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On ideals free of large prime factors

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In memory of Robert Rankin

Résumé. En 1989, E. Saias a établi une formule asymptotique pour $\Psi(x, y) = |\{n \le x : p \mid n \Rightarrow p \le y\}|$ avec un très bon terme d'erreur, valable si $\exp\left((\log\log x)^{(5/3)+\epsilon}\right) \le y \le x$, $x \ge x_0(\epsilon)$, $\epsilon > 0$. Nous étendons ce résultat à un corps de nombre K en obtenant une formule asymptotique pour la fonction analogue $\Psi_K(x, y)$ avec le même terme d'erreur et la même zone de validité. Notre objectif principal est de comparer les formules pour $\Psi(x, y)$ et $\Psi_K(x, y)$, en particulier comparer le second terme des développements.

Abstract. In 1989, E. Saias established an asymptotic formula for $\Psi(x, y) = |\{n \le x : p \mid n \Rightarrow p \le y\}|$ with a very good error term, valid for $\exp\left((\log\log x)^{(5/3)+\epsilon}\right) \le y \le x$, $x \ge x_0(\epsilon)$, $\epsilon > 0$. We extend this result to an algebraic number field K by obtaining an asymptotic formula for the analogous function $\Psi_K(x, y)$ with the same error term and valid in the same region. Our main objective is to compare the formulae for $\Psi(x, y)$ and $\Psi_K(x, y)$, and in particular to compare the second term in the two expansions.

1. Introduction

Many authors have studied the function $\Psi(x, y)$ defined to be the number of positive integers $n \le x$ with no prime factor exceeding y; see, for example, [1], [11], [12], [26] and other papers cited by these authors. Estimates (with various degrees of precision) for $\Psi(x, y)$ have been applied in certain types of investigations (for example, [5], [14], [15], [16], [18], [27]). Our objective in this paper is to extend the more precise result of Saias [26] for $\Psi(x, y)$ to an algebraic number field in order to compare the formulae obtained, and we apply our results to a sum analogous to one first considered by Ivić [14] for the rational field. We begin by giving a brief survey of two results on $\Psi(x, y)$ that we will need and the associated notation.

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First we give some definitions. The Dickman function $\rho(u)$ is defined by the differential-difference equation

$$\text{(1)}\qquad \begin{cases} \rho(u) = 0 & \text{for } u < 0, \\ \rho(u) = 1 & \text{for } 0 \le u \le 1, \\ u\rho'(u) + \rho(u-1) = 0 & \text{for } u > 1. \end{cases}$$

Define $\Lambda(x, y)$ for $x > 1$, $y \ge 2$ by

$$\text{(2)}\qquad \begin{cases} \Lambda(x,y) = x\int_0^\infty \rho\left(\frac{\log\frac{x}{t}}{\log y}\right) d\left(\frac{[t]}{t}\right) & \text{for } x \notin \mathbf{N} \\ \Lambda(x,y) = \frac{1}{2}\left(\Lambda(x-0,y) + \Lambda(x+0,y)\right) & \text{for } x \in \mathbf{N}. \end{cases}$$

Write $\log_2(x)$ for $\log(\log x)$ when $x > 1$. Let $\epsilon > 0$; define the region H_ϵ by

$$\text{(3)}\qquad H_\epsilon : (\log_2 x)^{\frac{5}{3}+\epsilon} \le \log y \le \log x,\ x \ge x_0(\epsilon).$$

When (3) holds we write $y \in H_\epsilon$. Let $u = \frac{\log x}{\log y}$; it is well known that

$$\text{(4)}\qquad \Psi(x,y) = x\rho(u)\left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right)$$

for $y \in H_\epsilon$; this range for y was established in [11]. Various other expressions for $\Psi(x, y)$ have been derived; we utilize one with a very good error term established by Saias in [26]:

$$\text{(5)}\qquad \Psi(x,y) = \Lambda(x,y)\left(1 + O_\epsilon\left(\exp\left(-(\log y)^{\frac{3}{5}-\epsilon}\right)\right)\right)$$

for $y \in H_\epsilon$.

The first goal of this paper is to establish a result comparable to (5) in the case when the rational field $\mathbf{Q}$ is replaced by an algebraic number field K. Let K be a number field with degree $n \ge 2$ and ring of integers $\mathfrak{O}_K$. For any ideal $\mathfrak{a}$ of $\mathfrak{O}_K$, define

$$\text{(6)}\qquad P(\mathfrak{a}) = \max\{N(\mathfrak{p}) : \mathfrak{p}|\mathfrak{a}\}$$

where $\mathfrak{p}$ denotes a prime ideal with norm $N(\mathfrak{p})$, and let $P(\mathfrak{O}_K) = 1$. Define $\Psi_K(x, y)$ by

$$\text{(7)}\qquad \Psi_K(x,y) = \left|\{\mathfrak{a} : N(\mathfrak{a}) \le x, P(\mathfrak{a}) \le y\}\right|.$$

Thus when $K = \mathbf{Q}$, $\Psi_K(x, y)$ reduces to $\Psi(x, y)$. For papers in the literature on $\Psi_K(x, y)$ see for example [3], [6], [7], [8]. [10], [19] and [22]. We establish in Theorem 1.1 an asymptotic formula for $\Psi_K(x, y)$ for $y \in H_\epsilon$ with an error term of the same order of magnitude as that in (5). We use this theorem to study the difference between $\Psi_K(x, y)$ and its leading term and derive our main result in Theorem 1.3. This enables us to compare the second term in the asymptotic formulae for $\Psi_K(x, y)$ and $\Psi(x, y)$.

In order to state our main results, we need some more notation. Let $\zeta_K(s)$ denote the Dedekind zeta-function for the field K , a well studied function. As we see from Lemma 2.3(i), $\zeta_K(s)$ has a simple pole at $s = 1$ with residue λ_K (given in (21) in terms of invariants of K). Let

$$g_K(s) = \zeta_K(s) - \lambda_K \zeta(s) \tag{8}$$

where $\zeta(s)$ is the Riemann zeta-function. Denote the Laplace transform of $\rho(u)$ (defined in (1)) by $\hat{\rho}(s)$ (see (43)). We define $\xi = \xi(u)$ to be the unique real solution of

$$e^{\xi} = 1 + u\xi \qquad (u > 1), \tag{9}$$

with $\xi(1) = 0$ by convention. Define $\alpha_0 = \alpha_0(x, y)$ by

$$\alpha_0 = 1 - \frac{\xi(u)}{\log y} \text{ where } u = \frac{\log x}{\log y}. \tag{10}$$

Let

$$J_0(x, y) = \frac{1}{2\pi i} \int_{\alpha_0 - i\infty}^{\alpha_0 + i\infty} g_K(s)(s-1)\log y \; \hat{\rho}((s-1)\log y) s^{-1} x^s ds. \tag{11}$$

We will see in Lemma 4.3 that the integral in (11) converges. For $\epsilon > 0$, write

$$L_\epsilon(y) = \exp\left((\log y)^{\frac{3}{5} - \epsilon}\right). \tag{12}$$

We can now state our result analogous to (5).

Theorem 1.1. *Let $\epsilon > 0$. For $y \in H_\epsilon$*

$$\Psi_K(x, y) = \lambda_K \Lambda(x, y) \left(1 + O\left(\frac{1}{L_\epsilon(y)}\right)\right) + J_0(x, y).$$

Using (4) and (5), we can compare $\Psi_K(x, y)$ with $\Psi(x, y)$, and we have:

Corollary 1.2. *For $y \in H_\epsilon$*

$$\Psi_K(x, y) - \lambda_K \Psi(x, y) = J_0(x, y) + O\left(\frac{x\rho(u)}{L_\epsilon(y)}\right).$$

Theorem 1.1 and its Corollary prompt us to ask what the magnitude of $J_0(x, y)$ is and how it compares with that of $\Psi(x, y)$.

Theorem 1.3. *Assume $y \in H_\epsilon$.*
(i) As $u = \frac{\log x}{\log y} \to \infty$,

$$J_0(x, y) = -\frac{x}{\log y} \rho(u)\xi(u) \left(g_K(1) + O\left(\frac{\log u}{\log y} + \frac{\log u}{\sqrt{u}}\right)\right). \tag{13}$$

(ii) If $g_K(1) \neq 0$, $J_0(x, y)$ and $\Psi(x, y) - x\rho(u)$ have the same order of magnitude as $u \to \infty$.

We see from (18) and (23) that $g_K(1) = \sum_{m=1}^{\infty} \frac{j(m)-\lambda_K}{m}$ which converges. The question of whether there are algebraic number fields $K \neq \mathbf{Q}$ for which $g_K(1) = 0$ is an interesting one. The author has consulted several experts in the area, but a definitive answer to this question does not seem to be known at present. However, at least for some fields K, there are other ways of looking at $g_K(1)$ that might help in deciding whether it is zero. The author would like to thank Professor B. Z. Moroz and the Referee for suggesting the following approaches. When K is a normal extension of $\mathbf{Q}$, $\zeta_K(s) = \zeta(s)F(s)$ where $F(1) = \lambda_K$ and $F(s)$ is known to be an entire function. Since $\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|)$ as $s \to 1$, where γ is Euler's constant, we deduce that as $s \to 1$

$$\zeta_K(s) = \frac{\lambda_K}{s-1} + \gamma\lambda_K + F'(1) + O(|s-1|),$$

and hence

$$g_K(1) = \lim_{s\to 1} (\zeta_K(s) - \lambda_K\zeta(s)) = F'(1).$$

For K an abelian extension of $\mathbf{Q}$, let G be the corresponding Galois group and G^* be the character group of G. The elements of G^* can be regarded as Dirichlet characters; let χ_o denote the principal character of G^*. It is known that

$$F(s) = \prod_{\substack{\chi\in G^* \\ \chi\neq\chi_o}} L(s,\chi)$$

where $L(s,\chi)$ denotes a Dirichlet L-function; see for example Theorem 9.2.2 and section 9.4 of [9] and also Theorem 8.1 of [24]. Hence, since $F(1) = \lambda_K$,

$$g_K(1) = F'(1) = \lambda_K \sum_{\substack{\chi\in G^* \\ \chi\neq\chi_o}} \frac{L'(1,\chi)}{L(1,\chi)}.$$

In particular when K is a quadratic field, $g_K(1) = F'(1) = L'(1,\chi)$ with χ a quadratic character; the results in [4] may enable one to calculate $g_K(1)$ with arbitrary precision. The techniques in [23] might also be useful in investigating $g_K(1)$ further in some cases. However we do not address these problems here.

We note that by (4) and (13) it follows from Theorem 1.1 that for $y \in H_\epsilon$

$$\Psi_K(x,y) \sim \lambda_K x\rho(u) \text{ as } u \to \infty, \tag{14}$$

a known result for suitable y; Krause [19] has shown that this holds for $y \in H_\epsilon$. Hence Theorem 1.3 (ii) tells us that provided $g_K(1) \neq 0$ the second term in $\lambda_K\Lambda(x,y)$ has the same order of magnitude as $J_0(x,y)$. In Theorem 6.4 in section 6, we show how to express a truncated version of the complex integral $J_0(x,y)$ (see (57)) in terms of real integrals. This representation may be more useful in some applications.

To prove Theorem 1.1, we adopt the method used to establish (5) (see [26] or chapter 3.5 of [31]) but with $\zeta(s)$ replaced by $\zeta_K(s)$. To do so requires properties of $\zeta_K(s)$ analogous to some of the strongest known for $\zeta(s)$, for example the zero free region given in [29] and consequential properties; these are described in section 2. Properties of the Dickman function are given in section 3. With these tools the proof of Theorem 1.1 in section 4 is standard.

The main work of this paper is to establish Theorem 1.3 in section 5. Our approach must take into account that we have only limited information on the partial sums of the coefficients of the Dirichlet series for $\zeta_K(s)$ (see Lemma 2.1(ii)), that the bounds for $\hat{\rho}(s)$ depend on the size of $t = \Im(s)$ (see Lemma 3.4(iii)), and that, as y increases in the range H_ϵ, u decreases from $(\log x)(\log_2 x)^{-\frac{5}{3}-\epsilon}$ to 1. These remarks suggest that we should split $J_0(x,y)$ into several integrals which we find we have to estimate by different methods. The main contribution (when $g_K(1) \neq 0$) comes from the small values of t (see Lemma 5.1).

We end the paper with an application of our Theorems. From (4), Ivić [14] derived the order of magnitude of the sum

$$S_{\mathbf{Q}}(x) = \sum_{n \le x} \frac{1}{P(n)} \text{ where } P(n) = \max\{p : p \mid n\} \text{ if } n > 1, P(1) = 1,$$

with as usual p denoting a rational prime. An asymptotic formula was obtained in [5], and a sharper asymptotic formula was obtained as a special case of Theorem 3 of [27]. In section 7, we consider a sum analogous to $S_{\mathbf{Q}}(x)$ for the field K and estimate it using our results. Let

$$S_K(x) = \sum_{\substack{\mathfrak{a} \\ N(\mathfrak{a}) \le x}} \frac{1}{P(\mathfrak{a})} \tag{15}$$

where $P(\mathfrak{a})$ is defined in (6). Let

$$\mathcal{L} = \mathcal{L}(x) = \exp\left(\left(\frac{1}{2}\log x \log_2 x\right)^{\frac{1}{2}}\right). \tag{16}$$

We establish the following result.

Theorem 1.4. *(i) If $g_K(1) \neq 0$,*

$$S_K(x) = x\left(\lambda_K + O\left(\frac{1}{L_\epsilon(\mathcal{L})}\right)\right)\int_2^x \frac{1}{v^2 \log v}\left\{\rho\left(\frac{\log \frac{x}{v}}{\log v}\right)\right.$$
$$\left. - \int_1^x \frac{w-[w]}{w^2 \log v}\rho'\left(\frac{\log \frac{x}{vw}}{\log v}\right)dw\right\}dv + \left(1 + O\left(\frac{1}{L_\epsilon(\mathcal{L})}\right)\right)\int_2^x \frac{J(\frac{x}{v}, v)}{v \log v}dv$$

where $J(x,y)$ is defined in (57).

(ii) As $x \to \infty$, $S_K(x) =$

$$x\int_2^x \frac{1}{v^2\log v}\left\{\lambda_K + \frac{\log_2 x}{2\log v}\left(\lambda_K(1-\gamma) - g_K(1) + o(1)\right)\right\}\rho\left(\frac{\log\frac{x}{v}}{\log v}\right)dv$$

where γ is Euler's constant and $g_K(1) = \lim_{s\to 1}\left(\zeta_K(s) - \lambda_K\zeta(s)\right)$.

We remark that other more general applications of the methods used to derive (5) can be found in the literature. For example, in [28], H. Smida studied the sum

$$\sum_{\substack{m\le x\\ P(m)\le y}} d_k(m), \tag{17}$$

where $d_k(m)$ denotes the number of representations of m as a product of k positive integers, its generating function being

$$\sum_{m=1}^{\infty} d_k(m)m^{-s} = \left(\zeta(s)\right)^k \qquad (\Re(s) > 1).$$

Similarly one could consider sums analogous to (17) with $d_k(m)$ replaced by another appropriate multiplicative function with a generating function involving one or more Dedekind zeta-functions, and we may return to this problem.

The author would like to thank the Referee for helpful comments, and in particular for those relating to the constant $g_K(1)$ and for a simplification in the quantity $S_K(x)$ investigated in Theorem 1.4.

Note added in proof: The author recently established an asymptotic expansion for the number defined by (7) that is analogous to the expansion obtained in [26] for K the rational field. It is hoped to include this result in a paper being prepared.

2. Properties of $\zeta_{\mathbf{K}}(\mathbf{s})$

As usual, we write $s = \sigma + it$.

Throughout this paper, K denotes a number field with degree $n \ge 2$ and ring of integers $\mathfrak{O}_K$. Write $\mathfrak{a}, \mathfrak{b}$ for ideals of $\mathfrak{O}_K$ and $\mathfrak{p}$ for a prime ideal, and let $N(\mathfrak{a})$ denote the norm of $\mathfrak{a}$.

For $\sigma > 1$, the Dedekind zeta-function $\zeta_K(s)$ is given by

$$\zeta_K(s) = \sum_{\mathfrak{a}} (N(\mathfrak{a}))^{-s} = \sum_{m=1}^{\infty} j(m)m^{-s} \tag{18}$$

where $j(m)$ is the number of ideals $\mathfrak{a}$ with $N(\mathfrak{a}) = m$. We require some properties of $\zeta_K(s)$ that are analogous to some of the strongest available for the Riemann zeta-function $\zeta(s)$ near the line $\sigma = 1$, and we embody those we need and related ones in the following Lemmas.

Lemma 2.1. *(i) Let $d_n(m)$ denote the number of representations of m as a product of n positive integers; then*

$$j(m) \le d_n(m). \tag{19}$$

(ii) Let λ_K be the residue of $\zeta_K(s)$ at $s = 1$ (given in (21) below); then

$$S(v) := \sum_{m \le v} j(m) = \lambda_K v + O(v^{1-\frac{1}{n}}). \tag{20}$$

These results are well known. For (i), see Corollary 3 of Lemma 7.1 of [24], and for (ii), see Theorem 6.3 of [21] from which we see that

$$\lambda_K = 2^{q+r}\pi^r Rh/m\,|\Delta|^{\frac{1}{2}} \tag{21}$$

where q is the number of real and r is the number of complex conjugate pairs of monomorphisms $K \to \mathbf{C}$, m is the number of roots of unity in K and R, h, Δ denote the regulator, class number, discriminant of K, respectively. For a stronger result, see Satz 210 of [20] or for recent results see [25] when $n \ge 3$ and [13] for $n = 2$.

Lemma 2.2. *For δ fixed with $0 < \delta < \frac{1}{2}$,*

$$\sum_{m \le x} d_n(m) m^{\delta - 1} \ll x^{\delta} (\log x)^n.$$

Proof. This follows by partial summation and the result (see (13.3) and Theorem 13.2 of [17])

$$\sum_{m \le x} d_n(m) \ll x (\log x)^{n-1}.$$

□

Lemma 2.3. *(i) $\zeta_K(s)$ is differentiable in the half plane $\sigma > 1 - \frac{1}{n}$ except for a simple pole at $s = 1$ with residue λ_K (given by (21)), and in this region*

$$\zeta_K(s) = \frac{\lambda_K s}{s-1} + s \int_1^{\infty} (S(v) - \lambda_K v) v^{-s-1} dv. \tag{22}$$

(ii) With $g_K(s) = \zeta_K(s) - \lambda_K\ \zeta(s)$ as in equation (8), we have for $\sigma > 1 - \frac{1}{n}$ that

$$g_K(s) = \sum_{m=1}^{\infty} b(m) m^{-s} = s \int_1^{\infty} (S(v) - \lambda_K [v]) v^{-s-1} dv \tag{23}$$

where $b(m) = j(m) - \lambda_K \ll d_n(m)$ and $\sum_{m \le v} b(m) = S(v) - \lambda_K [v] \ll v^{1-\frac{1}{n}}$.

(iii) For $\sigma > 1 - \frac{1}{n}$ and any $N \geq 1$

$$g_K(s) = \sum_{m \leq N} b(m) m^{-s} + O\left(N^{1-\frac{1}{n}-\sigma}\left(\frac{|s|}{\sigma - 1 + \frac{1}{n}} + 1\right)\right). \tag{24}$$

Proof. (i) (22) follows for $\sigma > 1$ from (18) and (20) on using partial summation, and the other properties follow by analytic continuation since by (20) the integral is absolutely convergent for $\sigma > 1 - \frac{1}{n}$. (If we used a stronger version of Lemma 2.1(ii), this range for σ could be extended, but we do not need this.)

(ii) Since for $\sigma > 0$

$$\zeta(s) = \frac{s}{s-1} + s \int_1^\infty ([v] - v) v^{-s-1} dv, \tag{25}$$

(23) follows from part (i), (8) and (20).

(iii) By partial summation

$$g_K(s) = \sum_{m \leq N} b(m) m^{-s} - (S(N) - \lambda_K N) N^{-s} + s \int_N^\infty (S(v) - \lambda_K [v]) v^{-s-1} dv$$

and the result then follows from (20). □

We remark that $\zeta_K(s)$ has more general properties in the whole complex plane that are analogous to those of $\zeta(s)$, but we do not require them as we are concerned only with the behaviour of $\zeta_K(s)$ in a region just to the left of the line $\sigma = 1$. The properties that we need depend on the zero free region of $\zeta_K(s)$, established in [29] by A.V.Sokolovskii, and related results:

Lemma 2.4. *(i) For suitable positive constants c, t_0, $\zeta_K(s) \neq 0$ in the region*

$$\sigma \geq 1 - c(\log |t|)^{-2/3} (\log_2 |t|)^{-1/3}, \qquad |t| \geq t_0. \tag{26}$$

(ii) Let $\pi_K(x)$ denote the number of prime ideals $\mathfrak{p}$ with $N(\mathfrak{p}) \leq x$; then

$$\pi_K(x) = li(x) + O\left(x \ \exp\left(-c(\log x)^{3/5} (\log_2 x)^{-1/5}\right)\right). \tag{27}$$

Part (ii) is the prime ideal theorem. By standard arguments $\zeta_K(1+it) \neq 0$; hence by taking c to be sufficiently small it follows that $\zeta_K(s) \neq 0$ in the region

$$\sigma \geq 1 - c(\log t_0)^{-2/3} (\log_2 t_0)^{-1/3}, \qquad |t| \leq t_0. \tag{28}$$

We require bounds for $\zeta_K(s)$ and for $\zeta_K'(s)/\zeta_K(s)$ in appropriate regions.

Lemma 2.5. *For $1 - \frac{1}{2n+1} < \sigma < 1, |t| \geq t_0$*

$$g_K(s) \ll |t|^{1/2}, \qquad \zeta_K(s) \ll |t|^{1/2}. \tag{29}$$

Proof. We apply (24) with $N = |t|^n$ and the property $b(m) \ll d_n(m) \ll m^\delta$ for any fixed $\delta > 0$ to obtain

$$g_K(s) \ll \sum_{m \le N} m^{-\sigma+\delta} + |t|\, N^{1-\frac{1}{n}-\sigma} \ll N^{\frac{1}{2n+1}+\delta} + |t|\, N^{\frac{1}{2n+1}-\frac{1}{n}} \ll |t|^{1/2}$$

by our choice of N if we take $\delta \le \frac{1}{2n(2n+1)}$. Since $\zeta(s) \ll |t|^{1/2}$ for $\frac{1}{2} < \sigma < 1$, the bound for $\zeta_K(s)$ follows from (8) and analytic continuation. □

Lemma 2.6. *For s in the region (26)*

$$\zeta_K(s) \ll (\log|t|)^{2/3} \log_2 |t| . \tag{30}$$

Proof. From the results in [30], when $\sigma \le 1$ in the region (26) we have

$$\zeta_K(s) \ll (\log|t|)^{2/3}, \tag{31}$$

and, when $\sigma \ge \frac{3}{2}$, $\zeta_K(s)$ is bounded. Hence we need only consider $1 \le \sigma \le \frac{3}{2}$, $t \ge t_0$; the case $t \le -t_0$ follows similarly. We apply Cauchy's integral formula twice using (29) and (31). Let $\eta = \frac{1}{\log t}$; suppose $\zeta_K(s) \ll h(t) = o(|t|)$ in the region (26), and let R be the rectangle with vertices

$$1 - \eta + i(t \pm h(t)), 2 + i(t \pm h(t)).$$

We can bound $\zeta_K(s)$ by (31) when $w = 1 - \eta + i(t + v)$ and $|v| \le h(t)$, and $\zeta_K(s)$ is bounded when $w = 2 + i(t + v)$ and $|v| \le h(t)$. By Cauchy's integral formula and since $2 - \sigma \ge 1/2$ we have

$$\begin{aligned}
\zeta_K(s) &= \frac{1}{2\pi i} \int_R \frac{\zeta_K(w)}{w - s} dw \\
&\ll \int_{-h(t)}^{h(t)} \frac{dv}{|2 - \sigma + iv|} + h(t) \int_{1-\eta}^{2} \frac{du}{|u - \sigma + ih(t)|} \\
&\quad + (\log t)^{2/3} \int_{-h(t)}^{h(t)} \frac{dv}{|1 - \eta - \sigma + iv|} + h(t) \int_{1-\eta}^{2} \frac{du}{|u - \sigma - ih(t)|} \\
&\ll \log h(t) + 1 + (\log t)^{2/3} \left(1 + \int_{\sigma-1+\eta}^{h(t)} v^{-1} dv \right) \\
&\ll \left(1 + (\log t)^{2/3} \right) \log h(t). \qquad (32)
\end{aligned}$$

By (29), (32) holds with $h(t) = t^{1/2}$, and so we obtain

$$\zeta_K(s) \ll (\log t)^{5/3} \tag{33}$$

when $1 \le \sigma \le 3/2$ in the region (26). Now by (33) we can apply (32) again with $h(t) = (\log t)^{5/3}$, and the result follows. □

Corollary 2.7. *In the region (26)*

$$g_K(s) \ll (\log|t|)^{2/3} \log_2 |t| . \tag{34}$$

Proof. Since $\zeta(s) \ll (\log|t|)^{2/3}$ in the region (26) (see Theorem 6.3 of [17]), the result follows from the lemma and (8). □

Lemma 2.8. *In the region (26) for a suitable choice of c,*

$$\frac{\zeta_K'(s)}{\zeta_K(s)} \ll (\log|t|)^{2/3}(\log_2|t|)^{4/3}. \tag{35}$$

Proof. For $\sigma > 1$,

$$\zeta_K(s) = \prod_{\mathfrak{p}} (1 - (N(\mathfrak{p}))^{-s})^{-1}$$

and hence

$$\frac{\zeta_K'(s)}{\zeta_K(s)} = -\sum_{\mathfrak{p}} \frac{\log N(\mathfrak{p})}{(N(\mathfrak{p}))^s} + O(1) \ll \sum_p j(p)p^{-\sigma}\log p + O(1) \ll \frac{1}{\sigma-1}. \tag{36}$$

Using (30) and (36), we follow the method used to prove a slight improvement of (35) when $K = \mathbf{Q}$ described in the proof of Lemma 12.3 of [17]. In the argument leading to equation (12.55) of that proof, take

$$h(t) = (\log|t|)^{-2/3}(\log_2|t|)^{-4/3}, \qquad r = h(t_0)\log_2 t_0$$

and use Lemma 2.4(i) above and then (35) follows. □

3. Properties of the Dickman function

The Dickman function $\rho(u)$ is defined as in (1) by the differential-difference equation

$$\begin{cases} \rho(u) = 0 & \text{for } u < 0, \\ \rho(u) = 1 & \text{for } 0 \le u \le 1, \\ u\rho'(u) + \rho(u-1) = 0 & \text{for } u > 1. \end{cases} \tag{37}$$

Lemma 3.1. *The function $\rho(u)$ has the following properties:*

(i) As $u \to \infty$

$$\rho(u) = \exp\left(-u\left(\log u + \log_2 u - 1 + O\left(\frac{\log_2 u}{\log u}\right)\right)\right).$$

(ii)

- *$\rho(u)$ is continuous except at $u = 0$.*
- *$\rho'(u)$ is defined for $u \neq 0$ and continuous except at $u = 1$.*
- *$0 < \rho(u) \le 1$ for $u \ge 0$, $-1 \le \rho'(u) < 0$ for $u > 1$.*
- *$\rho(u)$ decreases strictly and $\rho'(u)$ increases strictly on $u > 1$.*

Proof. A stronger form of (i) is due to de Bruijn [2], and (ii) follows from (37). □

In (9), we defined $\xi = \xi(u)$ to be the unique real solution of the equation

$$\xi(1) = 0, \qquad e^{\xi} = 1 + u\xi \qquad (u > 1). \tag{38}$$

Define $I(s), J(s)$ by

$$I(s) = \int_0^s \frac{e^v - 1}{v} dv \qquad (s \in \mathbf{C}), \tag{39}$$

$$J(s) = \int_0^\infty \frac{e^{-s-v}}{s+v} dv \qquad (s \in \mathbf{C} \setminus (-\infty, 0]). \tag{40}$$

Lemma 3.2. *(i)* $\xi(u) = \log u + \log_2 u + O\left(\frac{\log_2 u}{\log u}\right)$ *for* $u \geq 3$.

$$\xi'(u) \sim \frac{1}{u} \text{ as } u \to \infty.$$

(ii) $\rho^{(k)}(u) = (-\xi(u))^k \rho(u) \left(1 + O\left(\frac{1}{u}\right)\right)$ *for* $u > 1,\ u \neq 2, 3, ..., k,\ k \in \mathbf{N}$.
(iii) For $u \geq 1$

$$\rho(u) = \left(\frac{\xi'(u)}{2\pi}\right)^{1/2} \exp\left(\gamma - u\xi + I(\xi)\right) \left(1 + O\left(\frac{1}{u}\right)\right).$$

(iv) For $|v| \leq \frac{2}{3}u,\ u \geq 3,\ u - v \geq 3$

$$\rho(u - v) = \rho(u) \exp\left(v \left(\log u + \log_2 u + O(1)\right)\right).$$

Proof. For (i)$-$(iii), see equations (47), (59), (56), (51) of chapter 3.5 of [31] or Lemme 3 of [26]. Part (iv) follows by considering the integral

$$-\int_{u-v}^{u} \frac{\rho'(w)}{\rho(w)} dw.$$

Note that we can rewrite (iv) as

$$\rho(u - v) = \rho(u) \exp\left(v \left(\xi(u) + O(1)\right)\right). \tag{41}$$

□

Corollary 3.3.

$$e^{-u\xi} = \rho(u) \exp\left(-u\left(1 + O\left(\frac{\log_2 u}{\log u}\right)\right)\right). \tag{42}$$

This follows from (i) of Lemmas 3.1 and 3.2.

As usual, we denote the Laplace transform of $\rho(u)$ by $\hat{\rho}(s)$, so for all $s \in \mathbf{C}$

$$\hat{\rho}(s) = \int_0^\infty e^{-sv} \rho(v) dv. \tag{43}$$

By Lemma 3.1(i), the integral converges absolutely for all $s \in \mathbf{C}$. In our context, the inverse of this Laplace transform is given by

$$\rho(u) = \frac{1}{2\pi i}\int_{-\xi(u)-i\infty}^{-\xi(u)+i\infty} e^{us}\hat{\rho}(s)ds \tag{44}$$

for all real $u \geq 1$; see, for example, equation (3.5.45) of [31].

Lemma 3.4. *(i)* $I(-s) + J(s) + \gamma + \log s = 0$ *for* $s \in \mathbf{C} \setminus (-\infty, 0]$, *where* γ *is Euler's constant.*

(ii) $s\hat{\rho}(s) = \exp(-J(s))$ *for* $s \in \mathbf{C} \setminus (-\infty, 0]$, $\hat{\rho}(s) = \exp(\gamma + I(-s))$.

(iii) For $\sigma = -\xi(u)$, $u > 1$,

$$\hat{\rho}(s) \ll \exp\left(I(\xi) - \frac{t^2u}{2\pi^2}\right) \text{ for } |t| \leq \pi,$$

$$\hat{\rho}(s) \ll \exp\left(I(\xi) - \frac{u}{\xi^2 + \pi^2}\right) \text{ for } |t| > \pi,$$

$$s\hat{\rho}(s) = 1 + O\left(\frac{1+u\xi}{|s|}\right) \text{ for } |t| > 1 + u\xi.$$

(iv)

$$s\hat{\rho}(s) = 1 + \int_1^\infty e^{-sv}\rho'(v)dv,$$

the integral being absolutely convergent for all $s \in \mathbf{C}$.

Proof. For (i) – (iii), see equations (43), (40), (44), (48), (49) of chapter 3.5 of [31]. For (iv), we have using (43)

$$s\hat{\rho}(s) = -\int_0^\infty \rho(v)d(e^{-sv}) = \left[-\rho(v)e^{-sv}\right]_0^\infty + \int_0^\infty e^{-sv}\rho'(v)dv$$

on integrating by parts. The result now follows since $\rho(0) = 1$, $e^{-\sigma v}\rho(v) \to 0$ as $v \to \infty$ and $\rho'(v) = 0$ for $0 < v < 1$. $\square$

Lemma 3.5. *As* $u = \frac{\log x}{\log y} \to \infty$,

$$-\int_1^x \frac{v-[v]}{v^2}\rho'\left(u - \frac{\log v}{\log y}\right)dv$$
$$= C\rho(u)\xi(u)\left(1 + O\left(\frac{1}{\log u} + \frac{1}{(\log_2 x)^{1/2}} + \frac{y}{x}\right)\right) \tag{45}$$

where

$$C = \int_1^\infty \frac{v-[v]}{v^2}dv = 1 - \gamma. \tag{46}$$

Proof. We consider first the integral over the range $1 \le v \le x^{2/3}$, where $\frac{\log v}{\log y} \le \frac{2}{3}u$. By Lemma 3.2(i), (ii), (iv) and the mean value theorem applied to ξ we have for $v < \min\left(x^{2/3}, \frac{x}{y}\right)$, so $u - \frac{\log v}{\log y} > 1$, that

$$-\rho'\left(u - \frac{\log v}{\log y}\right) = \xi\left(u - \frac{\log v}{\log y}\right)\rho\left(u - \frac{\log v}{\log y}\right)\left(1 + O\left(\frac{1}{u}\right)\right)$$

$$= \left(\xi(u) + O\left(\frac{\log v}{u \log y}\right)\right)\rho(u)\exp\left(\frac{\log v}{\log y}(\xi(u) + O(1))\right)\left(1 + O\left(\frac{1}{u}\right)\right)$$

$$= \xi(u)\rho(u)\exp\left(\frac{\log v}{\log y}(\xi(u) + O(1))\right)\left(1 + O\left(\frac{1}{\log u}\right)\right). \tag{47}$$

Throughout this paper we are assuming that $y \in H_\epsilon$ given by (3), so using Lemma 3.2(i)

$$\frac{\xi(u) + O(1)}{\log y} \ll (\log_2 x)^{-\frac{2}{3}-\epsilon}. \tag{48}$$

Hence if $\log v = o\left((\log_2 x)^{+\frac{2}{3}+\epsilon}\right)$,

$$\exp\left(\frac{\log v}{\log y}(\xi(u) + O(1))\right) = 1 + O\left(\log v(\log_2 x)^{-\frac{2}{3}-\epsilon}\right). \tag{49}$$

Define $V = V(x)$ by $\log V = (\log_2 x)^{+\frac{1}{6}+\epsilon}$; we could replace the exponent $\frac{1}{6}$ by any positive number $< \frac{2}{3}$. For $v < \min(V, \frac{x}{y})$, it follows from (47) and (49) that

$$\rho'\left(u - \frac{\log v}{\log y}\right) = \xi(u)\rho(u)\left(1 + O\left(\frac{1}{\log u} + \frac{1}{(\log_2 x)^{1/2}}\right)\right).$$

Hence since $\rho'\left(u - \frac{\log v}{\log y}\right) = 0$ for $v > \frac{x}{y}$,

$$\begin{aligned} I_1 &:= -\int_1^V \frac{v - [v]}{v^2}\rho'\left(u - \frac{\log v}{\log y}\right) dv \\ &= \xi(u)\rho(u)\left(1 + O\left(\frac{1}{\log u} + \frac{1}{(\log_2 x)^{1/2}}\right)\right)\int_1^{\min(V, \frac{x}{y})} \frac{v - [v]}{v^2} dv \\ &= C\xi(u)\rho(u)\left(1 + O\left(\frac{1}{\log u} + \frac{1}{(\log_2 x)^{1/2}} + \max\left(\frac{1}{V}, \frac{y}{x}\right)\right)\right). \end{aligned} \tag{50}$$

Since $\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|)$, we have $C = 1 - \gamma$ by (25). By (47) again when $\frac{x}{y} > V$

$$0 \le I_2 := -\int_V^{\min\left(x^{2/3}, \frac{x}{y}\right)} \frac{v-[v]}{v^2} \rho'\left(u - \frac{\log v}{\log y}\right) dv$$
$$\le \xi(u)\rho(u)\left(1 + O\left(\frac{1}{\log u}\right)\right) \int_V^{\min\left(x^{2/3}, \frac{x}{y}\right)} v^{-2+\eta} dv$$

where $\eta = \frac{\xi(u)+O(1)}{\log y} = O\left((\log_2 x)^{-\frac{2}{3}-\epsilon}\right)$ by (48). Hence since $V^\eta \sim 1$ as $x \to \infty$

$$I_2 \ll \xi(u)\rho(u)V^{-1}. \tag{51}$$

Since $\rho'\left(u - \frac{\log v}{\log y}\right) = 0$ for $v > \frac{x}{y}$, we can extend the integral in I_2 up to $v = x^{2/3}$ in all cases.

It remains to deal with the range $x^{2/3} \le v \le x$ where we use Lemma 3.1(ii) to bound $\rho'\left(u - \frac{\log v}{\log y}\right)$. We have

$$0 \le I_3 := -\int_{x^{2/3}}^{x} \frac{v-[v]}{v^2} \rho'\left(u - \frac{\log v}{\log y}\right) dv \le \int_{x^{2/3}}^{x} v^{-2} dv \le x^{-2/3}. \tag{52}$$

Combining (50), (51), (52) we obtain

$$I_1 + I_2 + I_3 = C\xi(u)\rho(u)\left(1 + O\left(\frac{1}{\log u} + \frac{1}{(\log_2 x)^{1/2}} + \frac{y}{x}\right)\right)$$

since $\frac{1}{V} + \frac{x^{-2/3}}{\xi(u)\rho(u)} = o\left((\log_2 x)^{-1/2}\right)$ by (48) and Lemma 3.1(i). This gives the result. □

In (2) we defined $\Lambda(x,y)$ by

$$\Lambda(x,y) = x\int_0^\infty \rho\left(\frac{\log\frac{x}{v}}{\log y}\right) d\left(\frac{[v]}{v}\right) \text{ for } x \notin \mathbf{N},$$
$$\Lambda(x,y) = \frac{1}{2}(\Lambda(x+0,y) + \Lambda(x-0,y)) \text{ for } x \in \mathbf{N}.$$

Lemma 3.6. *For $x \notin \mathbf{N}$ and $u = \frac{\log x}{\log y}$*

$$\Lambda(x,y) = x\left\{\rho(u) - \int_1^x \frac{v-[v]}{v^2 \log y} \rho'\left(u - \frac{\log v}{\log y}\right) dv\right\} - (x - [x]).$$

See equation (80) of chapter 3.5 of [31], or Lemma 2.6 of [27] (where the last bracketed expression was missing).

From Lemma 3.5 or equation (104) of chapter 3.5 of [31], we deduce

Corollary 3.7. *As $u \to \infty$*

$$\Lambda(x,y) = x\left\{\rho(u) + \frac{\xi(u)\rho(u)}{\log y}(1-\gamma+o(1))\right\}.$$

Note that $\frac{\xi(u)}{\log y} = O\left(\frac{\log(u+1)}{\log y}\right) = o(1)$ as $u \to \infty$ by (48).

Lemma 3.8 (Saias)**.** *For $\epsilon > 0$ and $y \in H_\epsilon$ given by (3)*

$$\Psi(x,y) = \Lambda(x,y)\left(1 + O_\epsilon\left(\exp\left(-(\log y)^{\frac{3}{5}-\epsilon}\right)\right)\right).$$

See [26] or the proof of Theorem 3.5.9 in [31],

4. Proof of Theorem 1.1

Recall that throughout y lies in the region H_ϵ given by (3) and $L_\epsilon(y)$ is defined by (12).

With $P(\mathfrak{a})$ as in (6), define $\zeta_K(s,y)$ by

$$\zeta_K(s,y) = \prod_{N(\mathfrak{p})\le y}\left(1-(N(\mathfrak{p}))^{-s}\right)^{-1} = \sum_{\substack{\mathfrak{a}\\ P(\mathfrak{a})\le y}} (N(\mathfrak{a}))^{-s} \tag{53}$$

which is valid in $\sigma > 0$ since the product is finite.

Lemma 4.1. *To each $\epsilon > 0$, there exists $y_0(\epsilon)$ such that*

$$\zeta_K(s,y) = \zeta_K(s)(s-1)\log y\ \hat{\rho}((s-1)\log y)\left(1 + O((L_\epsilon(y))^{-1})\right) \tag{54}$$

uniformly for

$$y \ge y_0(\epsilon), \sigma \ge 1-(\log y)^{-\frac{2}{5}-\epsilon}, |t| \le L_\epsilon(y). \tag{55}$$

Proof. The proof is similar to that given in [31] for the case $K = \mathbf{Q}$ (see Lemma 9.1 of chapter 3.5); see also Lemme 6 and Proposition 1 of [26]. The properties of $\zeta_K'(s)/\zeta_K(s)$ required have been established in Lemma 2.8. □

Recall (see (10)) that $\alpha_0 = 1 - \frac{\xi(u)}{\log y}$. Let

$$T = L_{\epsilon/3}(y). \tag{56}$$

Define

$$J(x,y) := \frac{1}{2\pi i}\int_{\alpha_0-iT}^{\alpha_0+iT} g_K(s)(s-1)\log y\ \hat{\rho}((s-1)\log y)\ x^s s^{-1} ds \tag{57}$$

where $g_K(s) = \zeta_K(s) - \lambda_K\zeta(s)$ as in (8). Then (see (11))

$$\lim_{T\to\infty} J(x,y) = J_0(x,y).$$

Lemma 4.2. *For* $y \in H_\epsilon$

$$\Psi_K(x,y) = \lambda_K \Psi(x,y) + J(x,y) + O\left(\frac{x\rho(u)}{L_\epsilon(y)}\right). \tag{58}$$

Proof. By Perron's formula

$$\Psi_K(x,y) = \frac{1}{2\pi i}\int_{\alpha_0 - iT}^{\alpha_0+iT} \zeta_K(s,y)x^s s^{-1} ds + E \tag{59}$$

where

$$E \ll x^{\alpha_0} \sum_{m=1}^{\infty} \frac{j_y(m)}{m^{\alpha_0}(1+T\ \left|\log\frac{x}{m}\right|)},$$

with

$$j_y(m) = |\{\mathfrak{a} : N(\mathfrak{a}) = m, P(\mathfrak{a}) \le y\}|$$

so $0 \le j_y(m) \le j(m) \le d_n(m)$, and by (53)

$$\zeta_K(s,y) = \sum_{m=1}^{\infty} j_y(m) m^{-s}.$$

Following the method employed to bound the error term in the proof of Lemma 9.4 of chapter 3.5 of [31], but with T defined differently, and using Lemma 4.1 and appropriate results from sections 2 and 3, in particular noting that $\zeta_K(\alpha_0) \ll |\alpha_0 - 1|^{-1}$, we find that

$$E \ll x\rho(u)(L_\epsilon(y))^{-1}. \tag{60}$$

We now use Lemma 4.1 with ϵ replaced by $\epsilon/3$ to substitute for $\zeta_K(s,y)$ in the integral in (59). The conditions of (55) hold since

$$|t| \le T = L_{\epsilon/3}(y) \text{ and } \alpha_0 = 1 - \frac{\xi(u)}{\log y} \ge 1 - (\log y)^{-\frac{2}{5}-\frac{\epsilon}{3}} \text{ for } y \in H_\epsilon,$$

and we assume throughout that x and hence y are sufficiently large. We obtain

$$\begin{aligned}\Psi_K(x,y) &= \frac{1}{2\pi i}\int_{\alpha_0-iT}^{\alpha_0+iT} \zeta_K(s)(s-1)\log y\ \hat\rho((s-1)\log y)\ x^s s^{-1} ds \\ &\qquad + O\left(\frac{x\rho(u)}{L_\epsilon(y)}\right) \\ &= \frac{\lambda_K}{2\pi i}\int_{\alpha_0-iT}^{\alpha_0+iT} \zeta(s)(s-1)\log y\ \hat\rho((s-1)\log y)\ x^s s^{-1} ds \\ &\qquad + J(x,y) + O\left(\frac{x\rho(u)}{L_\epsilon(y)}\right)\end{aligned} \tag{61}$$

by (8) and since

$$(L_{\epsilon/3}(y))^{-1}\int_{\alpha_0-iT}^{\alpha_0+iT} \zeta_K(s,y)x^s s^{-1} ds \ll \zeta_K(\alpha_0,y)x^{\alpha_0}(L_{\epsilon/3}(y))^{-1}\log T$$

$$\ll \zeta_K(\alpha_0)\xi(u)\hat{\rho}(-\xi(u))xe^{-u\xi(u)}\log y(L_{\epsilon/3}(y))^{-1} \ll x\rho(u)(L_\epsilon(y))^{-1}$$

on using Lemma 4.1, (22), Lemmas 3.4(iii), 3.2(iii) and the fact that $\log u = o(\log L_{\epsilon/3}(y))$ for $y \in H_\epsilon$.

The first term on the right of (61) equals

$$\lambda_K\Lambda(x,y) + O(x\rho(u)(L_\epsilon(y))^{-1}); \tag{62}$$

see the proof of Theorem 3.5.9 in [31] with a slightly different range of integration or Proposition 2 of [26]. The lemma now follows from Lemma 3.8. □

To complete the proof of Theorem 1.1, we need to show that

$$|J_0(x,y) - J(x,y)| \ll x\rho(u)(L_\epsilon(y))^{-1};$$

this follows from

Lemma 4.3.

$$\int_{\substack{\sigma=\alpha_0\\|t|\geq T}} g_K(s)(s-1)\log y\hat{\rho}((s-1)\log y)\; x^s s^{-1}ds \ll \frac{x\rho(u)}{L_\epsilon(y)}.$$

Proof. It is sufficient to consider the range $t \geq T$. Let

$$J^* = \int_{\substack{\sigma=\alpha_0\\t\geq T}} g_K(s)(s-1)\log y\;\hat{\rho}((s-1)\log y)\; x^s s^{-1}ds.$$

Since $T\log y > 1 + u\xi$, we have by Lemma 3.4(iii) that

$$(s-1)\log y\;\hat{\rho}((s-1)\log y) = 1 + O\left(\frac{1+u\xi}{|t|\log y}\right).$$

Hence by Lemma 2.3(iii) with $N = t^{n+1}$

$$\begin{aligned} J^* &= \int_{\substack{\sigma=\alpha_0\\t\geq T}} \left(\sum_{m\leq t^{n+1}} b(m)m^{-s} + O\left(t^{(n+1)\frac{\xi(u)}{\log y}-\frac{1}{n}}\right)\right) \\ &\qquad \left(1 + O\left(\frac{1+u\xi}{|t|\log y}\right)\right) x^s s^{-1}ds \\ &= \sum_{m=1}^{\infty} b(m)\int_{\substack{\sigma=\alpha_0\\t\geq\max(T,m^{\frac{1}{n+1}})}} \left(\frac{x}{m}\right)^s s^{-1}ds + E_1 \end{aligned} \tag{63}$$

where, as $|b(m)| \ll d_n(m)$ and by Lemma 2.2 and Corollary 3.3,

$$E_1 \ll \frac{1+u\xi}{\log y} x^{\alpha_0} \int_T^\infty \left(\sum_{m \le t^{n+1}} d_n(m) m^{-\alpha_0} \right) t^{-2} dt$$

$$+ x^{\alpha_0} \int_T^\infty t^{(n+1)\frac{\xi(u)}{\log y} - \frac{1}{n} - 1} dt \ll x\rho(u)(L_\epsilon(y))^{-1} \tag{64}$$

since

$$x^{\alpha_0} = xe^{-u\xi}, \qquad \frac{\xi(u)}{\log y} = o(1), \qquad T = L_{\epsilon/3}(y).$$

It remains to estimate the main term in (63). We have

$$\int_{\max(m^{\frac{1}{n+1}}, T)}^\infty \frac{(x/m)^{\alpha_0+it}}{\alpha_0+it} dt \ll \frac{(x/m)^{\alpha_0}}{1 + \max(m^{\frac{1}{n+1}}, T) \left| \log \frac{x}{m} \right|}$$

(see Lemma 2.2.1.1 of [31]). Hence the main term of (63) is

$$\ll x^{\alpha_0} \sum_{m=1}^\infty \frac{d_n(m) m^{-\alpha_0}}{1 + \max(m^{\frac{1}{n+1}}, T) \left| \log \frac{x}{m} \right|}. \tag{65}$$

When $|m-x| > x^{1-\frac{1}{2(n+1)}}$, $\left|\log \frac{x}{m}\right| \gg m^{-\frac{1}{2(n+1)}}$. Hence the contribution of these terms to (65) is

$$\ll x^{\alpha_0} \sum_{|m-x| > x^{1-\frac{1}{2(n+1)}}} \frac{d_n(m) m^{-\alpha_0}}{(m^{\frac{1}{n+1}} + T) m^{-\frac{1}{2(n+1)}}}$$

$$\ll xe^{-u\xi} \sum_{m=1}^\infty \frac{d_n(m)}{m^{\alpha_0 + \frac{1}{2(n+1)}} + T} \ll x\rho(u)(L_\epsilon(y))^{-1} \tag{66}$$

by Corollary 3.3; for the series on the right converges since $\alpha_0 + \frac{1}{2(n+1)} > 1$, and its sum is $\ll T^{-1} = (L_{\epsilon/3}(y))^{-1}$.

When $|m-x| \le x^{1-\frac{1}{2(n+1)}}$, $(x/m)^{\alpha_0} \ll 1$ and $d_n(m) \ll x^\delta$ for any $\delta > 0$, and so the contribution of these terms to (65) is

$$\ll x^\delta \, x^{1-\frac{1}{2(n+1)}} \ll x\rho(u)(L_\epsilon(y))^{-1} \tag{67}$$

if we take $\delta \le \frac{1}{4(n+1)}$ (say) so $L_\epsilon(y)/\rho(u) \ll x^{\frac{1}{2(n+1)} - \delta}$. Combining equations (63) to (67), we obtain

$$J^* \ll x\rho(u)(L_\epsilon(y))^{-1}.$$

The result of the lemma now follows, for the integral over $t \le -T$ is just the complex conjugate of J^*. $\square$

The result of Theorem 1.1 now follows from Lemmas 4.2 and 4.3.

In the next two sections we investigate $J(x, y)$ further.

5. Asymptotic formula for $\mathbf{J}(\mathbf{x}, \mathbf{y})$

Define $J(x,y)$ by (57) with $T = L_{\epsilon/3}(y)$ and $y \in H_\epsilon$ given by (3). We split the integral into several parts depending on the size of $|t|$ and of u, and deal with each part in a separate lemma. Our aim is to show that the magnitude of $J(x,y)$ (when $g_K(1) \neq 0$) is the same as that of the second term in $\Lambda(x,y)$, given in Corollary 3.7. Provided $g_K(1) \neq 0$, the leading term comes from the range $|t| \leq \pi$ in (68).

By the change of variable $(s-1)\log y \longrightarrow s$, we can rewrite (57) as

$$J(x,y) = \frac{x}{2\pi i}\int_{-\xi(u)-iT\log y}^{-\xi(u)+iT\log y} \frac{g_K(1+\frac{s}{\log y})}{s+\log y}\, s\hat{\rho}(s)e^{us}ds. \tag{68}$$

Lemma 5.1. *For* $\xi(u) > 1$,

$$\begin{aligned} J_1 &:= \frac{1}{2\pi i}\int_{-\xi(u)-i\pi}^{-\xi(u)+i\pi} \frac{g_K(1+\frac{s}{\log y})}{s+\log y}\, s\hat{\rho}(s)e^{us}ds \\ &= -\frac{\rho(u)\xi(u)}{\log y}\left\{g_K(1) + O\left(\frac{\xi(u)}{\log y} + \frac{1}{\sqrt{u}}\right)\right\}. \end{aligned}$$

Proof. Let

$$F(w) = g_K(w)w^{-1} \qquad \left(\Re(w) > 1 - \frac{1}{n}\right) \tag{69}$$

so in this region $F(w)$ is differentiable and is bounded for bounded w. Hence for $|w-1| \leq \frac{1}{2n}$ (say),

$$F(w) = F(1) + O(|w-1|).$$

Putting $w = 1 + \frac{-\xi+it}{\log y}$, $|t| \leq \pi$, we obtain since $\xi = \xi(u) > 1$

$$F\left(1 + \frac{-\xi+it}{\log y}\right) = g_K(1) + O(\xi(u)/\log y). \tag{70}$$

Thus by Lemma 3.4(iii)

$$\begin{aligned} J_1 &= \frac{e^{-u\xi}}{2\pi\log y}\int_{-\pi}^{\pi} g_K(1)(-\xi+it)\hat{\rho}(-\xi+it)e^{iut}dt \\ &\quad + O\left(\left(\frac{\xi(u)}{\log y}\right)^2 e^{-u\xi}\int_{-\pi}^{\pi}\exp\left(I(\xi) - \frac{ut^2}{2\pi^2}\right)dt\right). \end{aligned} \tag{71}$$

The error term in (71) is

$$\ll \exp\left(I(\xi) - u\xi\right)\frac{1}{\sqrt{u}}\left(\frac{\xi(u)}{\log y}\right)^2 \ll \rho(u)\left(\frac{\xi(u)}{\log y}\right)^2 \tag{72}$$

by Lemma 3.2(i) and (iii).

It remains to investigate the integrals

$$J_1^{(1)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\rho}(-\xi+it)e^{u(-\xi+it)}dt, \tag{73}$$

$$J_1^{(2)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\hat{\rho}(-\xi+it)e^{u(-\xi+it)}dt; \tag{74}$$

for by (71) and (72)

$$J_1 = \frac{g_K(1)}{\log y}(-\xi J_1^{(1)} + iJ_1^{(2)}) + O\left(\rho(u)\left(\frac{\xi(u)}{\log y}\right)^2\right). \tag{75}$$

By (44)

$$J_1^{(1)} = \rho(u) - \frac{1}{2\pi}\int_{|t|>\pi} \hat{\rho}(-\xi+it)e^{u(-\xi+it)}dt. \tag{76}$$

Using Lemma 3.4(iii) and Lemma 3.2(i) and (iii)

$$\begin{aligned}\frac{1}{2\pi}\int_{\pi\le|t|\le 1+u\xi} \hat{\rho}(-\xi+it)e^{u(-\xi+it)}dt &\ll u\xi e^{-u\xi}\exp\left(I(\xi) - \frac{u}{\xi^2+\pi^2}\right)\\ &\ll u\xi\sqrt{u}\rho(u)\exp\left(-\frac{u}{\xi^2+\pi^2}\right).\end{aligned} \tag{77}$$

For any $U_1 > 1+u\xi$, by Lemma 3.4(iii) the contribution to the integral in (76) from the range $1+u\xi \le t \le U_1$ is

$$\begin{aligned}&\frac{1}{2\pi}\int_{1+u\xi\le t\le U_1} \frac{e^{u(-\xi+it)}}{-\xi+it}\left(1+O\left(\frac{1+u\xi}{t}\right)\right)dt\\ &= \frac{1}{2\pi i}\int_{-\xi+i(1+u\xi)}^{-\xi+iU_1} s^{-1}e^{us}ds + O\left(e^{-u\xi}(1+u\xi)\int_{1+u\xi}^{U_1}\frac{dt}{t^2}\right)\\ &= \frac{1}{2\pi i}\left\{\left[\frac{e^{us}}{us}\right]_{-\xi+i(1+u\xi)}^{-\xi+iU_1} + u^{-1}\int_{-\xi+i(1+u\xi)}^{-\xi+iU_1} s^{-2}e^{us}ds\right\} + O(e^{-u\xi})\\ &\ll e^{-u\xi} = \rho(u)\exp\left(-u\left(1+O\left(\frac{\log_2 u}{\log u}\right)\right)\right)\end{aligned} \tag{78}$$

by Corollary 3.3. The same estimate holds when $1+u\xi \le -t \le U_1$. Letting $U_1 \to \infty$, we obtain from (76), (77) and (78) that

$$J_1^{(1)} = \rho(u)\left\{1 + O\left(u^{3/2}\log u\ \exp\left(-\frac{u}{\xi^2+\pi^2}\right)\right)\right\} \tag{79}$$

since $\xi(u) > 1$.

By Lemmas 3.4(iii) and 3.2(iii) and (i)

$$J_1^{(2)} \ll e^{-u\xi} \int_0^{\pi} t \exp\left(I(\xi) - \frac{ut^2}{2\pi^2}\right) dt$$

$$\ll \sqrt{u}\rho(u)u^{-1} = \frac{1}{\sqrt{u}}\rho(u) < \frac{1}{\sqrt{u}}\rho(u)\xi(u) \tag{80}$$

since $\xi(u) > 1$.

We deduce from (75), (79) and (80) that since $1 < \xi(u) \sim \log u$

$$J_1 = -\frac{\rho(u)\xi(u)}{\log y}\left(g_K(1) + O\left(\frac{\xi(u)}{\log y} + \frac{1}{\sqrt{u}}\right)\right)$$

as required. □

Lemma 5.2. *For $\xi(u) > 1$*

$$J_2 := \frac{1}{2\pi i}\int_{\substack{\sigma=-\xi(u)\\ \pi\le|t|\le 1+u\xi}} \frac{g_K(1+\frac{s}{\log y})}{s+\log y}\, s\hat{\rho}(s)e^{us}ds$$

$$\ll \frac{\rho(u)\xi(u)}{\log y}\exp\left(\frac{-u}{\xi^2+\pi^2}\right)u^3(\log u)^{3/2}.$$

Proof. Let $U_2 = \min(1+u\xi, \frac{1}{n+1}\log y)$; then for $\pi \le |t| \le U_2$, $s = -\xi + it$ we have $\left|\frac{s}{\log y}\right| \le \frac{\xi}{\log y} + \frac{1}{n+1} < \frac{1}{n}$ for sufficiently large y, and so $g_K(1+\frac{s}{\log y})$ is bounded whilst $\left|1+\frac{s}{\log y}\right| \gg 1$. With $F(w)$ as in (69) it follows that

$$F\left(1+\frac{s}{\log y}\right) \ll 1.$$

Hence by Lemma 3.4(iii)

$$\frac{1}{2\pi i}\int_{\substack{\sigma=-\xi(u)\\ \pi\le|t|\le U_2}} \frac{F(1+\frac{s}{\log y})}{\log y}\, s\hat{\rho}(s)e^{us}ds$$

$$\ll \frac{U_2^2}{\log y}\exp\left(I(\xi) - u\xi - \frac{u}{\xi^2+\pi^2}\right)$$

$$\ll \frac{(u\xi)^2}{\log y}\sqrt{u}\rho(u)\exp\left(-\frac{u}{\xi^2+\pi^2}\right)$$

$$\ll \frac{\rho(u)\xi(u)}{\log y}\exp\left(-\frac{u}{\xi^2+\pi^2}\right)u^{5/2}\log u \tag{81}$$

by Lemma 3.2(i) and (iii).

Now suppose that $U_2 = \frac{1}{n+1}\log y < 1+u\xi$, $U_2 \le |t| \le 1+u\xi$. In this case, $g_K(1+\frac{s}{\log y}) \ll \left(\frac{|t|}{\log y}\right)^{1/2}$ by Lemma 2.5, and $\left|1+\frac{s}{\log y}\right| \gg 1$. Using

Lemma 3.4(iii) again

$$\frac{1}{2\pi i}\int_{\substack{\sigma=-\xi(u)\\ \frac{1}{n+1}\log y\le|t|\le 1+u\xi}}\frac{F(1+\frac{s}{\log y})}{\log y}s\hat{\rho}(s)e^{us}ds$$

$$\ll (\log y)^{-3/2}\exp\left(I(\xi)-u\xi-\frac{u}{\xi^2+\pi^2}\right)\int_{\frac{1}{n+1}\log y}^{1+u\xi}t^{3/2}dt$$

$$\ll (u\xi)^{5/2}(\log y)^{-3/2}\sqrt{u}\rho(u)\exp\left(-\frac{u}{\xi^2+\pi^2}\right)$$

$$\ll \frac{\rho(u)\xi(u)}{\log y}\exp\left(-\frac{u}{\xi^2+\pi^2}\right)u^3(\log u)^{3/2}. \tag{82}$$

The result of the lemma now follows from (81) and (82), the latter applying only when $\frac{1}{n+1}\log y<1+u\xi$. □

Lemma 5.3. *For $u\ge 5$ and $1+u\xi<\frac{1}{n+1}\log y$*

$$J_3:=\frac{1}{2\pi i}\int_{\substack{\sigma=-\xi(u)\\ 1+u\xi\le|t|\le\frac{1}{n+1}\log y}}\frac{g_K(1+\frac{s}{\log y})}{s+\log y}\,s\hat{\rho}(s)e^{us}ds\ll\frac{\rho(u)\xi(u)}{\log y}\frac{\log u}{\sqrt{u}}.$$

Proof. We can expand $F\left((1-\frac{\xi}{\log y})+i\frac{t}{\log y}\right)$ in a power series in $\frac{t}{\log y}$ since $\frac{|t|}{\log y}\le\frac{1}{n+1}$, and we obtain

$$\sum_{m=0}^{\infty}c(m)\left(\frac{t}{\log y}\right)^m \quad\text{where } c(0)=F\left(1-\frac{\xi}{\log y}\right)\ll 1. \tag{83}$$

For $m\ge 1$ we have by Cauchy's inequalities that

$$c(m)\ll(n+\frac{1}{2})^m \tag{84}$$

since $F(w)$ is analytic and bounded for $\left|w-(1-\frac{\xi}{\log y})\right|\le(n+\frac{1}{2})^{-1}<\frac{1}{n}$. Substituting in the integral J_3 and using Lemma 3.4(iii) and (iv) we see that

$$J_3=\frac{e^{-u\xi}}{2\pi\log y}\sum_{m=1}^{\infty}c(m)(\log y)^{-m}$$

$$\int_{1+u\xi\le|t|\le\frac{1}{n+1}\log y}t^m e^{iut}\left(1+O\left(\frac{1+u\xi}{|t|}\right)\right)dt$$

$$+\frac{e^{-u\xi}c(0)}{2\pi\log y}\int_{1+u\xi\le|t|\le\frac{1}{n+1}\log y}e^{iut}\left(1+\int_1^{\infty}e^{(\xi-it)v}\rho'(v)dv\right)dt. \tag{85}$$

For $m \geq 1$,

$$\int_{1+u\xi}^{\frac{1}{n+1}\log y} t^{m-1} dt \ll \left(\frac{1}{n+1}\log y\right)^m, \tag{86}$$

$$\int_{1+u\xi \leq |t| \leq \frac{1}{n+1}\log y} t^m e^{iut} dt \ll u^{-1}\left(\frac{1}{n+1}\log y\right)^m$$

by the second mean value theorem for real integrals. Hence by (84)

$$\sum_{m=1}^{\infty} c(m)(\log y)^{-m} \int_{1+u\xi \leq |t| \leq \frac{1}{n+1}\log y} t^m e^{iut}\left(1 + O\left(\frac{1+u\xi}{|t|}\right)\right) dt$$

$$\ll \sum_{m=1}^{\infty}\left(\frac{n+\frac{1}{2}}{n+1}\right)^m \left(u^{-1} + u\xi\right) \ll u\xi. \tag{87}$$

However, when $m = 0$, the right side of (86) becomes $O(\log_2 y)$ which is too big for our purposes. Hence we adopt a different approach for this case, as indicated in (85). We split the inner integral into sections, recalling that it is absolutely convergent. Since

$$\int_{1+u\xi \leq |t| \leq \frac{1}{n+1}\log y} e^{iut} dt \ll u^{-1}, \tag{88}$$

our main concern is to investigate (with $s = -\xi + it$)

$$\frac{1}{2\pi i \log y} \int_{\substack{\sigma=-\xi \\ 1+u\xi \leq |t| \leq \frac{1}{n+1}\log y}} e^{us}\left(\int_1^{\infty} e^{-sv}\rho'(v)dv\right) ds. \tag{89}$$

The first three derivatives of $\rho(v)$ are continuous on $v \geq 4$, so consider first

$$i^{-1}\int_{\substack{\sigma=-\xi \\ 1+u\xi \leq t \leq \frac{1}{n+1}\log y}} e^{us}\left(\int_1^4 e^{-sv}\rho'(v)dv\right) ds$$

$$= e^{-u\xi}\int_1^4 \rho'(v)e^{v\xi}\left(\int_{1+u\xi}^{\frac{1}{n+1}\log y} e^{i(u-v)t}dt\right) dv \ll \frac{1}{u\xi}e^{(4-u)\xi} \tag{90}$$

since $\rho'(v)$ is bounded and $u - v \geq u - 4 \geq 1$. (It would be enough here and below to have $u - 4 \geq \delta$ for any fixed $\delta > 0$.)

Let X be large (with $\log_2 X > \xi + 1$) where later we let $X \to \infty$. On integrating by parts twice

$$\int_4^X e^{-sv}\rho'(v)dv = \left[\left(-s^{-1}\rho'(v) - s^{-2}\rho''(v)\right)e^{-sv}\right]_4^X + s^{-2}\int_4^X e^{-sv}\rho'''(v)dv. \tag{91}$$

In order to determine what this contributes to (89), we need estimates of the following integrals for $v = 4, X$:

$$\int_{\substack{\sigma=-\xi \\ 1+u\xi\le t\le \frac{1}{n+1}\log y}} s^{-1}e^{(u-v)s}ds$$

$$= \left[\frac{1}{s(u-v)}e^{(u-v)s}\right]_{-\xi+i(1+u\xi)}^{-\xi+\frac{i}{n+1}\log y} + \frac{1}{u-v}\int_{\substack{\sigma=-\xi \\ 1+u\xi\le t\le \frac{1}{n+1}\log y}} s^{-2}e^{(u-v)s}ds$$

$$\ll \frac{e^{-(u-v)\xi}}{(1+u\xi)\,|u-v|} \tag{92}$$

and

$$\int_{\substack{\sigma=-\xi \\ 1+u\xi\le |t|\le \frac{1}{n+1}\log y}} s^{-2}e^{(u-v)s}ds \ll \frac{e^{-(u-v)\xi}}{(1+u\xi)}. \tag{93}$$

We need also to estimate

$$\int_{\substack{\sigma=-\xi \\ 1+u\xi\le |t|\le \frac{1}{n+1}\log y}} e^{us}s^{-2}\int_4^X e^{-sv}\rho'''(v)dvds$$

$$\ll e^{-u\xi}\int_{1+u\xi}^{\frac{1}{n+1}\log y} t^{-2}\left(\int_4^X e^{\xi v}\left|\rho'''(v)\right| dv\right) dt. \tag{94}$$

Since $\rho'''(v) < 0$, the inner integral is

$$\left[-e^{\xi v}(\rho''(v) - \xi\rho'(v) + \xi^2\rho(v))\right]_4^X + \xi^3\int_4^X e^{\xi v}\rho(v)dv$$

where on using Lemma 3.1(i), (43) and Lemma 3.4(iii)

$$\int_4^X e^{\xi v}\rho(v)dv = \hat{\rho}(-\xi) + O(\xi^{-1}e^{4\xi}) + O(e^{-X\log X}) \ll e^{I(\xi)} + \xi^{-1}e^{4\xi} \tag{95}$$

as $X \to \infty$. From (95) and since $\left|\rho^{(k)}(X)\right| e^{\xi X} \to 0$ as $X \to \infty$ for $k = 0, 1, 2$, (94) is

$$\ll e^{-u\xi}\left(\xi^3 e^{I(\xi)} + \xi^2 e^{4\xi}\right)(1+u\xi)^{-1} \ll \left(\xi^3\rho(u)\sqrt{u} + \xi^2 e^{(4-u)\xi}\right)(1+u\xi)^{-1}. \tag{96}$$

From (91), (92), (93) and (96) we obtain

$$\int_{\substack{\sigma=-\xi \\ 1+u\xi\le |t|\le \frac{1}{n+1}\log y}} e^{us}\left(\int_4^\infty e^{-sv}\rho'(v)dv\right) ds$$

$$\ll \left(\xi^3\rho(u)\sqrt{u} + \xi^2 e^{(4-u)\xi}\right)(1+u\xi)^{-1} \ll \rho(u)\left(\xi(u)\right)^2/\sqrt{u} \tag{97}$$

by Corollary 3.3. Combining (97) and (90), we see that the double integral in (89) is

$$\ll \frac{\rho(u)\xi(u)}{\log y}\frac{\xi(u)}{\sqrt{u}}. \tag{98}$$

It follows from (87), (88) and (98) that

$$J_3 \ll \frac{e^{-u\xi}u\xi}{\log y} + \frac{\rho(u)\,(\xi(u))^2}{\sqrt{u}\log y} \ll \frac{\rho(u)(\xi(u))^2}{\log y\,\sqrt{u}} \ll \frac{\rho(u)\xi(u)}{\log y}\frac{\log u}{\sqrt{u}}$$

by Corollary 3.3. □

So far we have evaluated the part of the integral (68) with $|t| \le \max(1+u\xi, \frac{1}{n+1}\log y) = U_3$ (say). To complete the estimate for the range $U_3 \le |t| \le T\log y$, we consider separately the two cases $u < (\log_2 y)^2$, when $U_3 = \frac{1}{n+1}\log y$, and $u \ge (\log_2 y)^2$.

Lemma 5.4. *For $u < (\log_2 y)^2, \xi(u) > 1$*

$$\begin{aligned} J_4 &:= \frac{1}{2\pi i}\int_{\substack{\sigma=-\xi(u)\\ \frac{1}{n+1}\log y \le |t| \le T\log y}} \frac{g_K(1+\frac{s}{\log y})}{s+\log y}\, s\hat{\rho}(s)e^{us}ds \\ &\ll \frac{\rho(u)\xi(u)}{\log y}\exp\left(-u\left(1+O\left(\frac{\log_2 u}{\log u}\right)\right)\right). \end{aligned}$$

Proof. By Lemma 4(iii),

$$\begin{aligned} J_4 &= \frac{1}{2\pi i}\int_{\substack{\sigma=-\xi(u)\\ \frac{1}{n+1}\log y \le |t| \le T\log y}} \frac{g_K(1+\frac{s}{\log y})}{s+\log y}\left(1+O\left(\frac{1+u\xi}{|t|}\right)\right)e^{us}ds \\ &= \frac{e^{-u\xi}}{2\pi}\int_{\frac{1}{n+1}\le|t|\le T} \frac{g_K(\alpha_0+it)}{\alpha_0+it}x^{it}\left(1+O\left(\frac{1+u\xi}{|t|\log y}\right)\right)dt \end{aligned} \tag{99}$$

by a change of variable. We verify that we can use Corollary 2.7 to bound $g_K(\alpha_0+it)$ in (99) by showing that (26) is satisfied. Thus we need to show that

$$\alpha_0 = 1 - \frac{\xi}{\log y} \ge 1 - c(\log T)^{-2/3}(\log_2 T)^{-1/3}. \tag{100}$$

Since $\xi(u) \sim \log u = \log_2 x - \log_2 y$ and $y \in H_\epsilon$ (see (3))

$$\frac{\xi}{\log y} < \frac{\log_2 x}{\log y} \le (\log y)^{\frac{3}{5+3\epsilon}-1} = (\log y)^{-\frac{2}{5}-\frac{9\epsilon}{5(5+3\epsilon)}},$$

and

$$(\log T)^{2/3}(\log_2 T)^{1/3} \ll (\log y)^{\frac{2}{3}(\frac{3}{5}-\frac{\epsilon}{3})}(\log_2 y)^{1/3}$$

and so (100) follows for sufficiently large y. Hence by Corollary 2.7

$$g_K(\alpha_0+it) \ll (\log|t|)^{2/3}\log_2|t| \ll \log|t| \tag{101}$$

for $t_0 \le |t| \le T$, and $g_K(\alpha_0 + it)$ is bounded for $\frac{1}{n+1} \le |t| \le t_0$.

By (101), the error term in (99) is

$$\begin{aligned} &\ll e^{-u\xi} \frac{(1+u\xi)}{\log y} \left(1 + \int_{t_0}^{T} t^{-2} \log t dt\right) \\ &\ll \frac{\rho(u)\xi(u)}{\log y} \exp\left(-u\left(1 + O\left(\frac{\log_2 u}{\log u}\right)\right)\right) \end{aligned} \tag{102}$$

by Corollary 3.3.

The main term in (99) may be written as

$$\frac{x^{-1}}{2\pi i} \int_{\substack{\sigma=\alpha_0 \\ \frac{1}{n+1} \le |t| \le T}} g_K(s) s^{-1} x^s ds.$$

We integrate this in the range $\frac{1}{n+1} \le t \le T$ by parts six times to obtain

$$\begin{aligned} \frac{x^{-1}}{2\pi i} \Bigg\{ &\left[\sum_{j=0}^{5} (-1)^j \frac{d^j}{ds^j} (g_K(s) s^{-1}) x^s (\log x)^{-j-1} \right]_{\alpha_0 + \frac{i}{n+1}}^{\alpha_0 + iT} \\ &+ (\log x)^{-6} \int_{\alpha_0 + \frac{i}{n+1}}^{\alpha_0 + iT} \frac{d^6}{ds^6} (g_K(s) s^{-1}) x^s ds \Bigg\}, \end{aligned} \tag{103}$$

with a corresponding expression when $\frac{1}{n+1} \le -t \le T$. Applying Cauchy's inequalities to $g_K(s)s^{-1}$ on a circle with centre s and radius of the form $c_1/\log|t|$ for $|t| \ge t_0$ and using (101) we have that for $t_0 \le |t| \le T$

$$\frac{d^j}{ds^j}(g_K(s)s^{-1}) \ll |t|^{-1} (\log|t|)^{j+1} \qquad (0 \le j \le 6). \tag{104}$$

For $\frac{1}{n+1} \le |t| \le t_0$, the left side of (104) is bounded. Hence the main term in (99) is

$$\begin{aligned} &\ll x^{\alpha_0 - 1} \left((\log x)^{-1} + (\log x)^{-6} \left(1 + \int_{t_0}^{T} t^{-1} (\log t)^7 dt \right) \right) \\ &\ll e^{-u\xi} \left((\log x)^{-1} + (\log x)^{-6} (\log T)^8 \right) \\ &\ll e^{-u\xi} \left((\log x)^{-1} + (\log x)^{-6} (\log y)^{8(\frac{3}{5} - \frac{\epsilon}{3})} \right) \ll (\log x)^{-1} e^{-u\xi} \\ &\ll \frac{\rho(u)\xi(u)}{\log y} \exp\left(-u\left(1 + O\left(\frac{\log_2 u}{\log u}\right)\right)\right) \end{aligned} \tag{105}$$

since $y \le x$, $\xi(u) > 1$ and by Corollary 3.3. The result of the lemma now follows from (99), (102) and (105). $\square$

Lemma 5.5. *For* $u \geq (\log_2 y)^2$

$$J_5 := \frac{1}{2\pi i} \int_{\substack{\sigma=-\xi(u) \\ U_3 \leq |t| \leq T \log y}} \frac{g_K(1+\frac{s}{\log y})}{s+\log y} s\hat{\rho}(s) e^{us} ds$$
$$\ll \frac{\rho(u)\xi(u)}{\log y} \exp\left(-u\left(1+O\left(\frac{\log_2 u}{\log u}\right)\right)\right).$$

Proof. Note that $\frac{1}{n+1} \leq U_3/\log y < T$ since $\log_2 x \leq (\log y)^{\frac{3}{5}-\frac{9\epsilon}{5(5+3\epsilon)}}$ and so

$$\frac{u\xi}{\log y} < \frac{\log x \log_2 x}{(\log y)^2} < T = L_{\epsilon/3}(y).$$

Using a modification of (99)

$$J_5 = \frac{e^{-u\xi}}{2\pi} \int_{\frac{U_3}{\log y} \leq |t| \leq T} \frac{g_K(\alpha_0+it)}{\alpha_0+it} x^{it} \left(1+O\left(\frac{1+u\xi}{|t|\log y}\right)\right) dt$$
$$\ll e^{-u\xi}\left\{1+\frac{u\xi}{\log y}+\int_{\max(t_0,U_3/\log y)}^{T} t^{-1}\log t\left(1+\frac{u\xi}{t\log y}\right)dt\right\}$$

by (101). The integral is $\ll (\log T)^2$ since $\frac{u\xi}{|t|\log y} \leq \frac{u\xi}{U_3} \leq 1$. Hence by Corollary 3.3 and the definition of T

$$J_5 \ll \frac{\rho(u)\xi(u)}{\log y} \exp\left(-u\left(1+O\left(\frac{\log_2 u}{\log u}\right)\right)\right)\left(\log y + (\log y)^{\frac{11}{5}-\frac{2\epsilon}{3}}\right)$$
$$\ll \frac{\rho(u)\xi(u)}{\log y} \exp\left(-u\left(1+O\left(\frac{\log_2 u}{\log u}\right)\right)\right)$$

since $\log_2 y \leq \sqrt{u}$ and so the positive powers of $\log y$ can be absorbed into the $O-$term of the exponential. This completes the proof. ☐

Collecting together the results of Lemmas 5.1 to 5.5, we obtain from (68)

Lemma 5.6. *For* $u \geq 5$, *so* $\xi(u) > 1$,

$$J(x,y) = -\frac{\rho(u)\xi(u)}{\log y} x \left\{g_K(1) + O\left(\frac{\log u}{\log y} + \frac{\log u}{\sqrt{u}}\right)\right\}.$$

Comparing this result with Corollary 3.7, we see that as $u \to \infty$, the two quantities $\Lambda(x,y) - x\rho(u)$ and $J(x,y)$ have the same order of magnitude provided $g_K(1) \neq 0$. Which error term dominates in Lemma 5.6 depends on the size of $u = \frac{\log x}{\log y}$ compared with $\log y$. The result of Theorem 1.3 now follows from Theorem 1.1 and Lemma 4.3.

6. $\mathbf{J(x,y)}$ in terms of real integrals

The definition of $J(x,y)$ in (57) is given in terms of a complex integral. Our aim in this section is to find an alternative way of expressing $J(x,y)$ as a combination of finite real integrals and an error term. In some situations it may be easier to manipulate this alternative form for $J(x,y)$.

Using Lemma 3.4(iv), we write (57) in the form

$$J(x,y) = \frac{1}{2\pi i}\int_{\alpha_0-iT}^{\alpha_0+iT} g_K(s)s^{-1}x^s \left(1+\int_1^\infty e^{-z(s-1)\log y}\rho'(z)dz\right)ds = I_1 + I_2 \tag{106}$$

where $T = L_{\epsilon/3}(y)$ and

$$I_1 = \frac{1}{2\pi i}\int_{\alpha_0-iT}^{\alpha_0+iT} g_K(s)s^{-1}x^s ds, \tag{107}$$

$$I_2 = \frac{1}{2\pi i}\int_{\alpha_0-iT}^{\alpha_0+iT} g_K(s)s^{-1}x^s \left(\int_1^\infty y^{-z(s-1)}\rho'(z)dz\right)ds. \tag{108}$$

By Lemma 2.3(ii),

$$g_K(s)s^{-1} = \int_{1-}^{\infty}(S(v)-\lambda_K[v])v^{-s-1}dv \qquad (\sigma > 1-\frac{1}{n}) \tag{109}$$

where by Lemma 2.1(ii)

$$S(v) = \lambda_K v + O(v^{1-\frac{1}{n}}). \tag{110}$$

Hence the integral in (109) is absolutely convergent when $\sigma = \alpha_0$. The idea is to use (109) and (110) to replace $g_K(s)s^{-1}$ in (107) and (108). It turns out (see Lemma 6.2) that, assuming $g_K(1) \neq 0$, the main term in Lemma 5.6 comes from I_2, with I_1 contributing to the error term.

Lemma 6.1.

$$I_1 = \frac{1}{\pi}x^{\alpha_0}\int_{1-}^{2x}(S(v)-\lambda_K[v])v^{-\alpha_0-1}(\log\frac{x}{v})^{-1}\sin(L_{\epsilon/3}(y)\log\frac{x}{v})dv + O(x\rho(u)(L_{\epsilon/3}(y))^{-1}).$$

Proof. Substituting (109) into (107), we have on interchanging the order of integration (valid by absolute convergence) that

$$I_1 = \frac{1}{2\pi}x^{\alpha_0}\int_{1-}^{\infty}(S(v)-\lambda_K[v])v^{-\alpha_0-1}\left(\int_{-T}^{T}(x/v)^{it}dt\right)dv.$$

When $v \neq x$, the inner integral equals

$$2(\log\frac{x}{v})^{-1}\sin(T\log\frac{x}{v}) \;\to\; 2T \text{ as } v \to x,$$

and hence is continuous at $v = x$. We deduce that

$$I_1 = \frac{1}{\pi} x^{\alpha_0} \int_{1-}^{\infty} (S(v) - \lambda_K[v]) v^{-\alpha_0 - 1} (\log \frac{x}{v})^{-1} \sin(L_{\epsilon/3}(y) \log \frac{x}{v}) dv$$

by the definition of T. The result of the lemma now follows since by (110)

$$\begin{aligned} x^{\alpha_0} \int_{2x}^{\infty} &(S(v) - \lambda_K[v]) v^{-\alpha_0 - 1} (\log \frac{x}{v})^{-1} \sin(L_{\epsilon/3}(y) \log \frac{x}{v}) dv \\ &\ll x^{\alpha_0} \int_{2x}^{\infty} v^{-\alpha_0 - \frac{1}{n}} dv \ll x^{1 - \frac{1}{n}} \ll x \rho(u) (L_{\epsilon/3}(y))^{-1}. \end{aligned}$$

□

Lemma 6.2. *For* $\xi(u) > 1$

$$I_1 \ll x \rho(u) \xi(u) (\log y)^{-1} \exp\left(-u\left(1 + O\left(\frac{\log_2 u}{\log u}\right)\right)\right).$$

Proof. We split the integral in Lemma 6.1 at the points $V := \exp(\sqrt{\log x})$ and $\sqrt{x}$. We have

$$\begin{aligned} x^{\alpha_0} \int_{1-}^{V} &(S(v) - \lambda_K[v]) v^{-\alpha_0 - 1} (\log \frac{x}{v})^{-1} \sin(L_{\epsilon/3}(y) \log \frac{x}{v}) dv \\ &\ll \frac{x^{\alpha_0}}{\log x} \int_{1}^{V} v^{-\alpha_0 - \frac{1}{n}} dv \ll \frac{x e^{-u\xi}}{\log x} \\ &\ll x \rho(u) \xi(u) (\log y)^{-1} \exp\left(-u\left(1 + O\left(\frac{\log_2 u}{\log u}\right)\right)\right) \end{aligned}$$

by Corollary 3.3. Also

$$\begin{aligned} x^{\alpha_0} \int_{V}^{\sqrt{x}} &(S(v) - \lambda_K[v]) v^{-\alpha_0 - 1} (\log \frac{x}{v})^{-1} \sin(L_{\epsilon/3}(y) \log \frac{x}{v}) dv \\ &\ll x^{\alpha_0} \int_{V}^{\sqrt{x}} v^{-\alpha_0 - \frac{1}{n}} dv \ll x^{\alpha_0} V^{\frac{\xi}{\log y} - \frac{1}{n}} \\ &\ll x \rho(u) \xi(u) (\log y)^{-1} \exp\left(-u\left(1 + O\left(\frac{\log_2 u}{\log u}\right)\right)\right) \end{aligned}$$

since $\exp\left(\left(\frac{\xi}{\log y} - \frac{1}{n}\right)\sqrt{\log x}\right)\log y = o(1)$. For $\sqrt{x} \le v \le 2x$, $\frac{\sin(T\log\frac{x}{v})}{T\log\frac{x}{v}}$ is bounded, and hence

$$\begin{aligned} x^{\alpha_0}\int_{\sqrt{x}}^{2x}(S(v)-\lambda_K[v])v^{-\alpha_0-1}(\log\frac{x}{v})^{-1}\sin(L_{\epsilon/3}(y)\log\frac{x}{v})dv \\ \ll x^{\alpha_0}L_{\epsilon/3}(y)\int_{\sqrt{x}}^{2x}v^{-\alpha_0-\frac{1}{n}}dv \ll x^{\alpha_0}L_{\epsilon/3}(y)x^{\frac{1}{2}(\frac{\xi}{\log y}-\frac{1}{n})} \\ \ll x\rho(u)\xi(u)(\log y)^{-1}\exp\left(-u\left(1+O\left(\frac{\log_2 u}{\log u}\right)\right)\right) \end{aligned}$$

since $L_{\epsilon/3}(y)\log y = o(x^{\frac{1}{2}(\frac{1}{n}-\frac{\xi}{\log y})})$. This completes the proof of the lemma. $\square$

Lemma 6.3. *Let* $X = \max(x, x^{-1}\exp((\log y)^{8/5}))$. *Then*

$$\begin{aligned} I_2 = \frac{x}{\pi\log y}\int_{1-}^{x}\frac{S(v)-\lambda_K[v]}{v^2} \\ \left(\int_{\frac{1}{Xv}}^{\frac{x}{yv}}\rho'\left(u-\frac{\log(vw)}{\log y}\right)\frac{\sin(L_{\epsilon/3}(y)\log w)}{w^{1+\frac{\xi}{\log y}}\log w}dw\right)dv \\ + O\left(x\rho(u)(L_{\epsilon/3}(y))^{-1}\right). \end{aligned}$$

Proof. The inner integral in (108) converges absolutely for $\sigma = \alpha_0$ since $\left|y^{-(s-1)z}\right| = e^{\xi z}$. Substituting (109) into (108) and rearranging the order of the integrals, we obtain

$$I_2 = \frac{x^{\alpha_0}}{2\pi}\int_{1-}^{\infty}\frac{S(v)-\lambda_K[v]}{v^{\alpha_0+1}}\int_{1}^{\infty}e^{\xi z}\rho'(z)\int_{|t|\le T}\left(\frac{x}{vy^z}\right)^{it}dtdzdv.$$

The inner integral is

$$\frac{2\sin(T\log\frac{x}{vy^z})}{\log\frac{x}{vy^z}} \text{ if } vy^z \ne x \text{ and } 2T \text{ if } vy^z = x,$$

and so is a continuous function of vy^z at x. Hence

$$I_2 = \frac{x^{\alpha_0}}{\pi}\int_{1-}^{\infty}\frac{S(v)-\lambda_K[v]}{v^{\alpha_0+1}}\left(\int_{1}^{\infty}e^{\xi z}\rho'(z)\frac{\sin(T\log\frac{x}{vy^z})}{\log\frac{x}{vy^z}}dz\right)dv. \tag{111}$$

Let $U = \max(2u, (\log y)^{3/5})$. We show that we can truncate the integral with respect to z at $z = U$ and the integral with respect to v at $v = x$ at the expense of a quantity covered by the error term of the lemma. For $z \ge U \ge 2u$, we have $y^z \ge x^2$ and so for $v \ge 1$, $\frac{x}{vy^z} \le \frac{1}{x} < 1$, whence

$\left|\log \frac{x}{vy^z}\right| \geq \log x$. Hence

$$\int_U^\infty e^{\xi z}\rho'(z)\frac{\sin(T\log\frac{x}{vy^z})}{\log\frac{x}{vy^z}}dz \ll \frac{1}{\log x}\int_U^\infty e^{\xi z}\left|\rho'(z)\right|dz \ll (L_{\epsilon/3}(y))^{-1}$$

since for $z \geq U > \log L_{\epsilon/3}(y)$

$$e^{\xi z}\rho'(z) \ll e^{\xi z}\rho(z)\log z \ll \exp(-z\log U)$$

by Lemma 3.1(i) and Lemma 3.2(ii). It follows that the contribution to (111) from the range $z \geq U$ is

$$\ll x^{\alpha_0}(L_{\epsilon/3}(y))^{-1}\int_1^\infty v^{-\alpha_0-\frac{1}{n}}dv \ll x\rho(u)(L_{\epsilon/3}(y))^{-1}. \tag{112}$$

For $v \geq x$, $z \geq 1$, we have $\frac{x}{vy^z} \leq \frac{1}{y}$ so $\left|\log\frac{x}{vy^z}\right| \geq \log y$. Hence

$$\begin{aligned} x^{\alpha_0}\int_x^\infty \frac{S(v)-\lambda_K[v]}{v^{\alpha_0+1}}&\left(\int_1^\infty e^{\xi z}\rho'(z)\frac{\sin(T\log\frac{x}{vy^z})}{\log\frac{x}{vy^z}}dz\right)dv \\ &\ll \frac{x^{\alpha_0}}{\log y}\int_x^\infty v^{-\alpha_0-\frac{1}{n}}dv\int_1^\infty e^{\xi z}\left|\rho'(z)\right|dz \\ &\ll \frac{x^{1-\frac{\xi}{\log y}}}{\log y}x^{\frac{\xi}{\log y}-\frac{1}{n}}\xi e^{I(\xi)} \ll x\rho(u)(L_{\epsilon/3}(y))^{-1} \qquad (113)\end{aligned}$$

since the integral over z is $\ll \xi e^{I(\xi)}+1$ and $\xi\sqrt{u}e^{u\xi}x^{-\frac{1}{n}} \ll (L_{\epsilon/3}(y))^{-1}$ for $y \in H_\epsilon$. Here we have used that

$$0 > \int_1^\infty e^{\xi z}\rho'(z)dz = \Big[e^{\xi z}\rho(z)\Big]_1^\infty - \xi\int_1^\infty e^{\xi z}\rho(z)dz = -1-\xi\hat{\rho}(-\xi),$$

by Lemma 3.4(iii), and Lemma 3.2(iii).

By (112) and (113), we can now write (111) in the form

$$\begin{aligned} I_2 = \frac{x^{\alpha_0}}{\pi}\int_{1-}^x \frac{S(v)-\lambda_K[v]}{v^{\alpha_0+1}}&\left(\int_1^U e^{\xi z}\rho'(z)\frac{\sin(L_{\epsilon/3}(y)\log\frac{x}{vy^z})}{\log\frac{x}{vy^z}}dz\right)dv \\ &+O\left(\frac{x\rho(u)}{L_{\epsilon/3}(y)}\right). \qquad (114)\end{aligned}$$

To obtain the result in the form given in the lemma, we change the variable in the inner integral of (114) by putting $w = \frac{x}{vy^z}$, so $z = \left(\log\frac{x}{vw}\right)/\log y = u - \frac{\log(vw)}{\log y}$, and $\frac{\partial z}{\partial w} = -\frac{1}{w\log y}$. Also $w = \frac{x}{vy}$ when $z=1$, and $w = \frac{x}{vy^U} = \frac{1}{xv}$ or $\frac{x}{v}\exp(-(\log y)^{8/5})$ according as $z = U = 2u$ or $z = U = (\log y)^{3/5}$;

hence when $z = U$, $w = \frac{1}{Xv}$ by definition of X. It follows that $e^{\xi z} = e^{\xi u} v^{-\frac{\xi}{\log y}} w^{-\frac{\xi}{\log y}}$ and so

$$I_2 = \frac{x}{\pi \log y} \int_{1-}^{x} \frac{S(v) - \lambda_K[v]}{v^2}$$
$$\left(\int_{\frac{1}{Xv}}^{\frac{x}{yv}} \rho' \left(u - \frac{\log(vw)}{\log y} \right) \frac{\sin(L_{\epsilon/3}(y) \log w)}{w^{1+\frac{\xi}{\log y}} \log w} dw \right) dv + O\left(\frac{x\rho(u)}{L_{\epsilon/3}(y)} \right)$$

as required. □

From Lemmas 6.1 and 6.3 we deduce

Theorem 6.4.

$$J(x,y) = \frac{x}{\pi \log y} \int_{1-}^{x} \frac{S(v) - \lambda_K[v]}{v^2}$$
$$\left(\int_{\frac{1}{Xv}}^{\frac{x}{yv}} \rho' \left(u - \frac{\log(vw)}{\log y} \right) \frac{\sin(L_{\epsilon/3}(y) \log w)}{w^{1+\frac{\xi}{\log y}} \log w} dw \right) dv$$
$$+ \frac{1}{\pi} x^{\alpha_0} \int_{1-}^{2x} (S(v) - \lambda_K[v]) v^{-\alpha_0 - 1} (\log \frac{x}{v})^{-1} \sin(L_{\epsilon/3}(y) \log \frac{x}{v}) dv$$
$$+ O\left(\frac{x\rho(u)}{L_{\epsilon/3}(y)} \right).$$

7. An application

Our aim in this section is to use our main results above and the methods of [27] to study the sum defined in (15):

$$S_K(x) = \sum_{\substack{\mathfrak{a} \\ N(\mathfrak{a}) \le x}} \frac{1}{P(\mathfrak{a})} \tag{115}$$

where $P(\mathfrak{a}) = \max\{N(\mathfrak{p}) : \mathfrak{p} \mid \mathfrak{a}\}$, $P(\mathfrak{O}_K) = 1$. When $K = \mathbf{Q}$ the sum in (115) becomes

$$S_{\mathbf{Q}}(x) = \sum_{n \le x} \frac{1}{P(n)} \text{ where } P(n) = \max\{p : p \mid n\}, \quad P(1) = 1$$

which, as stated in section 1, has been the subject of several papers (for example [5], [14], [15], [16], [18]). It follows from [27] that for a sufficiently small $\epsilon > 0$

$$S_{\mathbf{Q}}(x) = x \left(1 + O\left(\exp\left(-\left(\frac{1}{2} \log x \log_2 x \right)^{\frac{3}{10} - \epsilon} \right) \right) \right) H(x)$$

where

$$H(x)=\int_2^x \frac{1}{w^2\log w}\left\{\rho\left(\frac{\log x}{\log w}-1\right)-\int_1^x \frac{v-[v]}{v^2\log w}\rho'\left(\frac{\log \frac{x}{v}}{\log w}-1\right)dv\right\}dw.$$

It was shown in [14] that

$$S_{\mathbf{Q}}(x)=x\exp\left(-(2\log x\log_2 x)^{1/2}(1+o(1))\right).$$

We can obtain the corresponding results for a general number field K.

In (16) we defined

$$\mathcal{L}=\mathcal{L}(x)=\exp\left(\left(\frac{1}{2}\log x\log_2 x\right)^{\frac{1}{2}}\right). \tag{116}$$

Lemma 7.1.

(i) $$S_K(x)=\sum_{\mathcal{L}^{1/3}\le N(\mathfrak{p})\le\mathcal{L}^3}\frac{1}{N(\mathfrak{p})}\Psi_K\left(\frac{x}{N(\mathfrak{p})},N(\mathfrak{p})\right)+O(x\exp(-(3+o(1))\log\mathcal{L}(x))).$$

(ii) $S_K(x)=\lambda_K x\exp(-(2+o(1))\log\mathcal{L}(x))$.

Proof. Let $r(\mathfrak{a})$ denote the number of distinct prime ideals $\mathfrak{p}$ with $\mathfrak{p}\,|\,\mathfrak{a}$ and $N(\mathfrak{p})=P(\mathfrak{a})$, so $1\le r(\mathfrak{a})\le n$ where $n=[K:\mathbf{Q}]$. For each of these prime ideals $\mathfrak{p}$, $\mathfrak{a}=\mathfrak{p}\mathfrak{b}$ where $P(\mathfrak{b})\le P(\mathfrak{a})$, so

$$\sum_{\substack{\mathfrak{a}\\ N(\mathfrak{a})\le x}}\frac{r(\mathfrak{a})}{P(\mathfrak{a})}=\sum_{N(\mathfrak{p})\le x}\frac{1}{N(\mathfrak{p})}\sum_{\substack{\mathfrak{b}\\ N(\mathfrak{b})\le x/N(\mathfrak{p})\\ P(\mathfrak{b})\le N(\mathfrak{p})}}1=\sum_{N(\mathfrak{p})\le x}\frac{1}{N(\mathfrak{p})}\Psi_K\left(\frac{x}{N(\mathfrak{p})},N(\mathfrak{p})\right).$$

Hence

$$S_K(x)=\sum_{N(\mathfrak{p})\le x}\frac{1}{N(\mathfrak{p})}\Psi_K\left(\frac{x}{N(\mathfrak{p})},N(\mathfrak{p})\right)-\sum_{\substack{\mathfrak{a}\\ N(\mathfrak{a})\le x}}\frac{r(\mathfrak{a})-1}{P(\mathfrak{a})}. \tag{117}$$

When $\mathfrak{a}$ contributes to the last sum of (117), $\mathfrak{a}$ has two or more different prime ideal divisors with norm $P(\mathfrak{a})$, so $\mathfrak{a}=\mathfrak{p}_1\mathfrak{p}_2\mathfrak{b}$ with $N(\mathfrak{p}_1)=N(\mathfrak{p}_2)=P(\mathfrak{a}),P(\mathfrak{b})\le P(\mathfrak{a})$. It follows by a similar argument to above that

$$\sum_{\substack{\mathfrak{a}\\ N(\mathfrak{a})\le x}}\frac{r(\mathfrak{a})-1}{P(\mathfrak{a})}\ll\sum_{N(\mathfrak{p})\le x}\frac{1}{N(\mathfrak{p})}\Psi_K\left(\frac{x}{(N(\mathfrak{p}))^2},N(\mathfrak{p})\right).$$

Adapting the method used to prove Lemma 3.3(i) of [27], we find that the contribution to the first sum in (117) made by those $\mathfrak{p}$ with $N(\mathfrak{p})<\mathcal{L}^{1/3}$ or $\mathcal{L}^3<N(\mathfrak{p})\le x$ is

$$\ll x\exp\left(-(3+o(1))\log\mathcal{L}\right).$$

Since $\Psi_K\left(\frac{x}{(N(\mathfrak{p}))^2}, N(\mathfrak{p})\right) \leq \Psi_K\left(\frac{x}{N(\mathfrak{p})}, N(\mathfrak{p})\right)$, part (i) will be established if we show that

$$\sum_{\mathcal{L}^{1/3}\leq N(\mathfrak{p})\leq\mathcal{L}^3} \frac{1}{N(\mathfrak{p})}\Psi_K\left(\frac{x}{(N(\mathfrak{p}))^2}, N(\mathfrak{p})\right) \ll x\exp\left(-\left(3+o(1)\right)\log\mathcal{L}\right).$$

Then part (ii) follows on using the argument in the proof of Lemma 3.4 of [27].

From [19] or Theorems 1.1 and 1.3 and equation (4) above we see that for $y \in H_\epsilon$ and $\xi(u) > 1$

$$\Psi_K(x,y) = \lambda_K\, x\rho(u)\left(1 + O\left(\frac{\log u}{\log y}\right)\right). \tag{118}$$

When $u = \frac{\log x}{\log N(\mathfrak{p})} - 2, \mathcal{L}^{1/3} \leq N(\mathfrak{p}) \leq \mathcal{L}^3$, we have

$$\rho(u) = \exp\left(-\frac{1}{2}\frac{\log x\log_2 x}{\log N(\mathfrak{p})}(1+o(1))\right) = \exp\left(-\frac{(\log\mathcal{L})^2}{\log N(\mathfrak{p})}(1+o(1))\right)$$

on using equation (120), (116) and Lemma 3.1(i). Hence by (118)

$$\Psi_K\left(\frac{x}{(N(\mathfrak{p}))^2}, N(\mathfrak{p})\right) \ll \frac{x}{(N(\mathfrak{p}))^2}\exp\left(-\frac{(\log\mathcal{L})^2}{\log N(\mathfrak{p})}(1+o(1))\right).$$

Since $N(\mathfrak{p}) = p^m$ for some rational prime p and $m \leq n$,

$$\begin{aligned}\sum_{\mathcal{L}^{1/3}\leq N(\mathfrak{p})\leq\mathcal{L}^3} &\frac{1}{N(\mathfrak{p})}\Psi_K\left(\frac{x}{(N(\mathfrak{p}))^2}, N(\mathfrak{p})\right)\\ &\ll x\sum_{\mathcal{L}^{1/3}\leq p\leq\mathcal{L}^3}\frac{1}{p^3}\exp\left(-\frac{(\log\mathcal{L})^2}{\log p}(1+o(1))\right)\\ &\ll x\int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3}\frac{1}{w^3\log w}\exp\left(-\frac{(\log\mathcal{L})^2}{\log w}(1+o(1))\right)dw\end{aligned}$$

on using Lemma 2.9 of [27]. The required bound for the second sum in (117) now follows on integrating by parts. □

From (61) and (62) we have for $y \in H_\epsilon$

$$\Psi_K(x,y) = \lambda_K\Lambda(x,y) + J(x,y) + O(x\rho(u)/L_\epsilon(y)). \tag{119}$$

We substitute this in Lemma 7.1(i) and investigate the sums involved. $\Lambda(x,y)$ is given by Lemma 3.6 and $J(x,y)$ by Lemma 5.6. We need to estimate $J(\frac{x}{v}, v)$ for $\mathcal{L}^{1/3} \leq v \leq \mathcal{L}^3$.

Lemma 7.2. *(i) For* $\mathcal{L}^{1/3} \le v \le \mathcal{L}^3$,

$$J(\frac{x}{v}, v) = -\frac{1}{2}x(\log_2 x + \log_3 x + O(1))\Big(g_K(1) + O\left((\log_2 x)^{5/4}(\log x)^{-1/4}\right)\Big)\frac{\rho\left(\frac{\log x}{\log v} - 1\right)}{v \log v}.$$

(ii) Provided $g_K(1) \neq 0, -(2/g_K(1))v \log v J(\frac{x}{v}, v)$ *is positive and increases in magnitude as* v *increases from* $\mathcal{L}^{1/3}$ *to* $\mathcal{L}^3$.

Proof. (i)We apply Lemma 5.6 with $u = \frac{\log \frac{x}{v}}{\log v} = \frac{\log x}{\log v} - 1$. Then

$$\log u = \log_2 x - \log_2 v + O\left(\frac{\log v}{\log x}\right), \qquad \log_2 u = \log_3 x + O(1)$$

so for $\mathcal{L}^{1/3} \le v \le \mathcal{L}^3$,

$$\xi(u) = \log u + \log_2 u + O\left(\frac{\log_2 u}{\log u}\right) = \log_2 x - \log_2 \mathcal{L} + \log_3 x + O(1)$$

$$= \frac{1}{2}(\log_2 x + \log_3 x) + O(1) = \log_2 v + O(1). \tag{120}$$

Also

$$\frac{\log u}{\log v} + \frac{\xi(u)}{\sqrt{u}} \ll \frac{\log_2 x}{\log \mathcal{L}} + \sqrt{\frac{\log \mathcal{L}}{\log x}} \log_2 x \ll \left(\frac{\log_2 x}{\log x}\right)^{1/4} \log_2 x.$$

Hence by Lemma 5.6 when $\mathcal{L}^{1/3} \le v \le \mathcal{L}^3$ we have

$$J(\frac{x}{v}, v) = -\frac{x}{v \log v}\left(\frac{1}{2}(\log_2 x + \log_3 x) + O(1)\right)\Big(g_K(1) + O\left((\log_2 x)^{5/4}(\log x)^{-1/4}\right)\Big)\rho\left(\frac{\log x}{\log v} - 1\right)$$

which is the result stated.

(ii) As v increases from $\mathcal{L}^{1/3}$ to $\mathcal{L}^3$, $\rho\left(\frac{\log x}{\log v} - 1\right)$ increases and by (120) and Lemma 3.1(i) equals $\exp\left(-\frac{\log x}{\log v}(\log_2 v + O(1))\right)$. Hence if $g_K(1) \neq 0$, the result follows from (i) since $g_K(1)$ is real. $\square$

Lemma 7.3. *(i) Assume* $g_K(1) \neq 0$. *Then*

$$\sum_{\mathcal{L}^{1/3} \le N(\mathfrak{p}) \le \mathcal{L}^3} \frac{1}{N(\mathfrak{p})} J\left(\frac{x}{N(\mathfrak{p})}, N(\mathfrak{p})\right) = \left(1 + O\left(\frac{1}{L_\epsilon(\mathcal{L})}\right)\right)\int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3} \frac{J(\frac{x}{v}, v)}{v \log v} dv.$$

(ii) When $g_K(1) = 0$,

$$\sum_{\mathcal{L}^{1/3} \le N(\mathfrak{p}) \le \mathcal{L}^3} \frac{1}{N(\mathfrak{p})} J\left(\frac{x}{N(\mathfrak{p})}, N(\mathfrak{p})\right)$$
$$\ll x(\log_2 x)^{9/4} (\log x)^{-1/4} \int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3} \frac{\rho(\frac{\log x}{\log v} - 1)}{(v \log v)^2} dv.$$

Proof. (i) From the Prime Ideal Theorem in the form of Lemma 2.4(ii), we deduce that

$$\theta_K(x) := \sum_{N(\mathfrak{p}) \le x} \log N(\mathfrak{p}) = x(1 + E(x))$$

where

$$E(x) \ll (L_{\epsilon/2}(x))^{-1}.$$

By Lemma 7.2(ii), it follows as in Lemmas 2.8 and 2.9 of [27] with $g(v) = v \log v \left|J(\frac{x}{v}, v)\right|$ and $h(v) = v \log v$ that

$$\sum_{\mathcal{L}^{1/3} \le N(\mathfrak{p}) \le \mathcal{L}^3} \frac{1}{N(\mathfrak{p})} J\left(\frac{x}{N(\mathfrak{p})}, N(\mathfrak{p})\right) = \int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3} \frac{J(\frac{x}{v}, v)}{v \log v} (1 + O((L_{3\epsilon/4}(v))^{-1})) dv$$
$$+ O\left(\max_{v = \mathcal{L}^{1/3}, \mathcal{L}^3} \frac{\left|J(\frac{x}{v}, v)\right|}{L_{\epsilon/2}(v)}\right). \tag{121}$$

For $\mathcal{L}^{1/3} \le v \le \mathcal{L}^3$, $(L_{3\epsilon/4}(v))^{-1} \ll (L_\epsilon(\mathcal{L}))^{-1}$. Also for $v = \mathcal{L}^{1/3}, \mathcal{L}^3$,

$$\frac{1}{v \log v} \rho(\frac{\log x}{\log v} - 1) \ll \exp(-(3 + o(1)) \log \mathcal{L})$$

and so

$$\frac{J(\frac{x}{v}, v)}{L_{\epsilon/2}(v)} \ll x \exp(-(3 + o(1)) \log \mathcal{L}). \tag{122}$$

This error term is smaller than

$$\frac{1}{L_\epsilon(\mathcal{L})} \int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3} \frac{J(\frac{x}{v}, v)}{v \log v} dv; \tag{123}$$

for (123) is

$$\asymp x \log_2 x \frac{1}{L_\epsilon(\mathcal{L})} \int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3} \frac{\rho(\frac{\log x}{\log v} - 1)}{(v \log v)^2} dv$$

and from [5] (or by proofs analogous to those of Lemma 3.4 and 3.5 of [27])

$$x \int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3} \frac{\rho(\frac{\log x}{\log v} - 1)}{v^2 \log v} dv \sim \sum_{\mathcal{L}^{1/3} \le p \le \mathcal{L}^3} p^{-1} \Psi(\frac{x}{p}, p) = x \exp(-(2 + o(1)) \log \mathcal{L}).$$

The result now follows from (121) and (122).

(ii) This follows from Lemma 7.2(i) since

$$\sum_{\mathcal{L}^{1/3}\le N(\mathfrak{p})\le\mathcal{L}^3} \frac{1}{(N(\mathfrak{p}))^2\log N(\mathfrak{p})}\rho\left(\frac{\log x}{\log N(\mathfrak{p})}-1\right) \sim \int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3} \frac{\rho(\frac{\log x}{\log v}-1)}{(v\log v)^2}dv.$$

□

Although we do not need to do so to prove Theorem 1.4, we can use Theorem 6.4 in section 6 to express the integral on the right of Lemma 7.3(i) in terms of real integrals. Let $\eta(v)=\xi(\frac{\log x}{\log v}-1)/\log v$. From Theorem 6.4 we deduce

Lemma 7.4.

$$\int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3} \frac{J(\frac{x}{v},v)}{v\log v}dv = \frac{x}{\pi}\int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3} \frac{x^{-\eta(v)}}{v^{2-\eta(v)}\log v}$$
$$\left\{\int_{1-}^{2x/v} \frac{S(z)-\lambda_K[z]}{z^{2-\eta(v)}}(\log\frac{x}{vz})^{-1}\sin(L_{\epsilon/3}(v)\log\frac{x}{vz})dz\right\}dv$$
$$+\frac{x}{\pi}\int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3}\frac{1}{(v\log v)^2}$$
$$\left\{\int_{1-}^{x/v}\frac{S(z)-\lambda_K[z]}{z^2}\left\{\int_{\frac{1}{X(v)z}}^{x/v^2z}\rho'\left(\frac{\log\frac{x}{vzw}}{\log v}\right)\frac{\sin(L_{\epsilon/3}(v)\log w}{w^{1+\eta(v)}\log w}dw\right\}dz\right\}dv$$
$$+O\left(x\int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3}\rho\left(\frac{\log x}{\log v}-1\right)v^{-2}(L_{\epsilon/3}(v)\log v)^{-1}dv\right)$$

where $X(v)=\max(\frac{x}{v},\frac{v}{x}\exp((\log v)^{8/5}))$.

Note that the $O-$term is

$$\ll (L_\epsilon(\mathcal{L}))^{-1}\int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3}(v\log v)^{-1}J(\frac{x}{v},v)dv.$$

Lemma 7.5.

$$\lambda_K\sum_{\mathcal{L}^{1/3}\le N(p)\le\mathcal{L}^3}(N(\mathfrak{p}))^{-1}\left\{\Lambda\left(\frac{x}{N(\mathfrak{p})},N(\mathfrak{p})\right)\right.$$
$$\left.+O\left(x(N(\mathfrak{p})L_\epsilon(N(\mathfrak{p}))^{-1}\rho\left(\frac{\log x}{\log N(\mathfrak{p})}-1\right)\right)\right\}$$
$$=x(\lambda_K+O((L_\epsilon(\mathcal{L}))^{-1}))\int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3}\frac{1}{v^2\log v}\left\{\rho\left(\frac{\log x}{\log v}-1\right)\right.$$
$$\left.-\int_1^x\frac{w-[w]}{w^2\log v}\rho'\left(\frac{\log\frac{x}{w}}{\log v}-1\right)dw\right\}dv.$$

Proof. This follows on combining the method used to prove Lemma 7.3(i) with that used to establish Theorem 3 of [27] (and in particular with a result analogous to Lemma 4.1(i) of [27]) in the case $\nu = 1$, $\eta(w) = 1$. □

Proof. (Theorem 1.4.) (i) From Lemmas 7.1(i), 7.3(i), 7.5 and equation (119), we obtain when $g_K(1) \neq 0$

$$\begin{aligned} S_K(x) = x(\lambda_K + O((L_\epsilon(\mathcal{L}))^{-1})) \int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3} \frac{1}{v^2 \log v} \Bigg\{ \rho\left(\frac{\log x}{\log v} - 1\right) \\ - \int_1^x \frac{w-[w]}{w^2 \log v} \rho'\left(\frac{\log \frac{x}{w}}{\log v} - 1\right) dw \Bigg\} dv \\ + (1 + O((L_\epsilon(\mathcal{L}))^{-1}) \int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3} \frac{1}{v \log v} J(\frac{x}{v}, v) dv. \end{aligned} \tag{124}$$

We can extend the range of integration for v to $2 \leq v \leq x$ at the expense of an error term of the form (122) which we know can be absorbed in the $O-$term above.

(ii) From (119), (120), (124) and Lemmas 7.2, 7.3, 7.5, 3.5, we deduce (irrespective of the value of $g_K(1)$) that

$$\begin{aligned} S_K(x) &= x(\lambda_K + O((L_\epsilon(\mathcal{L}))^{-1})) \int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3} \frac{1}{v^2 \log v} \rho\left(\frac{\log x}{\log v} - 1\right) \\ &\qquad \Big\{ 1 + \frac{C}{2 \log v} (\log_2 x + \log_3 x + O(1)) \Big\} dv \\ &\quad - \frac{1}{2} x (\log_2 x + \log_3 x + O(1)) \Big(g_K(1) + O\left((\log_2 x)^{5/4} (\log x)^{-1/4}\right)\Big) \\ &\qquad \int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3} \frac{1}{v^2 (\log v)^2} \rho\left(\frac{\log x}{\log v} - 1\right) dv \\ &= x \int_{\mathcal{L}^{1/3}}^{\mathcal{L}^3} \frac{1}{v^2 \log v} \rho\left(\frac{\log x}{\log v} - 1\right) \\ &\qquad \Big\{ \lambda_K + \frac{1}{2 \log v} (C\lambda_K - g_K(1)) (\log_2 x + \log_3 x + O(1)) \Big\} dv \end{aligned}$$

where $C = 1 - \gamma$ and $g_K(1) = \lim_{s \to 1} (\zeta_K(s) - \lambda_K \zeta(s))$. As before the integral over v can be extended to the range $2 \leq v \leq x$ since the error involved is bounded by the right side of (122) and so is negligible. □

References

[1] N.G. de Bruijn, *On the number of positive integers $\leq x$ and free of prime factors $> y$.* Indag. Math. **13** (1951), 50–60.

[2] N.G. de Bruijn, *The asymptotic behaviour of a function occurring in the theory of primes.*J. Indian Math. Soc. (NS) **15** (1951), 25–32.

[3] J.A. Buchmann, C.S Hollinger, *On smooth ideals in number fields.* J. Number Theory **59** (1996), 82–87.

[4] K. Dilcher, *Generalized Euler constants for arithmetical progressions.* Math. Comp. **59** (1992), 259–282.

[5] P. Erdös, A. Ivić, C. Pomerance,*On sums involving reciprocals of the largest prime factor of an integer.* Glas. Mat. Ser. III **21 (41)** (1986), 283–300.

[6] J.H. Evertse, P. Moree, C.L. Stewart, R. Tijdeman, *Multivariate Diophantine equations with many solutions.* Acta Arith. **107** (2003), 103–125.

[7] J.B. Friedlander, *On the large number of ideals free from large prime divisors.* J. Reine Angew. Math. **255** (1972), 1–7.

[8] J.R. Gillett, *On the largest prime divisors of ideals in fields of degree n.* Duke Math. J. **37** (1970), 589–600.

[9] L.J. Goldstein, *Analytic Number Theory*, Prentice-Hall, Inc., New Jersey, 1971.

[10] D.G. Hazlewood,*On ideals having only small prime factors.* Rocky Mountain J. Math. **7** (1977), 753–768.

[11] A. Hildebrand, *On the number of positive integers $\leq x$ and free of prime factors $> y$.* J. Number Theory **22** (1986), 289–307.

[12] A. Hildebrand, G. Tenenbaum, *On integers free of large prime factors.* Trans. Amer. Math. Soc. **296** (1986), 265–290.

[13] M.N. Huxley, N. Watt, *The number of ideals in a quadratic field II.* Israel J. Math. **120** (2000), part A, 125–153.

[14] A. Ivić, *Sum of reciprocals of the largest prime factor of an integer.* Arch. Math. **36** (1981), 57–61.

[15] A. Ivić, *On some estimates involving the number of prime divisors of an integer.* Acta Arith. **49** (1987), 21–33.

[16] A. Ivić, *On sums involving reciprocals of the largest prime factor of an integer II.* Ibid **71** (1995), 229–251.

[17] A. Ivić, *The Riemann zeta-function,* Wiley, New York - Chichester - Brisbane - Toronto - Singapore, 1985.

[18] A. Ivić, C. Pomerance, *Estimates for certain sums involving the largest prime factor of an integer.* In: Topics in Classical Number Theory (Budapest,1981), Colloq. Math. Soc. János Bolyai **34**, North-Holland, Amsterdam, 1984, 769–789.

[19] U. Krause, *Abschätzungen für die Funktion $\Psi_K(x,y)$ in algebraischen Zahlkörpern.* Manuscripta Math. **69** (1990), 319–331.

[20] E. Landau, *Einführung in die elementare und analytische Theorie der algebraische Zahlen und Ideale.* Teubner, Leipzig, 1927, reprint Chelsea, New York, 1949.

[21] S. Lang, *Algebraic Number Theory,* Addison-Wesley, Reading, Mass. - Menlo Park, Calif. - London - Don Mills, Ont., 1970.

[22] P. Moree, *An interval result for the number field $\Psi(x,y)$ function.* Manuscripta Math. **76** (1992), 437–450.

[23] P. Moree, *On some claims in Ramanujan's 'unpublished' manuscript on the partition and tau functions,* arXiv:math.NT/0201265.

[24] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers.* PNW, Warsaw, 1974.

[25] W.G. Nowak, *On the distribution of integer ideals in algebraic number fields.* Math. Nachr. **161** (1993), 59–74.

[26] E. Saias, *Sur le nombre des entiers sans grand facteur premier.* J. Number Theory **32** (1989), 78–99.

[27] E.J. Scourfield, *On some sums involving the largest prime divisor of n, II.* Acta Arith. **98** (2001), 313–343.

[28] H. Smida, *Valeur moyenne des fonctions de Piltz sur entiers sans grand facteur premier.* Ibid **63** (1993), 21–50.

[29] A.V. Sokolovskii, *A theorem on the zeros of Dedekind's zeta-function and the distance between 'neighbouring' prime ideals,* (Russian). Ibid **13** (1968), 321–334.

[30] W. Staś, *On the order of Dedekind zeta-function in the critical strip.* Funct. Approximatio Comment. Math. **4** (1976), 19–26.

[31] G. Tenenbaum, *Introduction to analytic and probabilistic number theory,* CUP, Cambridge, 1995.

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