}{dq} \left(e^{-q} \sum_{j=0}^{k-1} \frac{1}{j!} q^j \right) \Big|_{q=m^2-(g^{-1}m)^2} \dot{q}(0) \\ &= -e^{(g^{-1}m)^2-m^2} \frac{1}{(k-1)!} (m^2 - (g^{-1}m)^2)^{k-1} \dot{q}(0). \end{aligned}$$

Here $\dot{q}(0)$ denotes the polynomial in m which is explicitly given by $\dot{q}(0) = -2(\langle m, a_j m \rangle - \langle g^{-1}m, g^{-1}a_j m \rangle)$. We get for the second summand the upper bound

$$C'' \frac{1}{(k-1)!} \sup_{m \in M} e^{-m^2/2} (\kappa_g m^2)^{k-1} = C'' \Sigma_{k-1, \lambda},$$

where $C'' = \sup_{m \in M} |(\varrho(g)p)(m)\dot{q}(0)e^{-m^2/2}|$, and $\lambda = 1/2\kappa_g$; $\Sigma_{k-1, \lambda}$ was defined in (7). By repeating the arguments given in the proof of Lemma 3.2, it follows that, for $g \in U$, U as above, the second summand also converges to zero for $k \rightarrow \infty$. We get

$$\|A_j(\pi(g)\varphi - \varphi_{k-1}^g)\| \rightarrow 0$$

for $k \rightarrow \infty$, $g \in U$, and $j = 1, \dots, d$. This proves the proposition. ■

Remark 3.4. Proposition 3.3 implies that, for $g \in G$ in a sufficiently small neighbourhood U of e , $\pi(g)\varphi$ has an expansion as an absolutely convergent series in $C_0(M)$ given by

$$\pi(g)\varphi = \sum_{l=0}^{\infty} \varrho(g)p(r^2 - \varrho(g)r^2)^l / l! \cdot e^{-r^2}, \tag{10}$$

where ϱ is the left regular representation of G on $\mathbb{C}[M]$. Indeed, it follows from the proof of Lemma 3.2 that

$$\begin{aligned} \left\| \varrho(g)p \frac{(r^2 - \varrho(g)r^2)^l}{l!} e^{-r^2} \right\| &\leq \frac{\kappa_g^l}{l!} \sup_M |p(g^{-1}(m))e^{-m^2/2}| \cdot \sup_M |m^{2l}e^{-m^2/2}| \\ &= \frac{\kappa_g^l}{l!} C \cdot \sup_{y \geq 0} e^{-y/2} y^l = \frac{\kappa_g^l}{l!} C (2l)^l e^{-l} \approx (2\kappa_g)^l C \frac{1}{\sqrt{l}}, \end{aligned}$$

where we set $C = \sup_M |p(g^{-1}(m))e^{-m^2/2}|$. For the definition of κ_g , see the proof of Proposition 3.3.

We are now in position to show that the generators $d\pi(X)$ are already completely determined by their restriction to the subspace $\mathcal{P} = \mathbb{C}[M] \cdot e^{-r^2}$. A similar dense graph theorem involving the Gårding-subspace was conjectured originally by Hille within the theory of strongly continuous semigroups [5], and afterwards proved by Langlands in his doctoral thesis [6]. Our proof follows essentially the one of Langlands, which, nevertheless, makes use of the G -invariance of the Gårding-subspace. The fact that \mathcal{P} is not $\pi(G)$ -invariant is overcome by means of the approximation argument established in Proposition 3.3.

Theorem 3.5. Let $X_1, \dots, X_k \in \mathfrak{g}$, and denote by Γ_{X_1, \dots, X_k} the graph of the generators $d\pi(X_1), \dots, d\pi(X_k)$, i. e. the set

$$\left\{ (\varphi, d\pi(X_1)\varphi, \dots, d\pi(X_k)\varphi) \in C_0(M) \times \dots \times C_0(M) : \varphi \in \bigcap_{i=1}^k \mathcal{D}(d\pi(X_i)) \right\}.$$

Write $\Gamma_{X_1, \dots, X_k|_{\mathcal{P}}}$ for its restriction to $\mathcal{P} \times \dots \times \mathcal{P}$. Then, with respect to the strong product topology on $C_0(M) \times \dots \times C_0(M)$,

$$\Gamma_{X_1, \dots, X_k} = \overline{\Gamma_{X_1, \dots, X_k|_{\mathcal{P}}}}.$$

In particular, $\Gamma_{a_1, \dots, a_d} = \overline{\Gamma_{a_1, \dots, a_d|_{\mathcal{P}}}}$.

Proof. We assume in the following that $\{a_q, \dots, a_d\}$ represents a maximal linear independent subset of $\{X_1, \dots, X_k\}$. It suffices to verify the assertion for this set, then. Let $(\varphi, \psi_q, \dots, \psi_d) \in \overline{\Gamma_{a_q, \dots, a_d|_{\mathcal{P}}}}$. There exists a series of functions $\varphi_n \in \mathcal{P}$ such that

$$(\varphi_n, A_q\varphi_n, \dots, A_d\varphi_n) \rightarrow (\varphi, \psi_q, \dots, \psi_d)$$

with respect to the strong topology in $C_0(M) \times \dots \times C_0(M)$. We obtain $\psi_i = A_i\varphi$ for all $i = q, \dots, d$, the A_i being norm-closed, and, thus, $(\varphi, \psi_q, \dots, \psi_d) \in \Gamma_{a_q, \dots, a_d}$. To prove the converse inclusion $\Gamma_{a_q, \dots, a_d} \subset \overline{\Gamma_{a_q, \dots, a_d|_{\mathcal{P}}}}$, we consider local regularizations of π . Consider the canonical coordinates of second type $\Phi : U_\varepsilon \rightarrow W_0$ introduced in (2), and let $Q_\varepsilon = [0, \varepsilon] \times \dots \times [0, \varepsilon]$ be a cube in W_0 of length ε . If dt stands for Lebesgue-measure on \mathbb{R}^d and χ_{Q_ε} for the characteristic function of Q_ε , then, by the weak continuity of the G -representation π , the map $\chi_{Q_\varepsilon}(t) \pi(\Phi^{-1}(t))\varphi$ is weakly measurable with respect to dt for $\varphi \in C_0(M)$, as well as separable-valued, and therefore, by Pettis, strongly measurable, see e. g. [5]. Because of

$$\int_{Q_\varepsilon} \|\pi(\Phi^{-1}(t))\varphi\| dt = \|\varphi\| \cdot \varepsilon^d,$$

it follows that $\chi_{Q_\varepsilon}(t) \pi(\Phi^{-1}(t))\varphi$ is Bochner-integrable, and we define on $C_0(M)$ the linear bounded operators $\pi(\chi_{Q_\varepsilon})\varphi = \int_{Q_\varepsilon} \pi(\Phi^{-1}(t))\varphi dt$, in accordance with the regularizations $\pi(f)$ of π already introduced. Clearly, $\|\varepsilon^{-d}\pi(\chi_{Q_\varepsilon})\varphi - \varphi\| \rightarrow 0$ for $\varepsilon \rightarrow 0$ and arbitrary $\varphi \in C_0(M)$, since, as a consequence of the dominated convergence theorem of Lebesgue for Bochner integrals,

$$\text{s-lim}_{\varepsilon \rightarrow 0} \varepsilon^{-d}\pi(\chi_{Q_\varepsilon})\varphi = \text{s-lim}_{\varepsilon \rightarrow 0} \int_{Q_1} \pi(\Phi^{-1}(\varepsilon t))\varphi dt = \varphi.$$

Let $\varphi \in C_0(M)_\infty$. From equation (3) one deduces that $\chi_{Q_\varepsilon}(t)A_j\pi(\Phi^{-1}(t))\varphi$ is Bochner integrable, so that using the theorem of Lebesgue, and integrating by

parts, we obtain

$$\begin{aligned} \text{s-lim}_{h \rightarrow 0} h^{-1}(\pi(e^{ha_j}) - \mathbf{1})\pi(\chi_{Q_\varepsilon})\varphi &= \text{s-lim}_{h \rightarrow 0} h^{-1} \int_{Q_\varepsilon} (\pi(e^{ha_j}) - \mathbf{1})\pi(\Phi^{-1}(t))\varphi dt \\ &= \int_{Q_\varepsilon} A_j \pi(\Phi^{-1}(t))\varphi dt = \sum_{k=1}^d \int_{Q_\varepsilon} \frac{\partial}{\partial t_k} \left(\pi(\Phi^{-1}(t))\varphi \right) d_{kj}(t) dt \\ &= \sum_{k=1}^d \int_{\hat{Q}_\varepsilon} \left[d_{kj}(t)\pi(\Phi^{-1}(t))\varphi \right]_{(t_1, \dots, 0, \dots, t_d)}^{(t_1, \dots, \varepsilon, \dots, t_d)} dt_1 \wedge \hat{\cdot} \wedge dt_d \\ &\quad - \sum_{k=1}^d \int_{Q_\varepsilon} \pi(\Phi^{-1}(t))\varphi \frac{\partial}{\partial t_k} d_{kj}(t) dt, \end{aligned}$$

where $j = 1, \dots, d$, and $d_{kj}(t) = \frac{d}{dh} s_k^j(0, t)$. The symbol $\hat{\cdot}$ indicates that integration over the variable t_k is suppressed. As a consequence, $\pi(\chi_{Q_\varepsilon})\varphi \in \mathcal{D}(A_i)$ for all $\varphi \in C_0(M)_\infty$. Let us denote the difference on the right hand side of the last equality by $F_j(\varphi)$. It is defined for arbitrary $\varphi \in C_0(M)$, and continuous in φ with respect to the strong topology in $C_0(M)$. Since $C_0(M)_\infty$ is dense, and $\pi(\chi_{Q_\varepsilon})$ is bounded, we obtain, by the closedness of the A_i , that $\pi(\chi_{Q_\varepsilon})\varphi \in \mathcal{D}(A_i)$ for arbitrary $\varphi \in C_0(M)$, and

$$A_j \pi(\chi_{Q_\varepsilon})\varphi = F_j(\varphi) \tag{11}$$

for all $\varphi \in C_0(M)$. Assume now that $\varphi \in \bigcap_{i=q, \dots, d} \mathcal{D}(A_i)$. As already observed, $\text{s-lim}_{\varepsilon \rightarrow 0} \varepsilon^{-d} \pi(\chi_{Q_\varepsilon})\varphi = \varphi$, and we show that, similarly,

$$\text{s-lim}_{\varepsilon \rightarrow 0} \varepsilon^{-d} A_j \pi(\chi_{Q_\varepsilon})\varphi = A_j \varphi$$

for all $j = q, \dots, d$. Writing $d_{kj}(t) = \delta_{kj} + \sum_{|\alpha| \geq 1} c_\alpha^{kj} t^\alpha$ with complex coefficients c_α^{kj} , we obtain with (11), by substitution of variables,

$$\begin{aligned} \frac{A_j \pi(\chi_{Q_\varepsilon})\varphi}{\varepsilon^d} &= \sum_{k=1}^d \int_{\hat{Q}_1} \delta_{kj} \left[\pi(\Phi^{-1}(t))\varphi \right]_{(t_1, \dots, 0, \dots, t_d)\varepsilon}^{(t_1, \dots, 1, \dots, t_d)\varepsilon} \frac{dt_1 \wedge \hat{\cdot} \wedge dt_d}{\varepsilon} \\ &\quad + \sum_{k=1}^d \int_{\hat{Q}_1} \sum_{l \neq k} \frac{\partial d_{kj}}{\partial t_l}(0) \left[t_l \pi(\Phi^{-1}(t))\varphi \right]_{(t_1, \dots, 0, \dots, t_d)\varepsilon}^{(t_1, \dots, 1, \dots, t_d)\varepsilon} dt_1 \wedge \hat{\cdot} \wedge dt_d \\ &\quad + \sum_{k=1}^d \int_{\hat{Q}_1} \left(\frac{\partial d_{kj}}{\partial t_k}(0) \varepsilon \pi(\Phi^{-1}(\varepsilon t_1, \dots, \varepsilon, \dots, \varepsilon t_d))\varphi - 0 \right) \frac{dt_1 \wedge \hat{\cdot} \wedge dt_d}{\varepsilon} \\ &\quad + \sum_{k=1}^d \int_{\hat{Q}_1} \sum_{|\alpha| \geq 2} c_\alpha^{kj} \left[t^\alpha \pi(\Phi^{-1}(t))\varphi \right]_{(t_1, \dots, 0, \dots, t_d)\varepsilon}^{(t_1, \dots, 1, \dots, t_d)\varepsilon} \frac{dt_1 \wedge \hat{\cdot} \wedge dt_d}{\varepsilon} \\ &\quad - \sum_{k=1}^d \int_{Q_1} \frac{\partial d_{kj}}{\partial t_k}(\varepsilon t) \pi(\Phi^{-1}(\varepsilon t))\varphi dt. \end{aligned}$$

As ε goes to zero, the second and fourth summand on the right hand vanish, while the third and fifth cancel each other. Hence, only the limit of the first summand

$$\varepsilon^{-1} \int_{\hat{Q}_1} \pi(e^{\varepsilon t_1 a_1}) \cdots [\pi(e^{\varepsilon a_j}) - \mathbf{1}] \cdots \pi(e^{\varepsilon t_d a_d})\varphi dt_1 \wedge \hat{\cdot} \wedge dt_d,$$

as $\varepsilon \rightarrow 0$, remains. In the following, we prove that

$$\text{s-lim}_{\varepsilon \rightarrow 0} \varepsilon^{-1} \pi(e^{\varepsilon t_1 a_1}) \cdots [\pi(e^{\varepsilon a_j}) - \mathbf{1}] \cdots \pi(e^{\varepsilon t_d a_d}) \varphi = A_j \varphi$$

for all $j = q, \dots, d$. For $j = d$, the assertion is clear. Let, therefore, j be equal $d - 1$. Then,

$$[\pi(e^{\varepsilon a_{d-1}}) - \mathbf{1}] \pi(e^{\varepsilon t_d a_d}) \varphi = [\pi(e^{\varepsilon a_{d-1}}) - \mathbf{1}] \varphi + [\pi(e^{\varepsilon a_{d-1}}) - \mathbf{1}] [\pi(e^{\varepsilon t_d a_d}) - \mathbf{1}] \varphi.$$

Now, since $\varepsilon^{-1} [\pi(e^{\varepsilon t_d a_d}) - \mathbf{1}] \varphi$ converges strongly to $t_d A_d \varphi$ as $\varepsilon \rightarrow 0$, and $\|\varepsilon^{-1} [\pi(e^{\varepsilon t_d a_d}) - \mathbf{1}] \varphi\|$ to $\|t_d A_d \varphi\|$, we obtain

$$\|[\pi(e^{\varepsilon a_{d-1}}) - \mathbf{1}] \varepsilon^{-1} [\pi(e^{\varepsilon t_d a_d}) - \mathbf{1}] \varphi\| \rightarrow 0,$$

and hence the assertion for $j = d - 1$. By iteration we get, for arbitrary $j = q, \dots, d$,

$$\begin{aligned} & \varepsilon^{-1} [\pi(e^{\varepsilon a_j}) - \mathbf{1}] \pi(e^{\varepsilon t_{j+1} a_{j+1}}) \cdots \pi(e^{\varepsilon t_d a_d}) \varphi \\ &= \frac{\pi(e^{\varepsilon a_j}) - \mathbf{1}}{\varepsilon} \varphi + [\pi(e^{\varepsilon a_j}) - \mathbf{1}] \sum_{i=j+1}^d \left(\prod_{m=j+1}^{i-1} \pi(e^{\varepsilon t_m a_m}) \right) \frac{\pi(e^{\varepsilon t_i a_i}) - \mathbf{1}}{\varepsilon} \varphi, \end{aligned}$$

and a repetition of the arguments above finally yields the desired statement for $j = q, \dots, d$. Taking all together, we conclude for $\varphi \in \bigcap_{i=q}^d \mathcal{D}(A_i)$ that

$$\varepsilon^{-d} (\pi(\chi_{Q_\varepsilon}) \varphi, A_q \pi(\chi_{Q_\varepsilon}) \varphi, \dots, A_d \pi(\chi_{Q_\varepsilon}) \varphi) \rightarrow (\varphi, A_q \varphi, \dots, A_d \varphi) \quad (12)$$

as $\varepsilon \rightarrow 0$ with respect to the strong product topology in $C_0(M) \times \cdots \times C_0(M)$. Now, for $\varphi \in C_0(M)_\infty$,

$$\begin{aligned} & (\pi(\chi_{Q_\varepsilon}) \varphi, A_q \pi(\chi_{Q_\varepsilon}) \varphi, \dots, A_d \pi(\chi_{Q_\varepsilon}) \varphi) \\ &= \int_{Q_\varepsilon} (\pi(\Phi^{-1}(t)) \varphi, A_q \pi(\Phi^{-1}(t)) \varphi, \dots, A_d \pi(\Phi^{-1}(t)) \varphi) dt. \end{aligned} \quad (13)$$

If ε is sufficiently small, and $\varphi \in \mathcal{P}$, then, by Proposition 3.3,

$$(\pi(\Phi^{-1}(t)) \varphi, A_q \pi(\Phi^{-1}(t)) \varphi, \dots, A_d \pi(\Phi^{-1}(t)) \varphi) \in \overline{\Gamma_{a_q, \dots, a_d} \mathcal{P}}, \quad t \in Q_\varepsilon,$$

so that the integral (13) also lies in $\overline{\Gamma_{a_q, \dots, a_d} \mathcal{P}}$. Since \mathcal{P} is dense, it follows with (11), F_j being continuous, that

$$(\pi(\chi_{Q_\varepsilon}) \varphi, A_q \pi(\chi_{Q_\varepsilon}) \varphi, \dots, A_d \pi(\chi_{Q_\varepsilon}) \varphi) \in \overline{\Gamma_{a_q, \dots, a_d} \mathcal{P}}$$

for arbitrary $\varphi \in C_0(M)$ and sufficiently small ε . With (12) we finally obtain $(\varphi, A_q \varphi, \dots, A_d \varphi) \in \overline{\Gamma_{a_q, \dots, a_d} \mathcal{P}}$ for $\varphi \in \bigcap_{i=q}^d \mathcal{D}(A_i)$. This proves the theorem. ■

4. Analytic elements of $(\pi, C_0(M))$

Let π be a continuous representation of a Lie group G on a Banach space \mathcal{B} , and denote by \mathcal{B}_ω the space of all analytic elements in \mathcal{B} , i. e. the space of all $\varphi \in \mathcal{B}$ for which $g \mapsto \pi(g)\varphi$ is an analytic map from G to \mathcal{B} . \mathcal{B}_ω is invariant under the G -action π and the action $d\pi$ of \mathfrak{g} , norm-dense in \mathcal{B} , and one has the inclusion $\mathcal{B}_\omega \subset \mathcal{B}_\infty$. Analytic elements of Banach representations were first studied by Harish–Chandra in [4], and their importance is due to the fact that the closure of every $d\pi(\mathfrak{U}(\mathfrak{g}_\mathbb{C}))$ -invariant subspace of \mathcal{B}_ω constitutes a $\pi(G)$ -invariant subspace of \mathcal{B} . In addition note that every closed, $\pi(G)$ -invariant subspace is also $\pi(C_c(G))$ -invariant, where $C_c(G)$ denotes the space of continuous, complex valued functions on G with compact support. Assume now that G is a real linear algebraic group acting regularly on a smooth, real affine variety M , and consider the left regular representation of G on $C_0(M)$ as introduced in the previous section. The following theorem states that the elements of $\mathcal{P} = \mathbb{C}[M] \cdot e^{-r^2}$ are contained in $C_0(M)_\omega$, and is a consequence of the approximation argument given in Proposition 3.3.

Theorem 4.1. *The elements of the \mathfrak{g} -module $\mathcal{P} = \mathbb{C}[M] \cdot e^{-r^2}$ are analytic vectors of the left regular representation π of G on $C_0(M)$.*

Proof. Let $\varphi = p \cdot e^{-r^2} \in \mathcal{P}$. Since $g \mapsto h^{-1}g$ is an analytic isomorphism for arbitrary $h \in G$, it suffices to show that $g \mapsto \pi(g)\varphi$ is analytic in a neighbourhood of the unit element, see e. g. [4]. According to (10), $\pi(g)\varphi$ is given by the series

$$\pi(g)\varphi = \sum_{l=0}^{\infty} \varrho(g)p(r^2 - \varrho(g)r^2)^l / l! \cdot e^{-r^2},$$

which converges absolutely in $C_0(M)$ provided that $g \in G$ is contained in a sufficiently small neighbourhood U of e . Since G acts regularly on the affine variety M , $(g, m) \mapsto gm$ is an analytic map from $G \times M$ to M , and therefore $(g, m) \mapsto (\varrho(g)p)(m)$ an analytic function, being also polynomial in m . Thus,

$$(\varrho(g)p)(m) = p(g^{-1}m) = \sum_{\Lambda, \beta} c_{\Lambda\beta} \theta^\Lambda(g^{-1}) m^\beta, \tag{14}$$

with constants $c_{\Lambda\beta}$ and multiindices Λ, β , where $\theta^\Lambda(g) = \theta_{11}^{\Lambda_{11}}(g) \dots \theta_{nn}^{\Lambda_{nn}}(g)$, and $m^\beta = m_1^{\beta_1} \dots m_n^{\beta_n}$, the sum being finite. The coefficient functions $\theta_{ij}(g) = g_{ij}$, as well as their powers, are real analytic. In a neighbourhood of the unit element, they are given by the absolute convergent Cauchy product series

$$\theta^\Lambda(g) = \sum_{|\gamma| \geq 0} b_\gamma^\Lambda \Theta^\gamma(g) = \sum_{\gamma_1, \dots, \gamma_d \geq 0} b_{\gamma_1, \dots, \gamma_d}^\Lambda \Theta_1^{\gamma_1}(g) \dots \Theta_d^{\gamma_d}(g), \tag{15}$$

where the $\Theta_1, \dots, \Theta_d$ are supposed to be coordinates near e with $\Theta_i(e) = 0$. By rearranging the sums we obtain

$$(\varrho(g)p)(m) = \sum_{|\gamma| \geq 0} p_\gamma(m) \Theta^\gamma(g^{-1}),$$

where we put $p_\gamma(m) = \sum_{\Lambda, \beta} c_{\Lambda\beta} b_\gamma^\Lambda m^\beta$. In a similar way, $(g, m) \mapsto (r^2 - \varrho(g)r^2)^l(m)$ is a real analytic function on $G \times M$ which is a polynomial expression in m . The coefficient functions have an expansion of the form $\theta_{ij}(g) = \delta_{ij} + \sum_{|\gamma| \geq 1} b_\gamma^{ij} \Theta^\gamma(g)$, and we put $\eta_{ij}(g) = \sum_{|\gamma| \geq 1} b_\gamma^{ij} \Theta^\gamma(g)$. In this way we get

$$\begin{aligned} & m^2 - (g^{-1}m)^2 \\ &= \sum_{i=1}^n (1 - \theta_{ii}^2(g^{-1}))m_i^2 - \sum_{i \neq j} \theta_{ij}^2(g^{-1})m_i^2 - 2 \sum_{k=1}^n \sum_{i < j} \theta_{ki}(g^{-1})\theta_{kj}(g^{-1})m_i m_j \\ &= - \sum_{i,j=1}^n \eta_{ij}^2(g^{-1})m_i^2 - 2 \sum_{i \neq j} \eta_{ij}(g^{-1})m_i m_j - 2 \sum_{k=1}^n \sum_{i < j} \eta_{ki}(g^{-1})\eta_{kj}(g^{-1})m_i m_j, \end{aligned}$$

thus obtaining

$$(m^2 - (g^{-1}m)^2)^l = \sum_{l \leq |\Lambda| \leq 2l, |\beta|=2l} d_{\Lambda\beta}^l \eta^\Lambda(g^{-1})m^\beta, \tag{16}$$

where the $d_{\Lambda\beta}^l$ are real numbers, and the $\eta^\Lambda(g)$ are given by the Cauchy product series

$$\eta^\Lambda(g) = \sum_{|\delta| \geq |\Lambda|} b_\delta^\Lambda \Theta^\delta(g). \tag{17}$$

As a consequence, we can represent $(m^2 - ((g^{-1}m)^2)^l$ near the unit as an infinite sum in the variables $\Theta_1(g^{-1}), \dots, \Theta_d(g^{-1})$ in which only summands of order $\geq l$ do appear. Explicitly, one has

$$(m^2 - (g^{-1}m)^2)^l = \sum_{|\delta| \geq l} q_\delta^l(m) \Theta^\delta(g^{-1}),$$

with $q_\delta^l(m) = \sum_{l \leq |\Lambda| \leq \min(2l, |\delta|), |\beta|=2l} d_{\Lambda\beta}^l b_\delta^\Lambda m^\beta$. For $|\delta| < l$ one has $q_\delta^l \equiv 0$. By choosing U sufficiently small, we may assume that the coordinates $\Theta_1, \dots, \Theta_d$ are defined on U . In the sequel, we shall show that, on U , $\pi(g)\varphi$ has the expansion

$$\pi(g)\varphi = \sum_{|\gamma|, |\delta| \geq 0} p_\gamma \left(\sum_{l=0}^{|\gamma+\delta|} q_\delta^l/l! \right) \cdot e^{-r^2} \Theta^{\gamma+\delta}(g^{-1}). \tag{18}$$

For this sake, we first compute

$$\sum_{|\delta| \geq 1} |q_\delta^1(m) \Theta^\delta(g^{-1})| \leq \sum_{1 \leq |\Lambda| \leq 2, |\beta|=2} |d_{\Lambda\beta}^1| \left(\sum_{|\delta| \geq |\Lambda|} |b_\delta^\Lambda \Theta^\delta(g^{-1})| \right) |m^\beta| \leq K_g m^2,$$

where K_g is a constant which depends only on g and goes to zero for $g \rightarrow e$. But then, because of $q_\delta^l(m) = \sum_{\delta^{(1)} + \dots + \delta^{(l)} = \delta} q_{\delta^{(1)}}^1(m) \cdots q_{\delta^{(l)}}^1(m)$, each summand of the

series appearing in (18) can be estimated according to

$$\begin{aligned} & \left\| p_\gamma \cdot \left(\sum_{l=0}^{|\gamma+\delta|} q_\delta^l / l! \right) \cdot e^{-r^2} \Theta^{\gamma+\delta}(g^{-1}) \right\| \\ = & \sup_M \left| \tilde{p}_\gamma(m) \Theta^\gamma(g^{-1}) \left(\sum_{l=0}^{|\gamma+\delta|} \sum_{\delta^{(1)}+\dots+\delta^{(l)}=\delta} q_{\delta^{(1)}}^1(m) \Theta^{\delta^{(1)}}(g^{-1}) \cdots q_{\delta^{(l)}}^1(m) \Theta^{\delta^{(l)}}(g^{-1}) / l! \right) \right| \\ \leq & \sup_M \left[|\tilde{p}_\gamma(m) \Theta^\gamma(g^{-1})| \sum_{l=0}^{|\gamma+\delta|} (K_g m^2)^l / l! \right] \leq \left(\sum_{\Lambda, \beta} |c_{\Lambda, \beta} b_\gamma^\Lambda \Theta^\gamma(g^{-1})| \sup_M |m^\beta e^{-m^2/2}| \right) \\ & \cdot \left(\sum_{l=0}^{|\gamma+\delta|} K_g^l / l! \sup_M |m^{2l} e^{-m^2/2}| \right) \leq \left(\sum_{\Lambda} C_\Lambda |b_\gamma^\Lambda \Theta^\gamma(g^{-1})| \right) \left(\sum_{l=0}^{|\gamma+\delta|} K_g^l (2l)^l e^{-l} / l! \right), \end{aligned}$$

where we put $\tilde{p}_\gamma(m) = p_\gamma(m) e^{-m^2}$, $C_\Lambda = \sum_{\beta} |c_{\Lambda, \beta}| \sup_M |m^\beta e^{-m^2/2}|$, and made use of (7). The sum over Λ appearing in the last line is a finite linear combination of summands of the convergent series $\sum_{|\gamma| \geq 0} |b_\gamma^\Lambda \Theta^\gamma(g^{-1})|$; the summands of the series over l behave as $(2K_g)^l / \sqrt{l}$ for big l . Since $K_g < 1/2$ for sufficiently small U , we deduce that the sum in (18) converges absolutely, by the comparison criterion and the convergence of the geometric series. Therefore, by Riemann's theorem on the rearrangement of terms of absolutely convergent series, we can write this sum also in the form

$$\sum_{l=0}^{\infty} \sum_{|\gamma|, |\delta| \geq 0} p_\gamma q_\delta^l / l! \cdot e^{-r^2} \Theta^{\gamma+\delta}(g^{-1}).$$

Now, by taking into account equations (14)-(17), and the definition of the polynomials p_γ, q_δ^l , one deduces that

$$\begin{aligned} & \left\| \varrho(g) p(r^2 - \varrho(g) r^2)^l \cdot e^{-r^2} - \sum_{0 \leq |\gamma|, |\delta| \leq N} p_\gamma q_\delta^l \cdot e^{-r^2} \Theta^{\gamma+\delta}(g^{-1}) \right\| \\ \leq & \sup_M \left| \sum_{\Lambda, \beta, \Lambda', \beta'} c_{\Lambda, \beta} d_{\Lambda', \beta'}^l m^{\beta+\beta'} e^{-m^2} \left(\theta^\Lambda(g^{-1}) \eta^{\Lambda'}(g^{-1}) - \sum_{0 \leq |\gamma|, |\delta| \leq N} b_\gamma^\Lambda b_\delta^{\Lambda'} \Theta^{\gamma+\delta}(g^{-1}) \right) \right| \end{aligned}$$

goes to zero as $N \rightarrow \infty$, and we obtain

$$\varrho(g) p^2(r^2 - \varrho(g) r^2)^l \cdot e^{-r^2} = \sum_{|\gamma|, |\delta| \geq 0} p_\gamma q_\delta^l \cdot e^{-r^2} \Theta^{\gamma+\delta}(g^{-1});$$

using equation (10) we get (18). Thus, for $\varphi \in \mathcal{P}$, $\pi(g)\varphi$ is analytic in a neighbourhood of the origin, as contended. ■

5. Subquotients and reducing series of \mathcal{P} and $C_0(M)$

Assume that G is a real reductive group as defined in [8]. Let K be a maximal compact subgroup of G with Lie-Algebra \mathfrak{k} , and

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

the corresponding Cartan decomposition of \mathfrak{g} . If π is a continuous representation of G on a Banach space \mathcal{B} , its restriction to K defines a continuous representation of K on \mathcal{B} . Denote by \hat{K} the set of all equivalence classes of finite dimensional irreducible representations of K . For each $\lambda \in \hat{K}$, let ξ_λ be the character of λ , $d(\lambda)$ the degree of λ , and $\chi_\lambda = d(\lambda)\xi_\lambda$. We define

$$P(\lambda) = \pi(\bar{\chi}_\lambda) = d(\lambda) \int_K \bar{\xi}_\lambda(k)\pi(k) dk,$$

dk normalized Haar measure on K . $P(\lambda)$ is a continuous projection of \mathcal{B} onto $\mathcal{B}(\lambda) = P(\lambda)\mathcal{B}$. Note that $\mathcal{B}(\lambda)$ is the isotypic K -submodule of \mathcal{B} of type λ , i. e. it consists of all vectors, the linear span of whose K -orbit is finite dimensional, and splits into irreducible K -submodules of type λ , see [9].

Let us resume the study of the left regular representation $(\pi, C_0(M))$, as introduced in section 3., of a real linear algebraic group G acting regularly on a smooth, real affine algebraic variety M , and of the \mathfrak{g} -module \mathcal{P} . From now on, we will assume that G is stable under transpose, so that G is real reductive in the sense specified above. Note that \mathbb{R}^n is endowed with a K -invariant scalar product in a natural way. By choosing appropriate coordinates, we can therefore assume that the distance function r^2 is invariant under the action of K . As in the previous sections, let ϱ be the left regular representation of G on $\mathbb{C}[M]$. It is locally regular, so that \mathcal{P} is contained in the subspace of differentiable, K -finite vectors of $(\pi, C_0(M))$.

Lemma 5.1. $\mathcal{P} = \mathbb{C}[M] \cdot e^{-r^2}$ is a (\mathfrak{g}, K) -module in the sense of Harish-Chandra and Lepowsky.

Proof. Plainly, \mathcal{P} is $\pi(K)$ - and $d\pi(\mathfrak{g})$ -invariant. Now, for general $\varphi \in C_0(M)_\infty$, $X \in \mathfrak{g}$, and $g \in G$ we have

$$\begin{aligned} \pi(g)d\pi(X)\varphi &= \lim_{h \rightarrow 0} h^{-1}\pi(g)[\pi(e^{hX}) - 1]\varphi \\ &= \lim_{h \rightarrow 0} h^{-1}[\pi(e^{\text{Ad}(g)hX}) - 1]\pi(g)\varphi = d\pi(\text{Ad}(g)X)\pi(g)\varphi. \end{aligned} \tag{19}$$

Let $\varphi \in \mathcal{P}$, $E_\varphi = \text{Span}\{\pi(K)\varphi\}$. Since $(\varrho, \mathbb{C}[M])$ is locally regular, E_φ is a finite dimensional subspace of \mathcal{P} . Thus, \mathcal{P} is a (\mathfrak{g}, K) -module in the sense of Harish-Chandra and Lepowsky, see e. g. [8]. ■

Lemma 5.2. Let $\mathcal{P}(\lambda) = \mathcal{P} \cap C_0(M)(\lambda)$. Then $\mathcal{P} = \sum_{\lambda \in \hat{K}} \mathcal{P}(\lambda)$, and $\overline{\mathcal{P}(\lambda)} = C_0(M)(\lambda)$.

Proof. Since the linear span of the K -orbit $\pi(K)[p \cdot e^{-r^2}]$ of an arbitrary element in \mathcal{P} is finite dimensional, we obtain the first assertion. The second assertion follows with Proposition 4.4.3.4. of [9]. ■

In what follows, we will call \mathcal{P} also the *underlying* (\mathfrak{g}, K) -module of $C_0(M)$. Note that in general Hilbert representation theory, this role is occupied by the space of differentiable, K -finite vectors. $\mathcal{P}(\lambda)$ is the λ -isotypic component for K of \mathcal{P} .

Remark 5.3. The fact that the sum $\sum_{\lambda \in \hat{K}} \mathcal{P} \cap C_0(M)(\lambda)$ is dense in $C_0(M)$ can also be seen by the following general argument. Fix $\varphi \in C_0(M)$, and $\varepsilon > 0$. Since \mathcal{P} is dense, there exists a polynomial $p = \sum_{\alpha} c_{\alpha} m^{\alpha} \in \mathbb{C}[M]$ such that $\|\varphi - p \cdot e^{-r^2}\| \leq \varepsilon/2$. For any finite subset F in \hat{K} we define $\bar{\chi}_F = \sum_{\lambda \in F} \bar{\chi}_{\lambda}$. As a consequence of the K -invariance of r^2 , we have

$$(P(\lambda) p \cdot e^{-r^2})(m) = \int_K \bar{\chi}_{\lambda}(k) p(k^{-1}m) dk \cdot e^{-r^2} = \sum_{\alpha} c_{\alpha} \int_K \bar{\chi}_{\lambda}(k) (k^{-1}m)^{\alpha} dk \cdot e^{-r^2},$$

so that $P(\lambda) p \cdot e^{-r^2} \in \mathcal{P} \cap C_0(M)(\lambda)$. Now, according to a theorem of Harish-Chandra, for any differentiable vector $\psi \in C_0(M)_{\infty}$, the series $\sum_{\lambda \in \hat{K}} P(\lambda) \psi$ converges absolutely to ψ , see Theorem 4.4.2.1 in [9]. For this reason, we can choose F in such a way that $\|(\pi(\bar{\chi}_F) - 1)p \cdot e^{-r^2}\| \leq \varepsilon/2$. Hence,

$$\|\pi(\bar{\chi}_F)p \cdot e^{-r^2} - \varphi\| \leq \|(\pi(\bar{\chi}_F) - 1)p \cdot e^{-r^2}\| + \|\varphi - p \cdot e^{-r^2}\| \leq \varepsilon.$$

Since for any F , $\pi(\bar{\chi}_F)p \cdot e^{-r^2} \in \sum_{\lambda \in \hat{K}} \mathcal{P} \cap C_0(M)(\lambda)$, we obtain the desired assertion.

Remark 5.4. Let T be a maximal torus of K . The equivalence classes of finite dimensional irreducible representations of K are in one-to-one correspondence with dominant integral and T -integral forms μ on the complexification $\mathfrak{t}_{\mathbb{C}}$ of the Lie-Algebra of T . Let λ_{μ} be the class corresponding to μ , and (π_{μ}, V^{μ}) a representative in λ_{μ} . Denote by $M(\Lambda)$ the Verma module of weight $\Lambda \in \mathfrak{t}_{\mathbb{C}}^*$, and $L(\Lambda)$ the unique, non zero irreducible subquotient of $M(\Lambda)$, see e. g. [8]. Then the differential of π_{μ} is equivalent to the $\mathfrak{k}_{\mathbb{C}}$ -module $L(\mu)$. On the other hand, every finite dimensional simple $\mathfrak{k}_{\mathbb{C}}$ -module has a highest weight Λ , and is equivalent to $L(\Lambda)$. Thus, every λ_{μ} -isotypic component for K is an isotypic component for $\mathfrak{k}_{\mathbb{C}}$ of type $L(\mu)$, and we get $\mathcal{P} = \sum_{\kappa \in \hat{\mathfrak{k}}_{\mathbb{C}}} \mathcal{P}_{\kappa}$, where $\hat{\mathfrak{k}}_{\mathbb{C}}$ denotes the set of all equivalence classes of finite dimensional simple $\mathfrak{k}_{\mathbb{C}}$ -modules. Hence \mathcal{P} is a Harish-Chandra module for $\mathfrak{g}_{\mathbb{C}}$ in the sense of [2].

As already noted at the beginning of the previous section, Theorem 4.1 allows us to derive $\pi(G)$ -invariant decompositions of $C_0(M)$ from algebraic decompositions of \mathcal{P} into $d\pi(\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}))$ -invariant subspaces. To begin with, note that the representation $d\pi$ of \mathfrak{g} on \mathcal{P} is equivalent to a representation $d\pi'$ of \mathfrak{g} on $\mathbb{C}[M]$ given by

$$d\pi'(X) = d\rho(X) - [d\rho(X)r^2], \quad X \in \mathfrak{g}. \tag{20}$$

Define now in \mathcal{P} the subspaces

$$W_p = d\pi(\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})) \text{ Span } \left\{ \pi(K)p \cdot e^{-r^2} \right\}, \quad p \in \mathbb{C}[M].$$

Lemma 5.5. W_p is a (\mathfrak{g}, K) -module in the sense of Harish-Chandra and Lepowsky.

Proof. Similar to the proof of Lemma 5.1. ■

Let $W_p(\lambda)$ be the λ -isotypic component for K of W_p . Although, in general, $\dim W_p(\lambda) = \infty$, so that W_p is not an admissible (\mathfrak{g}, K) -module, any subquotient of W_p turns out to be admissible.

Proposition 5.6. *The subquotients $W_p/W_{d\pi'(X)p}$ are admissible (\mathfrak{g}, K) -modules for any $p \in \mathbb{C}[M]$ and $X \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$.*

Proof. It suffices to consider the case $M = \mathbb{R}^n$, so that one has the grading $\mathbb{C}[\mathbb{R}^n] = \bigoplus_{l \geq 0} \mathbb{C}^l[\mathbb{R}^n]$ by the polynomial degree. Let $p \in \mathbb{C}[M]$, $X \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$. Since $d\pi(X)p \cdot e^{-r^2} = [d\pi'(X)p] \cdot e^{-r^2}$, one has $W_{d\pi'(X)p} \subset W_p$ by (19). Denote by Z the complement of $W_{d\pi'(X)p}(\lambda)$ in $W_p(\lambda)$. Since, for $Y \in \mathfrak{g}$, $d\pi'(Y) : \mathbb{C}^l[\mathbb{R}^n] \rightarrow \mathbb{C}^l[\mathbb{R}^n] \oplus \mathbb{C}^{l+2}[\mathbb{R}^n]$, one deduces $W_p(\lambda)/W_{d\pi'(X)p}(\lambda) \simeq Z \subset \bigoplus_{l=0}^{k-2} \mathbb{C}^l[\mathbb{R}^n] \cdot e^{-r^2}$, where k is the degree of $d\pi'(X)p$. Thus, for arbitrary $\lambda \in \hat{K}$, $(W_p/W_{d\pi'(X)p})(\lambda) \simeq W_p(\lambda)/W_{d\pi'(X)p}(\lambda)$ is finite dimensional, and the assertion follows. ■

By definition, $W_p \subset C_0(M)_\omega$ is invariant under $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$, so that its closure \overline{W}_p constitutes a G -invariant subspace. Therefore, by restricting π to \overline{W}_p , we obtain a Banach representation of G on \overline{W}_p . Let us introduce now some terminology from general representation theory.

Definition 5.7. Let π be a Banach representation of a Lie group G on a Banach space \mathcal{B} . Then by a *reducing series* for π we shall mean such a series for the integrated form of π , i. e. a nested family \mathfrak{B} of closed $\pi(C_c(G))$ -stable subspaces of \mathcal{B} such that $\{0\}$ and \mathcal{B} belong to \mathfrak{B} , and with the property that if $\mathfrak{A} \subset \mathfrak{B}$, then $\bigcap_{\mathcal{A} \in \mathfrak{A}} \mathcal{A}$ and $\overline{\bigcup_{\mathcal{A} \in \mathfrak{A}} \mathcal{A}}$ lie in \mathfrak{B} . A *Jordan-Hölder-series* for π is a maximal reducing series for π .

Let \mathfrak{B} be a reducing series for π . Two subspaces $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$ with $\mathcal{B}_1 \subset \mathcal{B}_2$ are said to be *adjacent* if there is no $\tilde{\mathcal{B}}$ in \mathfrak{B} such that $\mathcal{B}_1 \subset \tilde{\mathcal{B}} \subset \mathcal{B}_2$, where the inclusions are understood to be strict.

Definition 5.8. Let π be a Banach representation of G on \mathcal{B} . Then a reducing series \mathfrak{B} for π is said to be *discrete* if, whenever $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$ and $\mathcal{B}_1 \subset \mathcal{B}_2$, there are subspaces $\tilde{\mathcal{B}}_1, \tilde{\mathcal{B}}_2$ in \mathfrak{B} with $\mathcal{B}_1 \subset \tilde{\mathcal{B}}_1 \subset \tilde{\mathcal{B}}_2 \subset \mathcal{B}_2$ such that $\tilde{\mathcal{B}}_1$ and $\tilde{\mathcal{B}}_2$ are adjacent in \mathfrak{B} .

Now, as a consequence of Theorem 4.1, we have the following proposition. Note that a similar statement also holds in case that G is not reductive.

Proposition 5.9. *Let $p \in \mathbb{C}[M]$ and $\mathfrak{B} = \{\overline{W}_p, \overline{W}_{d\pi'(X)p}, \overline{W}_{d\pi'(YX)p}, \dots\}$, where X, Y, \dots are arbitrary elements in \mathfrak{g} . Then \mathfrak{B} is a discrete reducing series for (π, \overline{W}_p) .*

Proof. To avoid undue notation, let us denote the elements of \mathfrak{B} by $\mathcal{B}_1, \mathcal{B}_2, \dots$. By construction and Theorem 4.1, the elements of \mathfrak{B} are closed, $\pi(C_c(G))$ -invariant subspaces of \mathcal{B}_1 , and we have the inclusions $\mathcal{B}_1 \supset \mathcal{B}_2 \supset \mathcal{B}_3 \supset \dots$. Further, for $\mathfrak{A} \subset \mathfrak{B}$, $\bigcap_{\mathcal{B}_i \in \mathfrak{A}} \mathcal{B}_i$ and $\overline{\bigcup_{\mathcal{B}_i \in \mathfrak{A}} \mathcal{B}_i}$ lie in \mathfrak{B} . Thus \mathfrak{B} is a reducing series. Finally, by noting that every two $\mathcal{B}_i, \mathcal{B}_{i+1}$ are adjacent, we obtain the desired result. ■

Lemma 5.10. *Let $W_p(\lambda) = W_p \cap C_0(M)(\lambda)$. Then $W_p = \sum_{\lambda \in \hat{K}} W_p(\lambda)$ and $\overline{W_p(\lambda)} = \overline{W}_p(\lambda)$.*

Proof. Similar to the proof of Lemma 5.2. ■

Lemma 5.11. *Let $\mathfrak{B} = \{\mathcal{B}_1, \mathcal{B}_2, \dots\}$ be a reducing series for (π, \overline{W}_p) as in Proposition 5.9. Then the subquotients $\mathcal{B}_i/\mathcal{B}_j$, $i < j$, are K -finite Banach representations of G .*

Proof. Let (π, \mathcal{B}) be a Banach representation of a Lie group G , and V a closed, G -invariant subspace of \mathcal{B} . Then

$$P(\lambda)(\varphi + V) = \int \bar{\chi}_\lambda(k)(\pi(k)\varphi + V)dk,$$

so that $\varphi + V \in (\mathcal{B}/V)(\lambda)$ implies $\varphi \in \mathcal{B}(\lambda)$. One therefore gets an epimorphism from $(\mathcal{B}/V)(\lambda)$ onto $\mathcal{B}(\lambda)/V(\lambda)$ by setting $\varphi + V \mapsto \varphi + V(\lambda)$. Since $V(\lambda) \subset V$, this is clearly an isomorphism. Hence, $(\mathcal{B}_i/\mathcal{B}_j)(\lambda) \simeq \mathcal{B}_i(\lambda)/\mathcal{B}_j(\lambda)$, where $i < j$. Let Z be the algebraic complement of $\mathcal{B}_j(\lambda)$ in $\mathcal{B}_i(\lambda)$. Again, it suffices to consider the case $M = \mathbb{R}^n$. Then, by the preceding lemma,

$$\mathcal{B}_i(\lambda)/\mathcal{B}_j(\lambda) \simeq Z \subset \bigoplus_{l=0}^k \mathbb{C}^l[\mathbb{R}^n] \cdot e^{-r^2}$$

for some k . Hence $\dim(\mathcal{B}_i/\mathcal{B}_j)(\lambda) < \infty$ for all $\lambda \in \hat{K}$, as contended. ■

Assume now that \mathfrak{h} is a Cartan-subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let $\Phi = \Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ be a root system with respect to \mathfrak{h} , $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_{\mathbb{C}} : [Y, X] = \alpha(Y)X \text{ for all } Y \in \mathfrak{h}\}$ the root space corresponding to $\alpha \in \mathfrak{h}^*$, and Φ^+ a set of positive roots. As usual, we set

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}, \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+.$$

Consider the underlying (\mathfrak{g}, K) -module \mathcal{P} of $C_0(M)$, and define for $\lambda \in \mathfrak{h}^*$ the corresponding weight space

$$\mathcal{P}_\lambda = \{\varphi \in \mathcal{P} : d\pi(X)\varphi = \lambda(X)\varphi \text{ for all } X \in \mathfrak{h}\}.$$

Note that $d\pi(\mathfrak{g}_\alpha)\mathcal{P}_\lambda \subset \mathcal{P}_{\lambda+\alpha}$, and assume that \mathcal{P}_0 is not empty.

Proposition 5.12. *Let $p \cdot e^{-r^2} \in \mathcal{P}_\lambda$, and put $V_p = d\pi(\mathfrak{u}(\mathfrak{g}_{\mathbb{C}}))p \cdot e^{-r^2}$, as well as $V_{\mathfrak{n}^+, p} = \sum_{X \in \mathfrak{n}^+} d\pi(\mathfrak{u}(\mathfrak{g}_{\mathbb{C}})X)p \cdot e^{-r^2}$. Then $V_p/V_{\mathfrak{n}^+, p}$ is a highest weight module of weight λ .*

Proof. By construction, $d\pi(\mathfrak{n}^+)p \cdot e^{-r^2} \in V_{\mathfrak{n}^+, p}$, so that $p \cdot e^{-r^2} + V_{\mathfrak{n}^+, p}$ is annihilated by \mathfrak{n}^+ and therefore constitutes a highest weight vector. Since $V_p/V_{\mathfrak{n}^+, p}$ is generated by $p \cdot e^{-r^2} + V_{\mathfrak{n}^+, p}$ as a $\mathfrak{u}(\mathfrak{g}_{\mathbb{C}})$ -module, the assertion follows. ■

Let $M(\lambda) = \mathfrak{u}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathfrak{u}(\mathfrak{b})} \mathbb{C}_\lambda$ be the Verma module of weight λ given by the above root system. Then, by general theory [2], we have the following corollary.

- Corollary 5.13.** 1. $V_p/V_{\mathfrak{n}^+,p} = d\pi(\mathfrak{U}(\mathfrak{n}^-))(p \cdot e^{-r^2} + V_{\mathfrak{n}^+,p})$.
 2. $V_p/V_{\mathfrak{n}^+,p}$ equals the algebraic direct sum of finite dimensional weight spaces.
 3. $V_p/V_{\mathfrak{n}^+,p}$ has a central character.
 4. There exists precisely one \mathfrak{g} -homomorphism Ψ from $M(\lambda)$ onto $V_p/V_{\mathfrak{n}^+,p}$ such that $\Psi(1 \otimes 1) = p \cdot e^{-r^2} + V_{\mathfrak{n}^+,p}$.

Let us illustrate the above results by two examples, where in particular we obtain explicit decompositions of the (\mathfrak{g}, K) -module \mathcal{P} , and hence, of the representation $(\pi, C_0(M))$.

Example 5.14. Consider $G = \text{SL}(2, \mathbb{R})$, $K = \text{SO}(2)$, acting on $M = \mathbb{R}^2$ by matrices, and denote by $(\pi, C_0(\mathbb{R}^2))$ the corresponding regular representation of G on the Banach space $C_0(\mathbb{R}^2)$. As a basis for \mathfrak{g} , take the matrices

$$a_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\mathfrak{k} = \text{Span}_{\mathbb{R}} \{a_1 - a_2\}$. Let $\Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}) = \{\pm\alpha\}$ be a root system of $\mathfrak{g}_{\mathbb{C}}$ with respect to the Cartan subalgebra $\mathfrak{k}_{\mathbb{C}}$, where $\alpha \in \mathfrak{k}_{\mathbb{C}}^*$ is defined by $\alpha(Y) = i(Y_{12} - Y_{21})$. Let $\Phi^+ = \{\alpha\}$, so that $\mathfrak{n}^{\pm} = \mathfrak{g}_{\pm\alpha} = \text{Span}_{\mathbb{C}} \{X^{\pm}\}$, where $X^{\pm} = a_1 + a_2 \mp ia_3$. Put $H = a_1 - a_2$. Consider in $\mathbb{C}[\mathbb{R}^2]$ the polynomials

$$r^2 = m_1^2 + m_2^2, \quad q_{\pm} = m_1 \pm im_2, \tag{21}$$

as well as the constant polynomial $\mathbf{1}$. Let ϱ denote the regular representation of G on $\mathbb{C}[\mathbb{R}^2]$. Setting $\partial^{\pm} = \partial_{m_1} \pm i\partial_{m_2}$, one computes

$$d\pi(X^{\pm}) = \pm iq_{\pm} \partial^{\pm}, \quad d\pi(H) = m_1 \partial_{m_2} - m_2 \partial_{m_1}.$$

Since identical expressions hold for $d\varrho$, one has $d\varrho(\mathfrak{n}^+)q_+^k = 0$, so that with respect to $\mathfrak{k}_{\mathbb{C}}$, $(d\varrho, \mathbb{C}^k[\mathbb{R}^2])$ constitutes a $(k + 1)$ -dimensional, irreducible $\mathfrak{g}_{\mathbb{C}}$ -module of highest weight $\alpha k/2$ with highest weight vector q_+^k . The polynomials r^{2k} and q_-^k are of weight zero, respectively $-\alpha k/2$. Define now in $C_0(M)$ the infinite dimensional, indecomposable subspaces

$$W_{\mathbf{1}} = d\pi(\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}))e^{-r^2}, \quad W_{r^2} = d\pi(\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}))r^2 \cdot e^{-r^2}, \quad W_{q_{\pm}} = d\pi(\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}))q_{\pm} \cdot e^{-r^2}.$$

They are (\mathfrak{g}, K) -modules in the sense of Harish-Chandra and Lepowsky, and by taking suitable subquotients one gets admissible (\mathfrak{g}, K) -modules, and highest weight modules. Note that X^+ , X^- and H satisfy the the commutation relations of the complexification of $\mathfrak{sl}(2, \mathbb{C})$, so that, in particular, $[d\pi(X^+), d\pi(X^-)] = -4i d\pi(H)$. Since $d\pi(X^{\pm}) : \mathbb{C}^l[\mathbb{R}^2] \cdot e^{-r^2} \rightarrow \mathbb{C}^l[\mathbb{R}^2] \cdot e^{-r^2} \oplus \mathbb{C}^{l+2}[\mathbb{R}^2] \cdot e^{-r^2}$ maps \mathcal{P}_{λ} into $\mathcal{P}_{\lambda \pm \alpha}$, we obtain for the (\mathfrak{g}, K) -module $\mathcal{P} = \mathbb{C}[\mathbb{R}^2] \cdot e^{-r^2}$ the decomposition

$$\mathcal{P} = W_{\mathbf{1}} \oplus W_{r^2} \oplus W_{q_+} \oplus W_{q_-}.$$

Each of the summands is $d\pi(\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}))$ -invariant by construction, so that, by Theorem 4.1, their closures must be $\pi(G)$ -invariant. Since \mathcal{P} is dense, we get for $C_0(\mathbb{R}^2)$ a G -invariant decomposition according to

$$C_0(\mathbb{R}^2) = \overline{W_{\mathbf{1}} \oplus W_{r^2} \oplus W_{q_+} \oplus W_{q_-}}$$

into indecomposable closed subspaces, each of them having discrete reducing series. The representation $(\pi, C_0(\mathbb{R}^2))$ itself is not of finite K -type and therefore not admissible.

Example 5.15. Consider again $G = \mathrm{SL}(2, \mathbb{R})$, $K = \mathrm{SO}(2)$, but now acting on $M = \mathfrak{g} = \mathfrak{sl}(2, r) \simeq \mathbb{R}^3$ via the adjoint representation of G . Explicitly, we have the identification

$$\mathbb{R}^3 \ni m \mapsto X_m = \begin{pmatrix} m_2 & m_1 + m_3 \\ m_1 - m_3 & -m_2 \end{pmatrix} \in \mathfrak{g}.$$

Similarly, we write m_X for the point in \mathbb{R}^3 corresponding to $X \in \mathfrak{g}$ under this identification. Let $(\pi, C_0(\mathbb{R}^3))$ be the corresponding regular representation, and consider in $\mathbb{C}[\mathbb{R}^3]$ the polynomials

$$r^2 = m_1^2 + m_2^2 + m_3^2, \quad p_G = \det X_m = -m_1^2 - m_2^2 + m_3^2.$$

If not stated otherwise, we continue with the notation of the previous example. Since $(d/dh)(e^{hY} m)|_{h=0} = m_{[Y, X_m]}$ for $Y \in \mathfrak{g}$, one computes, using (6),

$$d\pi(X^\pm) = \pm 2i(m_3 \partial_{\mp} + q_{\mp} \partial_{m_3}), \quad d\pi(H) = 2(m_2 \partial_{m_1} - m_1 \partial_{m_2}),$$

as well as identical expressions for $d\rho$. Note that $d\rho(X)p_G = 0$ for all $X \in \mathfrak{g}$, and $d\rho(\mathfrak{n}^+)p_G^l q_-^k = 0$, where l, k are non negative integers. Hence, defining in $\mathbb{C}^k[\mathbb{R}^3]$ the finite dimensional subspaces

$$V^{k,l} = d\rho(\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}))[p_G^l q_-^{k-2l}], \quad 2l \leq k,$$

we get highest weight $\mathfrak{g}_{\mathbb{C}}$ -modules of weight $(k - 2l)\alpha$, see [2]. Since $\dim \mathbb{C}^k[\mathbb{R}^3] = \sum_{j=1}^{k+1} j$ and $\dim V^{k,l} = 2(k - 2l) + 1$, one deduces that $\mathbb{C}^k[\mathbb{R}^3]$ decomposes according to

$$\mathbb{C}^k[\mathbb{R}^3] = V^{k,0} \oplus V^{k,1} \oplus \dots \oplus V^{k, [k/2]}.$$

If we therefore set $W_p = d\pi(\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}))p \cdot e^{-r^2}$, $p \in \mathbb{C}[\mathbb{R}^3]$, we get for \mathcal{P} the decomposition

$$\mathcal{P} = \bigoplus_{l=0}^{\infty} p_G^l [W_1 \oplus W_{m_3} \oplus W_{r^2+m_3^2} \oplus \bigoplus_{k=1}^{\infty} (W_{q_+^k} \oplus W_{q_-^k})]$$

into $d\pi(\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}))$ -invariant subspaces of analytic vectors for $(\pi, C_0(\mathbb{R}^3))$, leading to a corresponding decomposition of $C_0(\mathbb{R}^3)$ into G -invariant subspaces. Consider further the smooth, affine G -varieties $N_\xi = \{m \in \mathbb{R}^3 : p_G = \xi\}$, where $\xi \neq 0$, and the underlying (\mathfrak{g}, K) -module $\mathcal{P}[\xi] = \mathbb{C}[N_\xi] \cdot e^{-r^2}$ of $C_0(N_\xi)$. One has $\mathbb{C}[N_\xi] \simeq \mathbb{C}[\mathbb{R}^3]/\mathcal{I}_{N_\xi}$, where \mathcal{I}_{N_ξ} denotes the vanishing ideal of N_ξ . Since \mathcal{I}_{N_ξ} is generated by the polynomial $p_G - \xi$, one obtains in this case the decomposition

$$\mathcal{P}[\xi] = W_1 \oplus W_{m_3} \oplus W_{r^2+m_3^2} \oplus \bigoplus_{k=1}^{\infty} (W_{q_+^k} \oplus W_{q_-^k}).$$

Note that in this example neither \mathcal{P} , nor $\mathcal{P}[\xi]$, are longer finitely generated as (\mathfrak{g}, K) -modules.

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