*Journal de Théorie des Nombres*
*de Bordeaux* **17** (2005), 217–236

# Systems of quadratic diophantine inequalities

par Wolfgang MÜLLER

Résumé. Soient $Q_1, \dots, Q_r$ des formes quadratiques avec des coefficients réels. Nous prouvons que pour chaque $\varepsilon > 0$ le système $|Q_1(x)| < \varepsilon, \dots, |Q_r(x)| < \varepsilon$ des inégalités a une solution entière non-triviale si le système $Q_1(x) = 0, \dots, Q_r(x) = 0$ a une solution réelle non-singulière et toutes les formes $\sum_{i=1}^r \alpha_i Q_i$, $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^s, \alpha \neq 0$ sont irrationnelles avec rang $> 8r$.

Abstract. Let $Q_1, \dots, Q_r$ be quadratic forms with real coefficients. We prove that for any $\epsilon > 0$ the system of inequalities $|Q_1(x)| < \epsilon, \dots, |Q_r(x)| < \epsilon$ has a nonzero integer solution, provided that the system $Q_1(x) = 0, \dots, Q_r(x) = 0$ has a nonsingular real solution and all forms in the real pencil generated by $Q_1, \dots, Q_r$ are irrational and have rank $> 8r$.

## 1. Introduction

Let $Q_1, \dots, Q_r$ be quadratic forms in $s$ variables with real coefficients. We ask whether the system of quadratic inequalities

$$|Q_1(x)| < \epsilon, \dots, |Q_r(x)| < \epsilon \tag{1.1}$$

has a nonzero integer solution for every $\epsilon > 0$. If some $Q_i$ is rational[1] and $\epsilon$ is small enough then for $x \in \mathbb{Z}^s$ the inequality $|Q_i(x)| < \epsilon$ is equivalent to the equation $Q_i(x) = 0$. Hence if all forms are rational then for sufficiently small $\epsilon$ the system (1.1) reduces to a system of equations. In this case W. Schmidt [10] proved the following result. Recall that the real pencil generated by the forms $Q_1, \dots, Q_r$ is defined as the set of all forms

$$Q_\alpha = \sum_{i=1}^r \alpha_i Q_i \tag{1.2}$$

where $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r$, $\alpha \neq 0$. The rational and complex pencil are defined similarly. Suppose that $Q_1, \dots, Q_r$ are rational quadratic forms. Then the system $Q_1(x) = 0, \dots, Q_r(x) = 0$ has a nonzero integer solution provided that

[1] A real quadratic form is called rational if its coefficients are up to a common real factor rational. It is called irrational if it is not rational.

(i) the given forms have a common nonsingular real solution, and either
(iia) each form in the complex pencil has rank $> 4r^2 + 4r$, or
(iib) each form in the rational pencil has rank $> 4r^3 + 4r^2$.

Recently, R. DIETMANN [7] relaxed the conditions (iia) and (iib). He replaced them by the weaker conditions

(iia') each form in the complex pencil has rank $> 2r^2 + 3r$, or
(iib') each form in the rational pencil has rank $> 2r^3$ if $r$ is even and rank $> 2r^3 + 2r$ if $r$ is odd.

If $r = 2$ the existence of a nonsingular real solution of $Q_1(x) = 0$ and $Q_2(x) = 0$ follows if one assumes that every form in the real pencil is indefinite (cf. SWINNERTON-DYER [11] and COOK [6]). As noted by W. SCHMIDT [10] this is false for $r > 2$.

We want to consider systems of inequalities (1.1) without hidden equalities. A natural condition is to assume that *all* forms in the real pencil are irrational. Note that if $Q_\alpha$ is rational and $\epsilon$ is small enough, then (1.1) and $x \in \mathbb{Z}^s$ imply $Q_\alpha(x) = 0$. We prove

**Theorem 1.1.** *Let $Q_1, \ldots, Q_r$ be quadratic forms with real coefficients. Then for every $\epsilon > 0$ the system* (1.1) *has a nonzero integer solution provided that*

(i) *the system $Q_1(x) = 0, \ldots, Q_r(x) = 0$ has a nonsingular real solution,*
(ii) *each form in the real pencil is irrational and has rank $> 8r$.*

In the case $r = 1$ much more is known. G.A. MARGULIS [9] proved that for an irrational nondegenerate form $Q$ in $s \geq 3$ variables the set $\{Q(x) \mid x \in \mathbb{Z}^s\}$ is dense in $\mathbb{R}$ (*Oppenheim conjecture*). In the case $r > 1$ all known results assume that the forms $Q_i$ are diagonal[2]. For more information on these results see E.D. FREEMAN [8] and J. BRÜDERN, R.J. COOK [4].

In 1999 V. BENTKUS and F. GÖTZE [2] gave a completely different proof of the Oppenheim conjecture for $s > 8$. We use a multidimensional variant of their method to count weighted solutions of the system (1.1). To do this we introduce for an integer parameter $N \geq 1$ the weighted exponential sum

$$S_N(\alpha) = \sum_{x \in \mathbb{Z}^s} w_N(x) e(Q_\alpha(x)) \qquad (\alpha \in \mathbb{R}^r)\,. \tag{1.3}$$

Here $Q_\alpha$ is defined by (1.2), $e(x) = \exp(2\pi i x)$ as usual , and

$$w_N(x) = \sum_{n_1+n_2+n_3+n_4=x} p_N(n_1)p_N(n_2)p_N(n_3)p_N(n_4) \tag{1.4}$$

[2]Note added in proof: Recently, A. GORODNIK studied systems of nondiagonal forms. In his paper *On an Oppenheim-type conjecture for systems of quadratic forms*, Israel J. Math. **149** (2004), 125–144, he gives conditions (different from ours) that guarantee the existence of a nonzero integer solution of (1.1). His Conjecture 13 is partially answered by our Theorem 1.1.

denotes the fourfold convolution of $p_N$, the density of the discrete uniform probability distribution on $[-N,N]^s \cap \mathbb{Z}^s$. Since $w_N$ is a probability density on $\mathbb{Z}^s$ one trivially obtains $|S_N(\alpha)| \leq 1$. The key point in the analysis of Bentkus and Götze is an estimate of $S_N(\alpha+\epsilon)S_N(\alpha-\epsilon)$ in terms of $\epsilon$ alone. Lemma 2.2 gives a generalization of their estimate to the case $r > 1$. It is proved via the double large sieve inequality. It shows that for $N^{-2} < |\epsilon| < 1$ the exponential sums $S_N(\alpha-\epsilon)$ and $S_N(\alpha+\epsilon)$ cannot be simultaneously large. This information is almost sufficient to integrate $|S_N(\alpha)|$ within the required precision. As a second ingredient we use for $0 < T_0 \leq 1 \leq T_1$ the uniform bound

$$\lim_{N\to\infty} \sup_{T_0 \leq |\alpha| \leq T_1} |S_N(\alpha)| = 0\,. \tag{1.5}$$

Note that (1.5) is false if the real pencil contains a rational form. The proof of (1.5) follows closely Bentkus and Götze [2] and uses methods from the geometry of numbers.

## 2. The double large sieve bound

The following formulation of the double large sieve inequality is due to Bentkus and Götze [2]. For a vector $T = (T_1, \ldots, T_s)$ with positive real coordinates write $T^{-1} = (T_1^{-1}, \ldots, T_s^{-1})$ and set

$$B(T) = \{(x_1,\ldots,x_s) \in \mathbb{R}^s \mid |x_j| \leq T_j \text{ for } 1 \leq j \leq s\}\,. \tag{2.1}$$

**Lemma 2.1** (Double large sieve)**.** *Let $\mu, \nu$ denote measures on $\mathbb{R}^s$ and let $S, T$ be $s$-dimensional vectors with positive coordinates. Write*

$$J = \int_{B(S)} \left( \int_{B(T)} g(x)h(y)e(\langle x,y\rangle)\, d\mu(x) \right) d\nu(y), \tag{2.2}$$

*where $\langle .,. \rangle$ denotes the standard scalar product in $\mathbb{R}^s$ and $g,h : \mathbb{R}^s \to \mathbb{C}$ are measurable functions. Then*

$$|J|^2 \ll A(2S^{-1}, g, \mu) A(2T^{-1}, h, \nu) \prod_{j=1}^{s} (1 + S_j T_j)\,,$$

*where*

$$A(S, g, \mu) = \int \left( \int_{y \in x + B(S)} |g(y)|\, d\mu(y) \right) |g(x)|\, d\mu(x)\,.$$

*The implicit constant is an absolute one. In particular, if* $|g(x)| \le 1$ *and* $|h(x)| \le 1$ *and* $\mu, \nu$ *are probability measures, then*

(2.3)
$$|J|^2 \ll \sup_{x\in\mathbb{R}^s} \mu(x + B(2S^{-1})) \sup_{x\in\mathbb{R}^s} \nu(x + B(2T^{-1})) \prod_{j=1}^{s} (1 + S_j T_j)\,.$$

**Remark.** This is Lemma 5.2 in [1]. For discrete measures the lemma is due to E. Bombieri and H. Iwaniec [3]. The general case follows from the discrete one by an approximation argument.

**Lemma 2.2.** *Assume that each form in the real pencil of* $Q_1, \dots, Q_r$ *has rank* $\ge p$. *Then the exponential sum* (1.3) *satisfies*

$$S_N(\alpha - \epsilon) S_N(\alpha + \epsilon) \ll \mu(|\epsilon|)^p \qquad (\alpha, \epsilon \in \mathbb{R}^r)\,, \tag{2.4}$$

*where*

$$\mu(t) = \begin{cases} 1 & 0 \le t \le N^{-2}\,, \\ t^{-1/2} N^{-1} & N^{-2} \le t \le N^{-1}\,, \\ t^{1/2} & N^{-1} \le t \le 1\,, \\ 1 & t \ge 1\ . \end{cases}$$

*Proof.* Set $S = S_N(\alpha - \epsilon) S_N(\alpha + \epsilon)$. We start with

$$\begin{aligned} S &= \sum_{x,y\in\mathbb{Z}^s} w_N(x) w_N(y) e(Q_{\alpha-\epsilon}(x) + Q_{\alpha+\epsilon}(y)) \\ &= \sum_{\substack{m,n\in\mathbb{Z}^s \\ m\equiv n(2)}} w_N\big(\tfrac{1}{2}(m-n)\big) w_N\big(\tfrac{1}{2}(m+n)\big) e\big(Q_{\alpha-\epsilon}\big(\tfrac{1}{2}(m-n)\big) + Q_{\alpha+\epsilon}\big(\tfrac{1}{2}(m+n)\big)\big) \\ &= \sum_{\substack{m\equiv n(2) \\ |m|_\infty, |n|_\infty \le 8N}} w_N\big(\tfrac{1}{2}(m-n)\big) w_N\big(\tfrac{1}{2}(m+n)\big) e\big(\tfrac{1}{2} Q_\alpha(m) + \tfrac{1}{2} Q_\alpha(n) + \langle m, Q_\epsilon n\rangle\big)\,. \end{aligned}$$

To separate the variables $m$ and $n$ in the weight function write

$$w_N(x) = \int_B h(\theta) e(-\langle \theta, x\rangle)\, d\theta\,, \tag{2.5}$$

where $B = (-1/2, 1/2]^s$ and $h$ denotes the (finite) Fourier series

$$h(\theta) = \sum_{k\in\mathbb{Z}^s} w_N(k) e(\langle \theta, k\rangle)\,.$$

Since $w = p_N * p_N * p_N * p_N$ we find $h(\theta) = h_N(\theta)^2$, where

$$h_N(\theta) = \sum_{k\in\mathbb{Z}^s} p_N * p_N(k) e(\langle \theta, k\rangle)\,.$$

Now set

$$a(m) = e(\tfrac{1}{2}(Q_\alpha(m) - \langle \theta_1 + \theta_2, m\rangle))\,,$$

$$b(n) = e(\tfrac{1}{2}(Q_\alpha(n) - \langle \theta_1 - \theta_2, n\rangle))\,.$$

Using (2.5) we find

$$\begin{aligned} |S| &= \Big| \int_B \int_B h(\theta_1)h(\theta_2) \sum_{\substack{m\equiv n(2)\\ |m|_\infty,|n|_\infty\le 8N}} a(m)b(n)e(\langle m, Q_\epsilon n\rangle)\, d\theta_1 d\theta_2 \Big| \\ &\le \Big(\int_B |h(\theta)|\, d\theta\Big)^2 \sup_{\theta_1,\theta_2\in B} \Big| \sum_{\substack{m\equiv n(2)\\ |m|_\infty,|n|_\infty\le 8N}} a(m)b(n)e(\langle m, Q_\epsilon n\rangle)\Big|\,. \end{aligned}$$

Note that $a(m)$ and $b(n)$ are independent of $\epsilon$. Furthermore, by Bessel's inequality

$$\begin{aligned} \int_B |h(\theta)|\, d\theta = \int_B |h_N(\theta)|^2\, d\theta &\le \sum_{k\in\mathbb{Z}^s} (p_N * p_N(k))^2 \\ &\le (2N+1)^{-s} \sum_{k\in\mathbb{Z}^s} p_N * p_N(k) \le (2N+1)^{-s}\,. \end{aligned}$$

Hence

$$S \ll N^{-2s} \sum_{\omega\in\{0,1\}^s} \sup_{\theta_1,\theta_2\in B} \Big| \sum_{\substack{m\equiv n\equiv\omega(2)\\ |m|_\infty,|n|_\infty\le 8N}} a(m)b(n)e(\langle m, Q_\epsilon n\rangle)\Big|\,.$$

We are now in the position to apply Lemma 2.1. Denote by $\lambda_1,\dots,\lambda_s$ the eigenvalues of $Q_\epsilon$ ordered in such a way that $|\lambda_1| \ge \cdots \ge |\lambda_s|$. Then $Q_\epsilon = U^T \Lambda U$, where $U$ is orthogonal and $\Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_s)$. Set $\Lambda^{1/2} = \mathrm{diag}(|\lambda_1|^{1/2},\dots,|\lambda_s|^{1/2})$, $E = \mathrm{diag}(\mathrm{sgn}(\lambda_1),\dots,\mathrm{sgn}(\lambda_s))$ and

$$\mathcal{M} = \{\Lambda^{1/2} U m \mid m \in \mathbb{Z}^s, m \equiv \omega(2), |m|_\infty \le 8N\},$$

$$\mathcal{N} = \{E\Lambda^{1/2} U m \mid m \in \mathbb{Z}^s, m \equiv \omega(2), |m|_\infty \le 8N\}.$$

Furthermore, let $\mu$ denote the uniform probability distribution on $\mathcal{M}$ and $\nu$ the uniform probability distribution on $\mathcal{N}$. Choose $S_j = T_j = 1 + 8\sqrt{s}|\lambda_j|^{1/2}N$. Then $x \in \mathcal{M}$ implies $x \in B(T)$ and $y \in \mathcal{N}$ implies $y \in B(S)$. If follows by (2.3) that

$$\begin{aligned} &\Big| N^{-2s} \sum_{\substack{m\equiv n\equiv\omega(2)\\ |m|_\infty,|n|_\infty\le 8N}} a(m)b(n)e(\langle m, Q_\epsilon n\rangle)\Big|^2 \\ &\qquad\qquad \ll N^{-2s}\Big(\sup_{x\in\mathbb{R}^s} A(x)\Big)^2 \prod_{j=1}^s (1 + |\lambda_j| N^2)\,, \end{aligned}$$

where

$$\begin{aligned} A(x) &= \#\{m \in \mathbb{Z}^s \mid |m|_\infty \le 8N, m \equiv \omega(2), \Lambda^{1/2}Um - x \in B(2S^{-1})\} \\ &\ll \#\{z \in U\mathbb{Z}^s \mid |z|_\infty \ll N\,,\ ||\lambda_j|^{1/2}z_j - x_j| \ll S_j^{-1}\} \\ &\ll \prod_{j=1}^{s} \min(N, 1 + |\lambda_j|^{-1}N^{-1})\,. \end{aligned}$$

Hence

$$S \ll \prod_{j=1}^{s} \tilde{\mu}(|\lambda_j|)$$

with $\tilde{\mu}(t) = N^{-1}(1 + t^{1/2}N)\min(N, 1 + t^{-1}N^{-1})$. To prove (2.4) we have to consider the case $N^{-2} \le |\epsilon| \le 1$ only. Otherwise the trivial bound $|S_N(\alpha)| \le 1$ is sufficient. Since $\lambda_j = \lambda_j(\epsilon)$ varies continuously on $\mathbb{R}^r \setminus \{0\}$ and $\lambda_j(c\epsilon) = c\lambda_j(\epsilon)$ for $c > 0$ there exist constants $0 < \underline{c}_j \le \overline{c}_j < \infty$ such that

$$\begin{aligned} \lambda_j(\epsilon) &\le \overline{c}_j|\epsilon| \qquad (1 \le j \le s), \\ \underline{c}_j|\epsilon| \le \lambda_j(\epsilon) &\le \overline{c}_j|\epsilon| \qquad (1 \le j \le p). \end{aligned} \tag{2.6}$$

If $N^{-2} \le |\epsilon| \le 1$ then $|\lambda_j| \ll 1$ and $\tilde{\mu}(|\lambda_j|) \ll 1$ for all $j \le s$. Furthermore, for $j \le p$ we find $|\lambda_j| \asymp |\epsilon|$ and $\tilde{\mu}(|\lambda_j|) \ll \max(|\epsilon|^{-1/2}N^{-1}, |\epsilon|^{1/2})$. Altogether this yields

$$S \ll \prod_{j=1}^{p} \tilde{\mu}(|\lambda_j|) \ll \max(|\epsilon|^{-1/2}N^{-1}, |\epsilon|^{1/2})^p \ll \mu(|\epsilon|)^p\,.$$

□

## 3. The uniform bound

**Lemma 3.1** (H. Davenport [5]). *Let $L_i(x) = \lambda_{i1}x_1 + \cdots + \lambda_{is}x_s$ be $s$ linear forms with real and symmetric coefficient matrix $(\lambda_{ij})_{1\le i,j\le s}$. Denote by $\|.\|$ the distance to the nearest integer. Suppose that $P \ge 1$. Then the number of $x \in \mathbb{Z}^s$ such that*

$$|x|_\infty < P \quad \text{and} \quad \|L_i(x)\| < P^{-1} \qquad (1 \le i \le s)$$

*is $\ll (M_1 \dots M_s)^{-1}$. Here $M_1, \dots, M_s$ denotes the first $s$ of the $2s$ successive minima of the convex body defined by $F(x,y) \le 1$, where for $x, y \in \mathbb{R}^s$*

$$F(x,y) = \max(P|L_1(x)-y_1|, \dots, P|L_s(x)-y_s|, P^{-1}|x_1|, \dots, P^{-1}|x_s|)\,.$$

**Lemma 3.2.** *Assume that each form in the real pencil of $Q_1,\dots,Q_r$ is irrational. Then for any fixed $0<T_0\le T_1<\infty$*

$$\lim_{N\to\infty}\sup_{T_0\le|\alpha|\le T_1}|S_N(\alpha)|=0\,.$$

*Proof.* We start with one Weyl step. Using the definition of $w_N$ we find

$$\begin{aligned}|S_N(\alpha)|^2&=\sum_{x,y\in\mathbb{Z}^s}w_N(x)w_N(y)e(Q_\alpha(y)-Q_\alpha(x))\\&=\sum_{\substack{z\in\mathbb{Z}^s\\|z|_\infty\le 8N}}\sum_{x\in\mathbb{Z}^s}w_N(x)w_N(x+z)e(Q_\alpha(z)+2\langle z,Q_\alpha x\rangle)\\&=(2N+1)^{-8s}\sum_{m_i,n_i,z}\sum_{x\in I(m_i,n_i,z)}e(Q_\alpha(z)+2\langle z,Q_\alpha x\rangle)\,.\end{aligned}$$

Here the first sum is over all $m_1,m_2,m_3,n_1,n_2,n_3,z\in\mathbb{Z}^s$ with $|m_i|_\infty\le N$, $|n_i|_\infty\le N$, $|z|_\infty\le 8N$ and $I(m_i,n_i,z)$ is the set

$$\{x\in\mathbb{Z}^s\mid|x-n_1-n_2-n_3|_\infty\le N,\ |x+z-m_1-m_2-m_3|_\infty\le N\}\,.$$

It is an $s$-dimensional box with sides parallel to the coordinate axes and side length $\ll N$. By Cauchy's inequality it follows that

$$\begin{aligned}|S_N(\alpha)|^4&\ll N^{-9s}\sum_{m_i,n_i,z}\Big|\sum_{x\in I(m_i,n_i,z)}e(2\langle x,Q_\alpha z\rangle)\Big|^2\\&\ll N^{-3s}\sum_{|z|_\infty\le 8N}\prod_{i=1}^{s}\min\left(N,\|2\langle e_i,Q_\alpha z\rangle\|^{-1}\right)^2\,.\end{aligned}$$

Here we used the well known bound

$$\sum_{x\in I_1\times\cdots\times I_s}e(\langle x,y\rangle)\ll\prod_{i=1}^{s}\min(|I_i|,\|\langle e_i,y\rangle\|^{-1})\,,$$

where $I_i$ are intervals of length $|I_i|\gg 1$ and $e_i$ denotes the $i$-th unit vector. Set

$$\mathcal{N}(\alpha)=\#\{z\in\mathbb{Z}^s\mid|z|_\infty\le 16N,\|2\langle e_i,Q_\alpha z\rangle\|<1/16N\ \text{for}\ 1\le i\le s\}.$$

We claim that

$$|S_N(\alpha)|^4\ll N^{-s}\mathcal{N}(\alpha)\,. \tag{3.1}$$

To see this set

$$\mathcal{D}_m(\alpha)=\#\{z\in\mathbb{Z}^s\mid|z|_\infty\le 8N,\tfrac{m_i-1}{16N}\le\{2\langle e_i,Q_\alpha z\rangle\}<\tfrac{m_i}{16N}\text{for}\ i\le s\},$$

where $\{x\}$ denotes the fractional part of $x$. Then $\mathcal{D}_m(\alpha)\le\mathcal{N}(\alpha)$ for all $m=(m_1,\dots,m_s)$ with $1\le m_i\le 16N$. Note that if $z_1$ and $z_2$ are counted in $\mathcal{D}_m(\alpha)$ then $z_1 - z_2$ is counted in $\mathcal{N}(\alpha)$. It follows that

$$\begin{aligned}|S_N(\alpha)|^4 &\ll N^{-3s} \sum_{1\le m_i\le 16N} \mathcal{D}_m(\alpha) \prod_{i=1}^{s} \min\Big(N, \frac{16N}{m_i-1} + \frac{16N}{16N-m_i}\Big)^2 \\ &\ll N^{-3s}\mathcal{N}(\alpha) \sum_{1\le m_i\le 8N} \prod_{i=1}^{s} \frac{N^2}{m_i^2} \\ &\ll N^{-s}\mathcal{N}(\alpha)\,.\end{aligned}$$

To estimate $\mathcal{N}(\alpha)$ we use Lemma 3.1 with $P = 16N$ and $L_i(x) = 2\langle e_i, Q_\alpha x\rangle$ for $1 \le i \le s$. This yields

$$\mathcal{N}(\alpha) \ll (M_{1,\alpha} \dots M_{s,\alpha})^{-1}\,, \tag{3.2}$$

where $M_{1,\alpha} \le \cdots \le M_{s,\alpha}$ are the first $s$ from the $2s$ successive minima of the convex body defined in Lemma 3.1.

Now suppose that there exists an $\epsilon > 0$, a sequence of real numbers $N_n \to \infty$ and $\alpha^{(n)} \in \mathbb{R}^r$ with $T_0 \le |\alpha^{(n)}| \le T_1$ such that

$$|S_{N_n}(\alpha^{(n)})| \ge \epsilon\,. \tag{3.3}$$

By (3.1) and (3.2) this implies

$$\epsilon^4 N_n^s \ll \Big(\prod_{i=1}^{s} M_{i,\alpha^{(n)}}\Big)^{-1}\,.$$

Since $(16N_n)^{-1} \le M_{1,\alpha^{(n)}} \le M_{i,\alpha^{(n)}}$ we obtain $\epsilon^4 N_n^s \ll N_n^{s-1} M_{s,\alpha^{(n)}}^{-1}$ and this proves

$$(16N_n)^{-1} \le M_{1,\alpha^{(n)}} \le \cdots \le M_{s,\alpha^{(n)}} \ll (\epsilon^4 N_n)^{-1}\,.$$

By the definition of the successive minima there exist $x_j^{(n)}, y_j^{(n)} \in \mathbb{Z}^s$ such that $(x_1^{(n)}, y_1^{(n)})$, $\dots$, $(x_s^{(n)}, y_s^{(n)})$ are linearly independent and $M_{j,\alpha^{(n)}} = F(x_j^{(n)}, y_j^{(n)})$. Hence for $1 \le i, j \le s$

$$|L_i(x_j^{(n)}) - y_{j,i}^{(n)})| \ll N_n^{-2}\,,$$
$$|x_{j,i}^{(n)}| \ll 1\,.$$

Since $|\alpha^{(n)}| \le T_1$ this inequalities imply $|y_{j,i}^{(n)}| \ll_{T_1} 1$. This proves that the integral vectors

$$W_n = (x_1^{(n)}, y_1^{(n)}, \dots, x_s^{(n)}, y_s^{(n)}) \qquad (n \ge 1)$$

are contained in a bounded box. Thus there exists an infinite sequence $(n'_k)_{k\ge1}$ with $W_{n'_1} = W_{n'_k}$ for $k \ge 1$. The compactness of $\{\alpha \in \mathbb{R}^s \mid T_0 \le |\alpha| \le T_1\}$ implies that there is a subsequence $(n_k)_{k\ge1}$ of $(n'_k)_{k\ge1}$ with

$\lim_{k\to\infty}\alpha^{(n_k)} = \alpha^{(0)}$ and $T_0 \leq |\alpha^{(0)}| \leq T_1$. Let $x_j = x_j^{(n_k)}$ and $y_j = y_j^{(n_k)}$ for $1 \leq j \leq s$. Then $x_j$ and $y_j$ are well defined and

$$y_j = (L_1(x_j), \ldots, L_s(x_j)) = 2Q_{\alpha^{(0)}} x_j \qquad (1 \leq j \leq s)\,. \tag{3.4}$$

We claim that $x_1, \ldots, x_s$ are linearly independent. Indeed, suppose that there are $q_j$ such that $\sum_{j=1}^s q_j x_j = 0$. Then $\sum_{j=1}^s q_j y_j = 0$ by (3.4). This implies $\sum_{j=1}^s q_j(x_j, y_j) = 0$ and the linear independence of $(x_j, y_j)$ yields $q_j = 0$ for all $j$. The matrix equation $2Q_{\alpha^{(0)}}(x_1, \ldots, x_s) = (y_1, \ldots, y_s)$ implies that $Q_{\alpha^{(0)}}$ is rational. By our assumptions this is only possible if $\alpha^{(0)} = 0$, contradicting $|\alpha^{(0)}| \geq T_0 > 0$. This completes the proof of the Lemma. □

**Lemma 3.3.** *Assume that each form in the real pencil of $Q_1, \ldots, Q_r$ is irrational and has rank $\geq 1$. Then there exists a function $T_1(N)$ such that $T_1(N)$ tends to infinity as $N$ tends to infinity and for every $\delta > 0$*

$$\lim_{N\to\infty} \sup_{N^{\delta-2} \leq |\alpha| \leq T_1(N)} |S_N(\alpha)| = 0\,.$$

*Proof.* We first prove that there exist functions $T_0(N) \leq T_1(N)$ such that $T_0(N) \downarrow 0$ and $T_1(N) \uparrow \infty$ for $N \to \infty$ and

$$\lim_{N\to\infty} \sup_{T_0(N) \leq |\alpha| \leq T_1(N)} |S_N(\alpha)| = 0\,. \tag{3.5}$$

From Lemma 3.2 we know that for each $m \in \mathbb{N}$ there exist an $N_m$ with

$$|S_N(\alpha)| \leq \frac{1}{m} \quad \text{for} \quad N \geq N_m \quad \text{and} \quad \frac{1}{m} \leq |\alpha| \leq m\,.$$

Without loss of generality we assume that $(N_m)_{m\geq 1}$ is increasing. For $N_m \leq N < N_{m+1}$ define $T_0(N) = \frac{1}{m}$, $T_1(N) = m$ and for $N < N_1$ set $T_0(N) = T_1(N) = 1$. Obviously this choice satisfies (3.5). Replacing $T_0(N)$ by $\max(T_0(N), N^{-1})$ we can assume that $N^{-1} \leq T_0(N) \leq 1$. Finally, Lemma 2.2 with $p \geq 1$ yields

$$\sup_{N^{\delta-2} \leq |\alpha| \leq T_0(N)} |S_N(\alpha)| \ll \sup_{N^{\delta-2} \leq |\alpha| \leq T_0(N)} \mu(|\alpha|)^p \ll \max(N^{-\delta/2}, T_0(N)^{1/2})^p \to 0\,.$$

□

## 4. The integration procedure

In this section we use Lemma 2.2 to integrate $|S_N(\alpha)|$. It is here where we need the assumption $p > 8r$.

**Lemma 4.1.** *For $0 < U \le T$ set $B(U,T) = \{\alpha \in \mathbb{R}^r \mid U \le |\alpha| \le T\}$ and define*

$$\gamma(U,T) = \sup_{\alpha \in B(U,T)} |S_N(\alpha)|\,.$$

*Furthermore, let $h$ be a measurable function with $0 \le h(\alpha) \le (1+|\alpha|)^{-k}$, $k > r$. If each form in the real pencil generated by $Q_1, \ldots, Q_r$ has rank $\ge p$ with $p > 8r$ and if $\gamma(U,T) \ge 4^{p/(8r)} N^{-p/4}$ then*

$$\int_{B(U,T)} |S_N(\alpha)| h(\alpha)\, d\alpha \ll N^{-2r} \min(1, U^{-(k-r)}) \gamma(U,T)^{1-\frac{8r}{p}}\,.$$

*Proof.* Set $B = B(U,T)$ and $\gamma = \gamma(U,T)$. For $l \ge 0$ define

$$B_l = \{\alpha \in B \mid 2^{-l-1} \le |S_N(\alpha)| \le 2^{-l}\}\,.$$

If $L$ denotes the least non negative integer such that $\gamma \ge 2^{-L-1}$ then $|S_N(\alpha)| \le \gamma \le 2^{-L}$ and for any $M \ge L$

$$B = \bigcup_{l=L}^{M} B_l \cup D_M\,,$$

where $D_M = \{\alpha \in B \mid |S_N(\alpha)| \le 2^{-M-1}\}$. By Lemma 2.2

$$|S_N(\alpha) S_N(\alpha+\epsilon)| \le C\mu(|\epsilon|)^p$$

with some constant $C$ depending on $Q_1,\ldots,Q_r$. By considering $C^{-1/2} S_N(\alpha)$ instead of $S_N(\alpha)$ we may assume $C = 1$. If $\alpha \in B_l$ and $\alpha + \epsilon \in B_l$ it follows that

$$4^{-l-1} \le |S_N(\alpha) S_N(\alpha+\epsilon)| \le \mu(|\epsilon|)^p\,.$$

If $|\epsilon| \le N^{-1}$ this implies $|\epsilon| \le N^{-2} 2^{4(l+1)/p} = \delta$, say, and if $|\epsilon| \ge N^{-1}$ this implies $|\epsilon| \ge 2^{-4(l+1)/p} = \rho$, say. Note that $\delta \le \rho$ if $2^{8(l+1)/p} \le N^2$, and this is true for all $l \le M$ if

$$M + 1 \le \log(N^{p/4})/\log 2\,. \tag{4.1}$$

We choose $M$ as the largest integer less or equal to $\log(N^{2r}\gamma^{\frac{8r}{p}-1})/\log 2 - 1$. Then the assumption $\gamma \ge 4^{p/(8r)} N^{-p/4}$ implies $L \le M$, (4.1) and

$$2^{-M} \ll N^{-2r} \gamma^{1-8r/p}\,. \tag{4.2}$$

To estimate the integral over $B_l$ we split $B_l$ in a finite number of subsets. If $B_l \ne \emptyset$ choose any $\beta_1 \in B_l$ and set $B_l(\beta_1) = \{\alpha \in B_l \mid |\alpha - \beta_1| \le \delta\}$. If $\alpha \in B_l \setminus B_l(\beta_1)$ then $|\alpha - \beta_1| \ge \rho$. If $B_l \setminus B_l(\beta_1) \ne \emptyset$ choose $\beta_2 \in B_l \setminus B_l(\beta_1)$ and set $B_l(\beta_2) = \{\alpha \in B_l \setminus B_l(\beta_1) \mid |\alpha - \beta_2| \le \delta\}$. Then $|\alpha - \beta_1| \ge \rho$ and $|\alpha - \beta_2| \ge \rho$ for all $\alpha \in B_l \setminus \{B_l(\beta_1) \cup B_l(\beta_2)\}$. Especially $|\beta_1 - \beta_2| \ge \rho$. In this way we construct a sequence $\beta_1, \ldots, \beta_m$ of points in $B_l$ with $|\beta_i - \beta_j| \ge \rho$ for $i \ne j$. This construction terminates after finitely many steps. To see this note that the balls $K_{\rho/2}(\beta_i)$ with center $\beta_i$ and radius $\rho/2$ are disjoint and contained in a ball with center 0 and radius $T+\rho/2$. Thus $m\mathrm{vol}(K_{\rho/2}) \le \mathrm{vol}(K_{T+\rho/2})$ and this implies $m \ll (1+T/\rho)^r$. Since $B_l \subseteq \biguplus_{i=1}^m B_l(\beta_i) \subseteq \biguplus_{i=1}^m K_\delta(\beta_i)$ we obtain

$$\begin{aligned}\int_{B_l} |S_N(\alpha)| h(\alpha)\, d\alpha &\le 2^{-l} \sum_{i=1}^m \int_{K_\delta(\beta_i)} (1+|\alpha|)^{-k} d\alpha \\ &\ll 2^{-l} \sum_{\substack{i\le m\\ |\beta_i|\le 1}} \delta^r + 2^{-l} \sum_{\substack{i\le m\\ |\beta_i|>1}} \left(\frac{\delta}{\rho}\right)^r \int_{K_{\rho/2}(\beta_i)} |\alpha|^{-k}\, d\alpha\,.\end{aligned}$$

Note that $|\alpha| \asymp |\beta_i|$ for $\alpha \in K_\rho(\beta_i)$ if $|\beta_i| \ge 1$. If $U > 1$ the first sum is empty and the second sum is $\ll (\delta/\rho)^r \int_{|\alpha|>U/2} |\alpha|^{-k}\, d\alpha \ll (\delta/\rho)^r U^{-(k-r)}$. If $U \le 1$ then the first sum contains $\ll \rho^{-r}$ summands; Thus both sums are bounded by $(\delta/\rho)^r$. This yields

$$\int_{B_l} |S_N(\alpha)| h(\alpha)\, d\alpha \ll 2^{-l} \left(\frac{\delta}{\rho}\right)^r \min(1, U^{-(k-r)})\,.$$

Altogether we obtain by (4.2) and the definition of $\delta$, $\rho$, $L$

$$\begin{aligned}\int_B |S_N(\alpha)| h(\alpha)\, d\alpha &\ll \sum_{l=L}^M 2^{-l} \left(\frac{\delta}{\rho}\right)^r \min(1, U^{-(k-r)}) + 2^{-M} \int_{|\alpha|\ge U} h(\alpha)\, d\alpha \\ &\ll \Big(N^{-2r} \sum_{l=L}^M 2^{-l(1-8r/p)} + 2^{-M}\Big) \min(1, U^{-(k-r)}) \\ &\ll \Big(N^{-2r} 2^{-L(1-8r/p)} + 2^{-M}\Big) \min(1, U^{-(k-r)}) \\ &\ll N^{-2r} \gamma^{1-8r/p} \min(1, U^{-(k-r)})\,.\end{aligned}$$

$\square$

## 5. Proof of Theorem 1.1

We apply a variant of the Davenport-Heilbronn circle method to count weighted solutions of (1.1). Without loss of generality we may assume $\epsilon = 1$. Otherwise apply Theorem 1.1 to the forms $\epsilon^{-1} Q_i$. We choose an even probability density $\chi$ with support in $[-1,1]$ and $\chi(x) \ge 1/2$ for $|x| \le 1/2$. By choosing $\chi$ sufficiently smooth we may assume that its Fourier transform satisfies $\widehat{\chi}(t) = \int \chi(x) e(tx)\, dx \ll (1+|t|)^{-r-3}$. Set

$$K(v_1, \dots, v_r) = \prod_{i=1}^r \chi(v_i)\,.$$

Then $\widehat{K}(\alpha) = \prod_{i=1}^{r} \widehat{\chi}(\alpha_i)$. By Fourier inversion we obtain for an integer parameter $N \geq 1$

$$\begin{aligned} A(N) &:= \sum_{x \in \mathbb{Z}^s} w_N(x) K(Q_1(x), \dots, Q_r(x)) \\ &= \sum_{x \in \mathbb{Z}^s} w_N(x) \int_{\mathbb{R}^r} e(\alpha_1 Q_1(x) + \dots + \alpha_r Q_r(x)) \widehat{K}(\alpha)\, d\alpha_1 \dots d\alpha_r \\ &= \int_{\mathbb{R}^r} S_N(\alpha) \widehat{K}(\alpha)\, d\alpha\,. \end{aligned}$$

Our aim is to prove for $N \geq N_0$, say,

$$A(N) \geq cN^{-2r} \tag{5.1}$$

with some constant $c > 0$. This certainly implies the existence of a non-trivial solution of (1.1), since the contribution of the trivial solution $x = 0$ to $A(N)$ is $\ll N^{-s}$ and $s \geq p > 8r$. To prove (5.1) we divide $\mathbb{R}^r$ in a major arc, a minor arc and a trivial arc. For $\delta > 0$ set

$$\begin{aligned} \mathfrak{M} &= \{\alpha \in \mathbb{R}^r \mid |\alpha| < N^{\delta-2}\}\,, \\ \mathfrak{m} &= \{\alpha \in \mathbb{R}^r \mid N^{\delta-2} \leq |\alpha| \leq T_1(N)\}\,, \\ \mathfrak{t} &= \{\alpha \in \mathbb{R}^r \mid |\alpha| > T_1(N)\}\,, \end{aligned}$$

where $T_1(N)$ denotes the function of Lemma 3.3. Using the bound $\widehat{K}(\alpha) \ll (1 + |\alpha|)^{-r-3}$, Lemma 4.1 (with the choice $U = T_1(N)$ and the trivial estimate $\gamma(T_1(N), \infty) \leq 1$) implies

$$\int_{\mathfrak{t}} S_N(\alpha) \widehat{K}(\alpha)\, d\alpha = O(N^{-2r} T_1(N)^{-3}) = o(N^{-2r})\,.$$

Furthermore, Lemma 4.1 with $U = N^{\delta-2}$ and $T = T_1(N)$, together with Lemma 3.3 yield

$$\int_{\mathfrak{m}} S_N(\alpha) \widehat{K}(\alpha)\, d\alpha = O(N^{-2r} \gamma(N^{\delta-2}, T_1(N))^{1-\frac{8r}{p}}) = o(N^{-2r})\,.$$

Thus (5.1) follows if we can prove that the contribution of the major arc is

$$\int_{\mathfrak{M}} S_N(\alpha) \widehat{K}(\alpha)\, d\alpha \gg N^{-2r}\,. \tag{5.2}$$

## 6. The major arc

**Lemma 6.1.** *Assume that each form in the real pencil of $Q_1, \dots, Q_r$ has rank $\geq p$. Let $g, h : \mathbb{R}^s \to \mathbb{C}$ be measurable functions with $|g| \leq 1$ and $|h| \leq 1$. Then for $N \geq 1$*

$$N^{-2s} \int_{[-N,N]^s} \int_{[-N,N]^s} g(x) h(y) e(\langle x, Q_\alpha y \rangle)\, dx\, dy \ll (|\alpha|^{-1/2} N^{-1})^p\,.$$

*Proof.* Note that the bound is trivial for $|\alpha| \le N^{-2}$. Hence we assume $|\alpha| \ge N^{-2}$. Denote by $\lambda_1, \dots, \lambda_s$ the eigenvalues of $Q_\alpha$ ordered in such a way that $|\lambda_1| \ge \dots \ge |\lambda_s|$. Then $Q_\alpha = U^T \Lambda U$, where $U$ is orthogonal and $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_s)$. Write $x = (\underline{x}, \overline{x})$, where $\underline{x} = (x_1, \dots, x_p)$ and $\overline{x} = (x_{p+1}, \dots, x_s)$. Then

$$\begin{aligned} &N^{-2s} \int_{[-N,N]^s} \int_{[-N,N]^s} g(x)h(y)e(\langle x, Q_\alpha y\rangle)\,dx\,dy \\ &= N^{-2s} \int_{U[-N,N]^s} \int_{U[-N,N]^s} g(U^{-1}x)h(U^{-1}y)e(\langle x, \Lambda y\rangle)\,dx\,dy \\ &= N^{-2(s-p)} \int_{\substack{|\overline{x}|_\infty \le \sqrt{s}N \\ |\overline{y}|_\infty \le \sqrt{s}N}} e\Big(\sum_{i=p+1}^{s} \lambda_i x_i y_i\Big) J(\overline{x}, \overline{y})\,d\overline{x}\,d\overline{y}\,, \end{aligned} \tag{6.1}$$

where

$$J(\overline{x}, \overline{y}) = N^{-2p} \int_{[-\sqrt{s}N, \sqrt{s}N]^p} \int_{[-\sqrt{s}N, \sqrt{s}N]^p} \tilde{g}(\underline{x})\tilde{h}(\underline{y}) e\Big(\sum_{i=1}^{p} \lambda_i x_i y_i\Big) d\underline{x}\,d\underline{y}\,.$$

Here $\tilde{g}(\underline{x}) = g(U^{-1}x) I_{A(\overline{x})}(\underline{x})$ with

$$A(\overline{x}) = \{\underline{x} \in \mathbb{R}^p \mid (\underline{x}, \overline{x}) \in U[-N,N]^s\} \subseteq [-\sqrt{s}N, \sqrt{s}N]^p\,,$$

and $\tilde{h}$ is defined similarly. If $|\alpha| \ge N^{-2}$ then by (2.6) $|\lambda_i| \asymp |\alpha| \gg N^{-2}$ for $i \le p$. Now we apply the double large sieve bound (2.3). For $1 \le j \le p$ set $S_j = T_j = \sqrt{s|\lambda_j|}N$. Let $\mu = \nu$ be the continuous uniform probability distribution on $\prod_{j=1}^{p}[-T_j, T_j]$ and set $\bar{g}(\underline{x}) = \tilde{g}(|\lambda_1|^{-1/2}x_1, \dots, |\lambda_p|^{-1/2}x_p)$ and $\bar{h}(\underline{x}) = \tilde{h}(\operatorname{sgn}(\lambda_1)|\lambda_1|^{-1/2}x_1, \dots, \operatorname{sgn}(\lambda_p)|\lambda_p|^{-1/2}x_p)$. Then

$$\begin{aligned} |J(\overline{x}, \overline{y})|^2 &\ll \Big| \int\int \bar{g}(\underline{x})\bar{h}(\underline{y})\,d\mu(\underline{x})\,d\nu(\underline{y}) \Big|^2 \\ &\ll \prod_{j=1}^{p} (1 + |\lambda_j| N^2)(|\lambda_j|^{-1} N^{-2})^2 \\ &\ll |\alpha|^{-p} N^{-2p}\,. \end{aligned}$$

Together with (6.1) this proves the lemma. $\square$

For $\alpha \in \mathfrak{M}$ we want to approximate $S_N(\alpha)$ by

$$G_0(\alpha) = \int \sum_{x \in \mathbb{Z}^s} w_N(x) e(Q_\alpha(x+z))\,d\pi(z)\,, \tag{6.2}$$

where $\pi = I_B * I_B * I_B * I_B$ is the fourfold convolution of the continuous uniform distribution on $B = (-1/2, 1/2]^s$. Set $g(u) = e(Q_\alpha(u))$. Denote by

$g_{u_1}$ the directional derivative of $g$ in direction $u_1$, and set $g_{u_1u_2} = (g_{u_1})_{u_2}$. We use the Taylor series expansions

$$
\begin{aligned}
f(1) &= f(0) + \textstyle\int_0^1 f'(\tau)\, d\tau\,,\\
f(1) &= f(0) + f'(0) + \textstyle\int_0^1 (1-\tau) f''(\tau)\, d\tau\,,\\
f(1) &= f(0) + f'(0) + \tfrac{1}{2} f''(0) + \tfrac{1}{2}\textstyle\int_0^1 (1-\tau)^2 f'''(\tau)\, d\tau\,.
\end{aligned}
$$

Applying the third of these relations to $f(\tau) = g(x+\tau u_1)$, the second to $f(\tau) = g_{u_1}(x+\tau u_2)$ and the first to $f(\tau) = g_{u_1u_i}(x+\tau u_3)$ we find for $u_1, u_2, u_3 \in \mathbb{R}^s$

$$
\begin{aligned}
g(x+u_1) &= g(x) + g_{u_1}(x) + \tfrac{1}{2} g_{u_1u_1}(x) + \tfrac{1}{2}\textstyle\int_0^1 (1-\tau)^2 g_{u_1u_1u_1}(x+\tau u_1) d\tau\,,\\
g_{u_1}(x+u_2) &= g_{u_1}(x) + g_{u_1u_2}(x) + \textstyle\int_0^1 (1-\tau) g_{u_1u_2u_2}(x+\tau u_2) d\tau\,,\\
g_{u_1u_i}(x+u_3) &= g_{u_1u_i}(x) + \textstyle\int_0^1 g_{u_1u_iu_3}(x+\tau u_3) d\tau\,.
\end{aligned}
$$

Together we obtain the expansion

$$
\begin{aligned}
g(x) = g(x+u_1) &- g_{u_1}(x+u_2) - \frac{1}{2} g_{u_1u_1}(x+u_3) + g_{u_1u_2}(x+u_3)\\
&+ \int_0^1 \big\{ - g_{u_1u_2u_3}(x+\tau u_3) + \frac{1}{2} g_{u_1u_1u_3}(x+\tau u_3)\\
&+ (1-\tau) g_{u_1u_2u_2}(x+\tau u_2) - \frac{1}{2}(1-\tau)^2 g_{u_1u_1u_1}(x+\tau u_1)\big\} d\tau\,.
\end{aligned}
$$

Multiplying with $w_N(x)$, summing over $x \in \mathbb{Z}^s$, and integrating $u_1, u_2, u_3$ with respect to the probability measure $\pi$ yields

$$
S_N(\alpha) = G_0(\alpha) + G_1(\alpha) + G_2(\alpha) + G_3(\alpha) + R(\alpha)\,,
$$

where $G_0(\alpha)$ is defined by (6.2),

$$
\begin{aligned}
G_1(\alpha) &= -\int\int \sum_{x\in\mathbb{Z}^s} w_N(x) g_u(x+z)\, d\pi(u)\, d\pi(z)\,,\\
G_2(\alpha) &= -\frac{1}{2}\int\int \sum_{x\in\mathbb{Z}^s} w_N(x) g_{uu}(x+z)\, d\pi(u)\, d\pi(z)\,,\\
G_3(\alpha) &= \int\int\int \sum_{x\in\mathbb{Z}^s} w_N(x) g_{uv}(x+z)\, d\pi(u)\, d\pi(v)\, d\pi(z)\,,
\end{aligned}
$$

and

$$
R(\alpha) \ll \sup_{|u|_\infty, |v|_\infty, |w|_\infty, |z|_\infty \le 1} \Big| \sum_{x\in\mathbb{Z}^s} w_N(x) g_{uvw}(x+z) \Big|\,.
$$

An elementary calculation yields

$$\begin{aligned} g_u(x) &= 4\pi i\, e(Q_\alpha(x))\langle x, Q_\alpha u\rangle\,, \\ g_{uv}(x) &= (4\pi i)^2 e(Q_\alpha(x))\langle x, Q_\alpha u\rangle\langle x, Q_\alpha v\rangle + 4\pi i\, e(Q_\alpha(x))\langle u, Q_\alpha v\rangle\,, \\ g_{uvw}(x) &= (4\pi i)^3 e(Q_\alpha(x))\langle x, Q_\alpha u\rangle\langle x, Q_\alpha v\rangle\langle x, Q_\alpha w\rangle + (4\pi i)^2 e(Q_\alpha(x))\times \\ &\qquad \Big(\langle x, Q_\alpha v\rangle\langle u, Q_\alpha w\rangle + \langle x, Q_\alpha u\rangle\langle v, Q_\alpha w\rangle + \langle x, Q_\alpha w\rangle\langle u, Q_\alpha v\rangle\Big)\,. \end{aligned}$$

Since $g_u$ and $g_{uv}$ are sums of odd functions (in at least one of the components of $u$) we infer $G_1(\alpha) = 0$ and $G_3(\alpha) = 0$. Furthermore, the trivial bound $g_{uvw}(x) \ll |\alpha|^3 N^3 + |\alpha|^2 N$ for $|x|_\infty \ll N$ yields

$$R(\alpha) \ll |\alpha|^3 N^3 + |\alpha|^2 N\,.$$

This is sharp enough to prove

$$\begin{aligned} \int_{\mathfrak{M}} |R(\alpha)\widehat{K}(\alpha)|\, d\alpha &\ll \int_{|\alpha|\le N^{\delta-2}} |\alpha|^3 N^3 + |\alpha|^2 N\, d\alpha \\ &\ll \int_0^{N^{\delta-2}} u^{r+2} N^3 + u^{r+1} N\, du \\ &\ll N^{3-(2-\delta)(r+3)} + N^{1-(2-\delta)(r+2)} \\ &\ll N^{-2r-3+\delta(r+3)} = o(N^{-2r})\,. \end{aligned}$$

To deal with $G_0$ and $G_2$ we need a bound for

$$\widetilde{G}_j(\alpha, u) = \int_{\mathbb{R}^s} \sum_{x\in\mathbb{Z}^s} w_N(x) L(x+z)^j e(Q_\alpha(x+z))\, d\pi(z)\,,$$

where $L(x) = \langle x, Q_\alpha u\rangle$ and $0 \le j \le 2$. Using the definition of $w_N$ and $\pi$ we find that $\widetilde{G}_j(\alpha, u)$ is equal to

$$\begin{aligned} &\int_{B^4} \sum_{x_1,\ldots,x_4\in\mathbb{Z}^s} \prod_{i=1}^4 p_N(x_i) L\Big(\sum_{i=1}^4 (x_i+z_i)\Big)^j e\Big(Q_\alpha\Big(\sum_{i=1}^4 (x_i+z_i)\Big)\Big) dz_1\ldots dz_4 \\ &= (2N+1)^{-4s} \int_{|x_1|_\infty,\ldots,|x_4|_\infty \le N+1/2} L\Big(\sum_{i=1}^4 x_i\Big)^j e\Big(Q_\alpha(\sum_{i=1}^4 x_i)\Big)\, dx_1\ldots dx_4. \end{aligned}$$

Expanding $L(x_1+x_2+x_3+x_4)$ and $Q_\alpha(x_1+x_2+x_3+x_4)$ this can be bounded by

$$\begin{aligned} &\max_{l_1+l_2+l_3+l_4=j} N^{-4s} \left| \int \Big\{\prod_{i=1}^4 L(x_i)^{l_i} e(Q_\alpha(x_i))\Big\} e\big(2\sum_{i<j}\langle x_i, Q_\alpha x_j\rangle\big) dx_1 \ldots dx_4 \right| \\ &\ll \max_{l_1+l_2+l_3+l_4=j} \frac{(|\alpha|N)^j}{N^{4s}} \left| \int \Big\{ \prod_{i=1}^4 h_i(x_i) \Big\} e(2\sum_{i<j}\langle x_i, Q_\alpha x_j\rangle) dx_1 \ldots dx_4 \right|. \end{aligned}$$

Here

$$h_i(x_i) = L(x_i)^{l_i} e(Q_\alpha(x_i))(|\alpha|N)^{-l_i} I_{\{|x_i|\le N+1/2\}} \ll 1 \,.$$

Applying Lemma 6.1 to the double integral over $x_1$ and $x_2$ and estimating the integral over $x_3$ and $x_4$ trivially we obtain uniformly in $|u| \ll 1$

$$\widetilde{G}_j(\alpha, u) \ll (|\alpha|N)^j |\alpha|^{-p/2} N^{-p} \,.$$

Setting

$$H_j(N) = \int_{\mathbb{R}^r} G_j(\alpha)\widehat{K}(\alpha)\, d\alpha$$

we conclude for sufficiently small $\delta > 0$ and $p > 8r$ $(G_0(\alpha) = \widetilde{G}_0(\alpha, 0))$

$$\begin{aligned}
\int_{\mathfrak{M}} G_0(\alpha)\widehat{K}(\alpha)\, d\alpha &= H_0(N) - \int_{|\alpha|\ge N^{\delta-2}} \widetilde{G}_0(\alpha,0)\widehat{K}(\alpha)\, d\alpha \\
&= H_0(N) + O(N^{-p}(\int_{N^{\delta-2}\le|\alpha|\le 1} |\alpha|^{-p/2}\, d\alpha + 1)) \\
&= H_0(N) + O(N^{-p-(2-\delta)(r-p/2)}) + O(N^{-p}) \\
&= H_0(N) + o(N^{-2r}) \,.
\end{aligned}$$

Similarly, the explicit expression of $g_{uu}(x)$ and the definition of $\tilde{G}_j(\alpha, u)$ yield

$$\begin{aligned}
&\int_{\mathfrak{M}} G_2(\alpha)\widehat{K}(\alpha)\, d\alpha \\
&\quad = H_2(N) + O\Big(\sup_{|u|_\infty\le 2} \int_{|\alpha|\ge N^{\delta-2}} |\widetilde{G}_2(\alpha,u)\widehat{K}(\alpha)| + |\alpha||\tilde{G}_0(\alpha,u)\widehat{K}(\alpha)| d\alpha\Big) \\
&\quad = H_2(N) + O\Big(N^{2-p}(\int_{N^{\delta-2}\le|\alpha|\le 1} |\alpha|^{2-p/2}\, d\alpha + 1)\Big) \\
&\qquad\qquad + O\Big(N^{-p}(\int_{N^{\delta-2}\le|\alpha|\le 1} |\alpha|^{1-p/2}\, d\alpha + 1)\Big) \\
&\quad = H_2(N) + o(N^{-2r}) \,.
\end{aligned}$$

Hence

$$\int_{\mathfrak{M}} S_N(\alpha)\widehat{K}(\alpha)\, d\alpha = H_0(N) + H_2(N) + o(N^{-2r}) \,.$$

Altogether we have proved that for $p > 8r$

$$A(N) = H_0(N) + H_2(N) + o(N^{-2r}) \,. \tag{6.3}$$

## 7. Analysis of the terms $H_0(N)$ and $H_2(N)$

**Lemma 7.1.** *Denote by $\pi_N$ the fourfold convolution of the continuous uniform probability distribution on $B_N = (-N-1/2, N+1/2]^s$ and by $f_N$ the density of $\pi_N$. Then*

$$H_0(N) = \int K(Q_1(x), \dots, Q_r(x)) f_N(x)\, dx$$

*and*

$$H_2(N) = -\frac{1}{6}\int K(Q_1(x), \dots, Q_r(x)) \Delta f_N(x)\, dx\,,$$

*where $\Delta f_N(x) = \sum_{i=1}^s \frac{\partial^2 f_N}{\partial x_i^2}(x)$. Furthermore, $\Delta f_N(x) \ll N^{-s-2}$.*

*Proof.* By Fourier inversion and the definition of $w_N$ and $\pi = \pi_0$ we find

$$\begin{aligned}
H_0(N) &= \int_{\mathbb{R}^r} G_0(\alpha)\widehat{K}(\alpha)\, d\alpha \\
&= \int \sum_{x\in\mathbb{Z}^s} w_N(x) \int_{\mathbb{R}^r} e(Q_\alpha(x+z))\widehat{K}(\alpha)\, d\alpha\, d\pi(z) \\
&= \int \sum_{x\in\mathbb{Z}^s} w_N(x) K(Q_1(x+z), \dots, Q_r(x+z))\, d\pi(z) \\
&= \int K(Q_1(x), \dots, Q_r(x))\, d\pi_N(x)\,.
\end{aligned}$$

This proves the first assertion of the Lemma. Similarly,

$$-2G_2(\alpha) = \int\!\!\int g_{uu}(x)\, d\pi(u) d\pi_N(x)\,.$$

This implies

$$-2H_2(N) = -2\int G_2(\alpha)\widehat{K}(\alpha) d\alpha = \int\!\!\int\!\!\int_{\mathbb{R}^r} g_{uu}(x)\widehat{K}(\alpha) d\alpha\, d\pi(u) d\pi_N(x)\,.$$

With the abbreviations $L_m = 2\langle x, Q_m u\rangle$ and $\widetilde{L}_m = 2\langle u, Q_m v\rangle$ the innermost integral can be calculated as

$$\begin{aligned}
&\int_{\mathbb{R}^r} g_{uu}(x)\widehat{K}(\alpha)\, d\alpha \\
&\quad= \int_{\mathbb{R}^r} e(Q_\alpha(x)) \left\{ \sum_{m,n=1}^r L_m L_n \widehat{\frac{\partial^2 K}{\partial v_m \partial v_n}}(\alpha) + \sum_{m=1}^r \widetilde{L}_m \widehat{\frac{\partial K}{\partial v_m}}(\alpha) \right\} d\alpha \\
&\quad= \sum_{m,n=1}^r L_m L_n \frac{\partial^2 K}{\partial v_m \partial v_n}(Q_1(x), \dots, Q_r(x)) + \sum_{m=1}^r \widetilde{L}_m \frac{\partial K}{\partial v_m}(Q_1(x), \dots, Q_r(x))\,.
\end{aligned}$$

Here we used the relations

$$\widehat{\frac{\partial K}{\partial v_m}}(\alpha) = 2\pi i \ \alpha_m \widehat{K}(\alpha)\,,$$
$$\widehat{\frac{\partial^2 K}{\partial v_m \partial v_n}}(\alpha) = (2\pi i)^2 \alpha_m \alpha_n \widehat{K}(\alpha)\,.$$

Since

$$\sum_{i,j=1}^{s} u_i u_j \frac{\partial^2}{\partial x_i \partial x_j}(K(Q_1(x),\ldots,Q_r(x)))$$
$$= \sum_{m,n=1}^{r} L_m L_n \frac{\partial^2 K}{\partial v_m \partial v_n}(Q_1(x),\ldots,Q_r(x)) + \sum_{m=1}^{r} \widetilde{L}_m \frac{\partial K}{\partial v_m}(Q_1(x),\ldots,Q_r(x))$$

we find

$$\int_{\mathbb{R}^r} g_{uu}(x)\widehat{K}(\alpha)\,d\alpha = \sum_{i,j=1}^{s} u_i u_j \frac{\partial^2}{\partial x_i \partial x_j}(K(Q_1(x),\ldots,Q_r(x))\,.$$

Altogether we conclude

$$\begin{aligned}
-2H_2(N) &= \int\!\!\int \sum_{i,j=1}^{s} u_i u_j \frac{\partial^2}{\partial x_i \partial x_j}(K(Q_1(x),\ldots,Q_r(x)))\,d\pi(u)\,d\pi_N(x)\\
&= \sum_{i=1}^{s}\int\!\!\int u_i^2 \frac{\partial^2}{\partial x_i^2}(K(Q_1(x),\ldots,Q_r(x)))\,d\pi(u)\,d\pi_N(x)\\
&= \left(\int u_1^2\,d\pi(u)\right)\sum_{i=1}^{s}\int \frac{\partial^2}{\partial x_i^2}(K(Q_1(x),\ldots,Q_r(x))\,d\pi_N(x)\,.
\end{aligned}$$

Since $\pi_N$ has compact support and $f_N$ is two times continuously differentiable, partial integration yields

$$\int \frac{\partial^2}{\partial x_i^2}(K(Q_1(x),\ldots,Q_r(x))f_N(x)\,dx = \int K(Q_1(x),\ldots,Q_r(x))\frac{\partial^2 f_N}{\partial x_i}(x)\,dx\,.$$

This completes the proof of the second assertion of the Lemma, since $\int u_1^2\,d\pi(u) = 1/3$.

Finally, we prove

$$\frac{\partial^2 f_N}{\partial x_i^2}(x) \ll N^{-s-2}\,.$$

Note that

$$\widehat{f_N}(t) = \prod_{i=1}^{s}\left(\frac{\sin(\pi t_i(2N+1))}{\pi t_i(2N+1)}\right)^4 = \widehat{f_0}((2N+1)t)\,.$$

Hence, by Fourier inversion

$$\begin{aligned}\frac{\partial^2 f_N}{\partial x_i^2}(x) &= (-2\pi i)^2 \int \widehat{f_N}(t) t_i^2 e(-\langle t, x\rangle)\, dt \\ &= -(2\pi)^2 (2N+1)^{-s-2} \int \widehat{f}_0(t) t_i^2 e(-(2N+1)\langle t, x\rangle)\, dt \\ &\ll N^{-s-2}\,.\end{aligned}$$

This completes the proof of Lemma 7.1. We remark that we used the fourfold convolution in the definition of $w_N$, $\pi_N$, $f_N$ for the above treatment of $H_2(N)$ only. At all other places of the argument a twofold convolution would be sufficient for our purpose. □

**Lemma 7.2.** *Assume that the system $Q_1(x) = 0, \dots, Q_r(x) = 0$ has a nonsingular real solution, then*

$$\lambda(\{x \in \mathbb{R}^s \mid |Q_i(x)| \le N^{-2}, |x|_\infty \le 1\}) \gg N^{-2r}\,,$$

*where $\lambda$ denotes the $s$-dimensional Lebesgue measure.*

*Proof.* This is proved in Lemma 2 of [10]. Note that if a system of homogeneous equations $Q_1(x) = 0, \dots, Q_r(x) = 0$ has a nonsingular real solution, then it has a nonsingular real solution with $|x|_\infty \le 1/2$.

Now we complete the proof of Theorem 1.1 as follows. For $c > 0$ and $N > 0$ set

$$A(c, N) = \lambda(\{x \in \mathbb{R}^s \mid |Q_i(x)| \le N^{-2}, |x|_\infty \le c\})\,.$$

Then

$$A(c, N) = c^s A(1, cN)\,.$$

By Lemma 7.1

$$\begin{aligned}H_0(N) &\gg N^{-s} \int_{|x|_\infty \le 2N} K(Q_1(x), \dots, Q_r(x))\, dx \\ &\gg \int_{|y|_\infty \le 2} K(N^2 Q_1(y), \dots, N^2 Q_r(y))\, dy \\ &\gg A(2, 2N) \\ &\gg A(1, 5N)\end{aligned}$$

and

$$\begin{aligned} H_2(N) &\ll N^{-s-2} \int_{|x|_\infty \le 5N} K(Q_1(x), \dots, Q_r(x))\, dx \\ &\ll N^{-2} \int_{|y|_\infty \le 5} K(N^2 Q_1(y), \dots, N^2 Q_r(y))\, dy \\ &\ll N^{-2} A(5, N) \\ &\ll N^{-2} A(1, 5N)\,. \end{aligned}$$

With Lemma 7.2 this yields

$$H_0(N) + H_2(N) \gg A(1, 5N) \gg N^{-2r}$$

for $N \ge N_0$, say. Together with (6.3) this completes the proof of Theorem 1.1. □

## References

[1] V. Bentkus, F. Götze, *On the lattice point problem for ellipsoids.* Acta Arith. **80** (1997), 101–125.
[2] V. Bentkus, F. Götze, *Lattice point problems and distribution of values of quadratic forms.* Annales of Mathematics **150** (1999), 977–1027.
[3] E. Bombieri, H. Iwaniec, *On the order of* $\zeta(1/2 + it)$. Ann. Scuola Norm Sup. Pisa (4) **13** (1996), 449–472.
[4] J. Brüdern, R.J. Cook, *On simultaneous diagonal equations and inequalities.* Acta Arith. **62** (1992), 125–149.
[5] H. Davenport, *Indefinite quadratic forms in many variables (II).* Proc. London Math. Soc. **8** (1958), 109–126.
[6] R. J. Cook, *Simultaneous quadratic equations.* Journal London Math. Soc. (2) **4** (1971), 319–326.
[7] R. Dietmann, *Systems of rational quadratic forms.* Arch. Math. (Basel) **82** (2004), no. 6, 507–516.
[8] E. D. Freeman, *Quadratic Diophantine Inequalities.* Journal of Number Theory **89** (2001), 269–307.
[9] G. A. Margulis, *Discrete subgroups and ergodic theory.* In "Number Theory, Trace Fromulas and Discrete Groups (Oslo, 1987)", 377–398, Academic Press, Boston, 1989.
[10] W. M. Schmidt, *Simultaneous rational zeros of quadratic forms.* Séminaire Delange-Pisot-Poitou (Théorie des Nombres), Paris 1980-1981, Progr. Math. **22** (1982), 281–307.
[11] H. P. F. Swinnerton-Dyer, *Rational zeros of two quadratic forms.* Acta Arith. **9** (1964), 261–270.

Wolfgang Müller
Institut für Statistik
Technische Universität Graz
8010 Graz, Austria
*E-mail* : `w.mueller@tugraz.at`