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On a mixed Littlewood conjecture for quadratic numbers

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Résumé. Nous étudions un problème diophantien simultané relié à la conjecture de Littlewood. En utilisant des minorations connues de formes linéaires de logarithmes p-adiques, nous montrons qu'un résultat que nous avons précédemment obtenu, concernant les nombres quadratiques, est presque optimal.

Abstract. We study a simultaneous diophantine problem related to Littlewood's conjecture. Using known estimates for linear forms in p-adic logarithms, we prove that a previous result, concerning the particular case of quadratic numbers, is close to be the best possible.

1. Introduction

In a joint paper, with O. Teulié [5], we have considered the following problem. Let $\mathcal{B} = (b_k)_{k\geq 1}$ be a sequence of integers greater than 1. Consider the sequence $(r_n)_{n\geq 0}$, where $r_0 = 1$ and $r_n = \prod_{0<k\leq n} b_k$ for $n > 0$. For $q \in \mathbb{Z}$, set

$$w_{\mathcal{B}}(q) = \sup\{n \in \mathbb{N}\ ;\ q \in r_n\mathbb{Z}\}$$

and

$$|q|_{\mathcal{B}} = \inf\{1/r_n\ ;\ q \in r_n\mathbb{Z}\}.$$

Notice that $|.|_{\mathcal{B}}$ is not necessarily an absolute value, but when $\mathcal{B}$ is the constant sequence p, where p is a prime number, then $|.|_{\mathcal{B}}$ is the usual p-adic value.

For $x \in \mathbb{R}$, we denote by $\{x\}$ the number in $[-1/2, 1/2[$ such that $x - \{x\} \in \mathbb{Z}$. As usual, we put $\|x\| = |\{x\}|$.

Let α be a real number. Given a positive integer M, Dirichlet's Theorem asserts that for any n, there exists an integer q, with $0 < q \leq Mr_n$, satisfying simultaneously the approximation condition $\|q\alpha\| < 1/M$ and the divisibility condition $r_n|q$, i. e. $|q|_{\mathcal{B}} \leq 1/r_n$. Indeed, it is enough to apply Dirichlet's Theorem to the number $r_n\alpha$. We thus find positive integers q with

$$q\|q\alpha\||q|_{\mathcal{B}} < 1.$$

By analogy with Littlewood's conjecture, we ask whether

$$\inf_{q\in\mathbb{N}^*} q\|q\alpha\||q|_{\mathcal{B}} = 0 \tag{1}$$

holds. The problem is trivial for α rational, and for an irrational number α, one can easily see [5] that condition (1) is equivalent to the following: for each $n \in \mathbb{N}$, consider the continued fraction expansion

$$r_n\alpha = [a_{0,n}; a_{1,n}, ..., a_{k,n} ...].$$

We have (1) if and only if

$$\sup_{n\geq 0, k\geq 1} a_{k,n} = +\infty.$$

However, we shall not use this characterization here.

We do not know whether (1) is satisfied for any real number α. In [5], we have proved that if we assume that the sequence $\mathcal{B} = (b_k)_{k\geq 1}$ is bounded, (1) is true for every quadratic number α. More precisely:

Theorem 1.1. (de Mathan and Teulié [5]) *Suppose that the sequence $\mathcal{B}$ is bounded. Let α be a quadratic real number. Then there exists an infinite set of integers $q > 1$ with*

$$\|q\alpha\| \ll 1/q \tag{2}$$

and

$$|q|_{\mathcal{B}} \ll 1/\ln q. \tag{3}$$

In particular, we have

$$\liminf_{q\longrightarrow+\infty} q\ln q\|q\alpha\||q|_{\mathcal{B}} < +\infty.$$

As usual, for positive functions x and y, the notation $x \ll y$ means that there exists a positive constant C such that $x \leq Cy$.

In our lecture at Graz, for the "Journées Arithmétiques 2003", it was discussed whether the factor $\ln q$ in (3) is best possible. We do not know the answer to this question, but we shall prove:

Theorem 1.2. *Assume that the sequence $\mathcal{B}$ is bounded. Let α be a real quadratic number, and let $\mathcal{S}$ be a set of integers $q > 1$ with*

$$\|q\alpha\| \ll 1/q. \tag{2}$$

Then there exists a constant $\lambda = \lambda(\mathcal{S})$ such that

$$|q|_{\mathcal{B}} \gg \frac{1}{(\ln q)^{\lambda}} \tag{4}$$

for any $q \in \mathcal{S}$.

One may expect that (4) holds for any $\lambda > 1$, but we are not able to prove this. We do not even know whether there exists a real number λ for which (4) holds for any set $\mathcal{S}$ of integers $q > 1$ satisfying (2). Indeed, Theorem 1.2 does not ensure that $\sup_{\mathcal{S}} \lambda(\mathcal{S}) < +\infty$.

There is some analogy between this problem, and the classical simultaneous Diophantine approximation. For instance, let us recall Peck's Theorem. Let n be an integer greater than 1, and let α_1,..., α_n, be n numbers in a real algebraic number field of degree $n+1$ over $\mathbb{Q}$. Then it was proved by Peck [7] that there exists an infinite set of integers $q > 1$ with

$$\|q\alpha_k\| \ll (\ln q)^{-1/(n-1)} q^{-1/n}$$

for $1 \leq k < n$, and

$$\|q\alpha_n\| \ll q^{-1/n}.$$

Assume that $1, \alpha_1, ..., \alpha_n$ are linearly independent over $\mathbb{Q}$, and let $\mathcal{S}$ be an infinite set of integers $q > 1$, with

$$\|q\alpha_k\| \ll q^{-1/n}$$

for each $1 \leq k \leq n$. Then we have proved in [3] that there exists a constant $\kappa = \kappa(\mathcal{S})$ such that

$$\max_{1 \leq k < n} \|q\alpha_k\| \gg (\ln q)^{-\kappa} q^{-1/n}.$$

Theorem 1.2 can be regarded as an analogue of this result with $n = 1$, and its proof is similar.

2. Proof of the result

2.1. Some rational approximations of α.

In the quadratic field $\mathbb{Q}(\alpha)$, there exists a unit ω of infinite order. Replacing, if necessary, ω by ω^2 or $1/\omega^2$, we may suppose $\omega > 1$. In his original work, Peck uses units which are "large" and whose other conjugates are "small" and close to be equal. Here, Peck's units are just the ω^m's, with $m \in \mathbb{N}$. We shall use these units in order to describe the rational approximations of α which satisfy (2).

Denote by $\sigma_0 = \mathrm{id}$ and $\sigma_1 = \sigma$ the automorphisms of $\mathbb{Q}(\alpha)$. As usual, we denote by Tr the trace form $\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}} = \sigma_0 + \sigma_1$. The basis $(1, \alpha)$ of $\mathbb{Q}(\alpha)$ admits a dual basis (β_0, β_1) for the non-degenerate $\mathbb{Q}$-bilinear form $(x, y) \longmapsto \mathrm{Tr}(xy)$ on $\mathbb{Q}(\alpha)$. That means that, if we set $\alpha_0 = 1$ and $\alpha_1 = \alpha$, we have $\mathrm{Tr}(\alpha_k \beta_l) = \delta_{kl}$, for $k = 0, 1$ and $l = 0, 1$, where $\delta_{ll} = 1$, and $\delta_{kl} = 0$ if $k \neq l$. Here it is easy to calculate $\beta_0 = -\frac{\sigma(\alpha)}{\alpha - \sigma(\alpha)}$ and $\beta_1 = \frac{1}{\alpha - \sigma(\alpha)}$. Hence, if we put

$$\eta = \frac{-q\sigma(\alpha) + q'}{\alpha - \sigma(\alpha)},$$

where q and q' are rational numbers, we have

$$q = \mathrm{Tr}\eta \tag{5}$$

and

$$q' = \mathrm{Tr}(\alpha\eta). \tag{6}$$

Also notice that (5) and (6) imply that

$$q\alpha - q' = (\alpha - \sigma(\alpha))\sigma(\eta)\cdot \tag{7}$$

Let D be a positive integer such that $D\alpha$, $\frac{D}{\alpha - \sigma(\alpha)}$, and $\frac{D\alpha}{\alpha - \sigma(\alpha)}$ are algebraic integers.

The notation $A \asymp B$, where A and B are positive quantities, means that $B \ll A \ll B$.

Lemma 2.1. *Let γ be a positive number in $\mathbb{Q}(\alpha)$. Let Δ be a positive integer such that $\Delta\gamma$ is an algebraic integer. For each $m \in \mathbb{N}$, define the rational number*

$$q = q(m) = \mathrm{Tr}(\gamma\omega^m). \tag{8}$$

Then Δq is a rational integer, one has $q > 0$ when m is large, and the integers $D\Delta q$ satisfy (2).

Proof. Also define

$$q' = q'(m) = \mathrm{Tr}(\alpha\gamma\omega^m).$$

As $\Delta\gamma\omega^m$ and $D\Delta\alpha\gamma\omega^m$ are algebraic integers, Δq and $D\Delta q'$ are rational integers. As $\sigma(\omega) = 1/\omega$, we have $q = \gamma\omega^m + \sigma(\gamma)\omega^{-m}$, hence $q > 0$ as soon as $\omega^{2m} > -\sigma(\gamma)/\gamma$, and then

$$q \asymp \omega^m. \tag{9}$$

From (7), we get $q\alpha - q' = (\alpha - \sigma(\alpha))\sigma(\gamma)\omega^{-m}$, hence

$$|q\alpha - q'| \asymp \omega^{-m}. \tag{10}$$

As $D\Delta q$ and $D\Delta q'$ are integers, it follows from (10) that for large m we have $\|D\Delta q\alpha\| = D\Delta|q\alpha - q'|$, and by (9) and (10), the integers $D\Delta q$ satisfy (2).

Conversely:

Lemma 2.2. *Let $\mathcal{S}$ be a set of positive integers q satisfying (2) . Then there exists a finite set Γ of numbers $\gamma \in \mathbb{Q}(\alpha)$, $\gamma \neq 0$, such that for any $q \in \mathcal{S}$, there exist $\gamma \in \Gamma$ and $m \in \mathbb{N}$ such that*

$$q = \mathrm{Tr}(\gamma\omega^m)\cdot \tag{8}$$

Proof. For $q \in \mathcal{S}$, let $m(q) = m$ be the positive integer such that $\omega^{m-1} \leq q < \omega^m$. We thus have $\omega^m \asymp q$. Let q' be the rational integer such that $\{q\alpha\} = q\alpha - q'$. Set

$$\gamma = \frac{-q\sigma(\alpha) + q'}{\alpha - \sigma(\alpha)}\omega^{-m}.$$

First, notice that $D\gamma$ is an algebraic integer. From (5), we get (8). Writing

$$\gamma\omega^m = q - \frac{q\alpha - q'}{\alpha - \sigma(\alpha)}$$

we see that $\gamma > 0$ when q is large, and $\gamma\omega^m \asymp q$. As we have $\omega^m \asymp q$, we thus get $\gamma \asymp 1$. We also have

$$\sigma(\gamma) = \frac{q\alpha - q'}{\alpha - \sigma(\alpha)}\omega^m,$$

hence, by (2), $|\sigma(\gamma)| \ll \omega^m/q$, and thus, $|\sigma(\gamma)| \ll 1$. Then, as $D\gamma$ is an algebraic integer in $\mathbb{Q}(\alpha)$, and $\max(|\gamma|, |\sigma(\gamma)|) \ll 1$, the set of the γ's is finite.

2.2. End of proof.

Denote by P the set of all prime numbers dividing one of the b_k. Since we assume that the sequence (b_k) is bounded, this set is finite. For $p \in P$, we extend the p-adic absolute value to $\mathbb{Q}(\alpha)$. The completion of this field is $\mathbb{Q}_p(\alpha)$. As above, let ω be a unit in $\mathbb{Q}(\alpha)$ with $\omega > 1$. Note that $|\omega|_p = 1$. The ball $\{x \in \mathbb{Q}_p(\alpha); |x-1|_p < p^{-1/(p-1)}\}$ is a subgroup of finite index in the multiplicative group $\{x \in \mathbb{Q}_p(\alpha); |x|_p = 1\}$. Hence, replacing ω by ω^n, where n is a suitable positive integer, we may also suppose that $|\omega - 1|_p < p^{-1/(p-1)}$ for every $p \in P$.

We shall use the p-adic logarithm function, which is defined on the multiplicative group $\{x \in \mathbb{C}_p; |x-1|_p < 1\} \subset \mathbb{C}_p$ by

$$\log x = \sum_{n=1}^{+\infty} (-1)^{n-1}\frac{(x-1)^n}{n}.$$

This function satisfies

$$\log xy = \log x + \log y,$$

and, for $|x-1|_p < p^{-1/(p-1)}$, $|\log x|_p = |x-1|_p$. Hence, for $|x-1|_p < p^{-1/(p-1)}$ and $|y-1|_p < p^{-1/(p-1)}$, we have

$$|\log x - \log y|_p = |\log \frac{x}{y}|_p = |\frac{x}{y} - 1|_p = |x-y|_p. \tag{11}$$

We prove:

Lemma 2.3. *Let p be a number of P. Let γ be a positive number of $\mathbb{Q}(\alpha)$. For $m \in \mathbb{N}$, set*

$$q = q(m) = \mathrm{Tr}(\gamma\omega^m). \tag{8}$$

Then, if

$$|\frac{\sigma(\gamma)}{\gamma} + 1|_p \geq p^{-1/(p-1)},$$

we have

$$|q|_p \asymp 1$$

for large m; if

$$|\frac{\sigma(\gamma)}{\gamma} + 1|_p < p^{-1/(p-1)},$$

then

$$|q|_p \asymp |2m \log \omega - \log(-\sigma(\gamma)/\gamma)|_p. \tag{12}$$

Proof. Recall that $q > 0$ when m is large (Lemma 2.1). From the definition, we get for each $p \in P$, $|q|_p = |\gamma\omega^m + \sigma(\gamma)\omega^{-m}|_p = |\gamma|_p|\omega^{2m} - \delta|_p$, where $\delta = -\sigma(\gamma)/\gamma$. If $|\delta - 1|_p \geq p^{-1/(p-1)}$, we have $|\omega^{2m} - \delta|_p \geq p^{-1/(p-1)}$, since $|\omega - 1|_p < p^{-1/(p-1)}$ and $|\omega^{2m} - 1|_p < p^{-1/(p-1)}$. Then we get

$$|q|_p \asymp 1.$$

If $|\delta - 1|_p < p^{-1/(p-1)}$, then, by (11), we write $|\omega^{2m} - \delta|_p = |2m \log \omega - \log \delta|_p$, and we obtain (12).

Accordingly, in order to achieve the proof of the result, we shall use known lower bounds for linear forms in p-adic logarithms. For instance, it follows from [8] that:

Lemma 2.4. (K. Yu [8]**)** *Let x and y be algebraic numbers in $\mathbb{C}_p$, with $|x-1|_p < p^{-1/(p-1)}$ and $|y-1|_p < p^{-1/(p-1)}$. Then there exists a real constant κ such that for any pair (k, ℓ) of rational integers with $k \log x + \ell \log y \neq 0$, one has*

$$|k \log x + \ell \log y|_p \gg (\max(|k|, |\ell|))^{-\kappa}.$$

Note that this result is trivial, with $\kappa = 1$, if $\log x$ and $\log y$ are not linearly independent over $\mathbb{Q}$, and $\log x \neq 0$, i.e, $x \neq 1$. Indeed, if $a \log x = b \log y$, where a and b are rational integers with $b \neq 0$, then we write $|k \log x + \ell \log y|_p = \frac{1}{|b|_p}|bk + a\ell|_p|x-1|_p$. Hence we get $|k \log x + \ell \log y|_p \gg |bk + a\ell|_p \geq |bk + a\ell|^{-1} \gg (\max(|k|, |\ell|)^{-1}$, when $k \log x + \ell \log y \neq 0$.

We can then achieve the proof of Theorem 1.2. Applying Lemma 2.2, we can suppose that the set Γ contains a unique element $\gamma > 0$, i.e., for any $q \in \mathcal{S}$, there exists $m \in \mathbb{N}$ such that we have (8). It follows from Lemma 2.3 and 2.4 that there exists a constant κ such that $|q|_p \gg m^{-\kappa}$ (one may take $\kappa = 0$ if $|\frac{\sigma(\gamma)}{\gamma} + 1|_p \geq p^{-1/(p-1)}$). As $q \asymp \omega^m$, hence $m \asymp \ln q$, we get $|q|_p \gg (\ln q)^{-\kappa}$. Now set $\kappa = \kappa_p$ (the constant κ_p may depend upon $p \in P$). Note that $|q|_{\mathcal{B}} \geq \prod_{p \in P} |q|_p$. Indeed, putting $|q|_{\mathcal{B}} = 1/r_n$, we have $q \in r_n\mathbb{Z}$, hence $|q|_p \leq |r_n|_p$ and $\prod_{p\in P} |q|_p \leq \prod_{p \in P} |r_n|_p = 1/r_n$. We thus get (4) with $\lambda = \sum_{p\in P} \kappa_p$, and Theorem 1.2 is proved.

2.3. A remark.

Note that one may also use Lemma 2.3 for solving the opposite problem. For simplicity, consider the case where $|.|_{\mathcal{B}}$ is the p-adic value for a prime number p. If we take a positive number $\gamma \in \mathbb{Q}(\alpha)$ such that $\sigma(\gamma) = -\gamma$, for instance, $\gamma = \alpha - \sigma(\alpha)$ (one may replace α by $-\alpha$, and so, we can suppose $\alpha - \sigma(\alpha) > 0$), then we have $\log(-\sigma(\gamma)/\gamma) = 0$, and by (12), we get $|\mathrm{Tr}(\gamma\omega^m)|_p \asymp |m|_p$. By Lemma 2.1, there exists a positive integer A such that for every large m, the numbers $q = q(m) = A\mathrm{Tr}(\gamma\omega^m)$ are positive integers satisfying (2). For $m = p^s$ with $s \in \mathbb{N}$, we get $|m|_p = 1/m$, hence $|q|_p \asymp 1/m$. Since $m \asymp \ln q$, we have thus proved that there exists an infinite set of integers $q > 1$ satisfying (2) and (3) (which is Theorem 1.1). In this way we obtain integers $q > 1$ satisfying (2) and such that $|q|_p \asymp 1/\ln q$.

One can ask whether there exists an infinite set of integers $q > 1$ satisfying (2), with

$$\inf |q|_p \ln q = 0. \tag{3'}$$

Given a positive decreasing sequence (ϵ_m) with $\sum_{m=0}^{+\infty} \epsilon_m = +\infty$, a p-adic version [4] of Khintchine's Theorem ensures that for almost all $x \in \mathbb{Z}_p$, there exist infinitely many positive integers m such that $|x - m|_p \leq \epsilon_m$. One often considers as reasonable the hypothesis that a given "special" irrational number $x \in \mathbb{Z}_p$ satisfies this condition, with $\epsilon_m = 1/(m \ln m)$ for $m > 1$ (which is false if $x \in \mathbb{Z}_p \cap \mathbb{Q}$, since in this case, we have $|x - m|_p \gg 1/m$ for m large). Let us prove that we can choose $\gamma > 0$ in $\mathbb{Q}(\alpha)$, with $|\frac{\sigma(\gamma)}{\gamma} + 1|_p < |\omega - 1|_p$, such that $\frac{\log(-\sigma(\gamma)/\gamma)}{\log \omega}$ is an irrational number in $\mathbb{Z}_p$. In order to make this obvious, we prove:

Lemma 2.5. *There exists $\xi \in \mathbb{Q}(\alpha)$ such that ξ is not a unit, $N_{\mathbb{Q}(\alpha):\mathbb{Q}}\xi = 1$, and $|\xi|_p = 1$.*

Proof. The number ω is a root of the equation $\omega^2 - S\omega + 1 = 0$, where S is a rational integer, $S = \mathrm{Tr}\,\omega$. The number ξ must be a root of an equation $\xi^2 - t\xi + 1 = 0$, where t is a rational number for which there exists a positive rational number ρ such that $t^2 - 4 = \rho^2(S^2 - 4)$. Such pairs (t, ρ) can be expressed by using a rational parameter θ:

$$t = \frac{2(S^2-4)\theta^2 + 2}{(S^2-4)\theta^2 - 1} = 2 + \frac{4}{(S^2-4)\theta^2 - 1}$$

$$\rho = \frac{4\theta}{(S^2-4)\theta^2 - 1}.$$

Let us show that we can choose $\theta \in \mathbb{Q}^*$ such that $t \notin \mathbb{Z}$ and $|t|_p \leq 1$. It is enough to take $\theta = p$. As we have $S^2 > 4$, hence $S^2 \geq 9$ and $(S^2-4)p^2 - 1 > 4$, t cannot be an integer for this choice of θ. But we have $|t|_p \leq 1$, since $|(S^2-4)p^2 - 1|_p = 1$. Then there exists a number $\xi \in \mathbb{Q}(\alpha)$ such that $\xi^2 - t\xi + 1 = 0$, and ξ is neither a rational number, since $\rho > 0$, nor an algebraic integer, since $t \notin \mathbb{Z}$. Then we have $N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\xi) = 1$, and $|\xi|_p = 1$ because either condition $|\xi|_p < 1$ or $|\xi|_p > 1$ would imply $|t|_p = |\xi + \xi^{-1}|_p > 1$.

Replacing ξ by ξ^n, where n is a suitable positive integer, we thus may find a ξ satisfying Lemma 2.5, with moreover $|\xi - 1|_p < |\omega - 1|_p$. Then we have $|\log \xi|_p < |\log \omega|_p$. Further let us prove that $\frac{\log \xi}{\log \omega} \in \mathbb{Q}_p$. Indeed that is trivial if $\alpha \in \mathbb{Q}_p$, since in this case ξ and ω lie in $\mathbb{Q}_p$, hence so do $\log \xi$ and $\log \omega$. If $\mathbb{Q}_p(\alpha)$ has degree 2 over $\mathbb{Q}_p$, then $\log \xi$ and $\log \omega$ lie in $\mathbb{Q}_p(\alpha)$. But σ can be extended into a continuous $\mathbb{Q}_p$-automorphism of $\mathbb{Q}_p(\alpha)$, and we get $\sigma(\frac{\log \xi}{\log \omega}) = \frac{\log \sigma(\xi)}{\log \sigma(\omega)} = \frac{-\log \xi}{-\log \omega} = \frac{\log \xi}{\log \omega}$, since $\xi\sigma(\xi) = \omega\sigma(\omega) = 1$. That proves that $\frac{\log \xi}{\log \omega} \in \mathbb{Q}_p$, and since $|\log \xi|_p < |\log \omega|_p$, we conclude that $\frac{\log \xi}{2\log \omega} \in \mathbb{Z}_p$. Lastly, $\frac{\log \xi}{\log \omega}$ is not a rational number, since ξ is not a unit. Now, by Hilbert's Theorem, there exists $\gamma \in \mathbb{Q}(\alpha)$, with $\gamma > 0$, such that $\xi = -\sigma(\gamma)/\gamma$. We thus have found $\gamma > 0$ in $\mathbb{Q}(\alpha)$, such that $|\frac{\sigma(\gamma)}{\gamma} + 1|_p < p^{-1/(p-1)}$ and $\frac{\log(-\sigma(\gamma)/\gamma)}{2\log \omega}$ is an irrational element of $\mathbb{Z}_p$. Under the above hypothesis, it would exist infinitely many integers $m > 1$ with $|\frac{\log(-\sigma(\gamma)/\gamma)}{2\log \omega} - m|_p \ll 1/(m \log m)$, and, by (12), we could obtain an infinite set of integers $q > 1$, $q = A\mathrm{Tr}(\gamma\omega^m)$ where A is a positive integer, satisfying (2) and such that $|q|_p \ll \frac{1}{\ln q \ln \ln q}$. In particular, (3') would be satisfied.

3. Conclusion

For a sequence $\mathcal{B}$ bounded, the Roth-Ridout Theorem [6] allows us to see that for any irrational algebraic real number α, thus in particular for α quadratic, we have:

$$\inf_{q>0} q^{1+\epsilon} \|q\alpha\| |q|_{\mathcal{B}} > 0$$

(see [5]). Of course, our method is far from enabling us to prove that there exists a real constant λ such that

$$\inf_{q>1} q(\ln q)^{\lambda}\|q\alpha\||q|_{\mathcal{B}} > 0.$$

We can only study the approximations with $q\|q\alpha\| \ll 1$. It seems difficult to study approximations in the "orthogonal direction" $q|q|_{\mathcal{B}} \ll 1$, with for instance, $q = p^n$, for a prime number p. For such approximations, it is not known whether $\inf_{n\in\mathbb{N}} \|p^n\alpha\| = 0$ holds, neither if there exists λ such that $\inf_{n>0} n^{\lambda}\|p^n\alpha\| > 0$. It is very difficult to obtain more precise results than the Roth-Ridout Theorem (see [1]).

Even for rational approximations satisfying (2), we are not able to prove that the constants $\lambda(\mathcal{S})$ are bounded. This is related to Lemma 2.4. It would be necessary to prove that there exists a real constant κ for which this Lemma holds for $x = \omega$ and for any $y \in \mathbb{Q}(\alpha)$ with $|y-1|_p < p^{-1/(p-1)}$ and $N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(y) = 1$. There exist many effective estimates of $|k\log x + \ell \log y|_p$ (see for instance [2] and [8]), but they do not provide the needed result. It seems difficult to take the particular conditions required into account.

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