Journal de Théorie des Nombres
de Bordeaux **19** (2007), 763–797

Local ε_0-characters in torsion rings

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Résumé. Soit p un nombre premier et K un corps, complet pour une valuation discrète, à corps résiduel de caractéritique positive p. Dans le cas où k est fini, généralisant la théorie de Deligne [1], on construit dans [10] et [11] une théorie des ε_0-constantes locales pour les représentations, sur un anneau local complet à corps résiduel algébriquement clos de caractéristique $\neq p$, du groupe de Weil W_K de K. Dans cet article, on généralise les résultats de [10] et [11] au cas où k est un corps arbitraire parfait.

Abstract. Let p be a rational prime and K a complete discrete valuation field with residue field k of positive characteristic p. When k is finite, generalizing the theory of Deligne [1], we construct in [10] and [11] a theory of local ε_0-constants for representations, over a complete local ring with an algebraically closed residue field of characteristic $\neq p$, of the Weil group W_K of K. In this paper, we generalize the results in [10] and [11] to the case where k is an arbitrary perfect field.

1. Introduction

Let K be a complete discrete valuation field whose residue field k is of characteristic p. When k is a finite field, the author defines in [10] local ε_0-constants $\varepsilon_{0,R}(V,\psi)$ for a triple $(R,(\rho,V),\psi)$ where R is a strict p'-coefficient ring (see Section 2 for the definition), (ρ,V) is an object in $\mathrm{Rep}(W_K,R)$, and $\psi : K \to R^\times$ is a non-trivial continuous additive character. In [10] the author proved several properties including the formula for induced representations. In the present paper, we generalize the results of two papers [10] and [11] to the case where k is an arbitrary perfect field of characteristic p. More precisely, we define an object $\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi})$ in $\mathrm{Rep}(W_k,\widetilde{\psi})$ of rank one (where W_k is a dense subgroup of the absolute Galois group of k defined in 3.1), called the *local ε_0-character*, for any triple $(R,(\rho,V),\widetilde{\psi})$ where R is a strict p'-coefficient ring, (ρ,V) an object in $\mathrm{Rep}(W_K,R)$ and $\widetilde{\psi}$ is a non-trivial invertible additive character sheaf on K. When k is finite of order q, this $\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi})$ and the local ε_0-constants

Manuscrit reçu le 12 mai 2006.

$\varepsilon_{0,R}(V,\psi)$ are related by

$$\mathrm{Tr}(\mathrm{Fr}_q\,;\,\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi})) = (-1)^{\mathrm{rank}\,V+\mathrm{sw}(V)}\varepsilon_{0,R}(V,\psi),$$

where $\widetilde{\psi}$ is the invertible character sheaf associated to ψ.

We generalize the properties of local ε_0-constants stated in [10] to those of local $\widetilde{\varepsilon}_0$-characters by using the specialization argument. We also prove the product formula which describes the determinant of the etale cohomology of a R_0-sheaf on a curve over a perfect field k as a tensor product of local $\widetilde{\varepsilon}_0$-characters.

2. Notation

Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers respectively.

Let $\mathbb{Z}_{>0}$ (resp. $\mathbb{Z}_{\geq 0}$) be the ordered set of positive (resp. non-negative) integers. We also define $\mathbb{Q}_{\geq 0}$, $\mathbb{Q}_{>0}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ in a similar way. For $\alpha \in \mathbb{R}$, let $\lfloor \alpha \rfloor$ (resp. $\lceil \alpha \rceil$) denote the maximum integer not larger than α (resp. the minimum integer not smaller than α).

For a prime number ℓ, let $\mathbb{F}_\ell$ denote the finite field of ℓ elements, $\mathbb{F}_{\ell^n}$ the unique extension of $\mathbb{F}_\ell$ of degree n for $n \in \mathbb{Z}_{>0}$, $\overline{\mathbb{F}}_\ell$ the algebraic closure of $\mathbb{F}_\ell$, $\mathbb{Z}_\ell = W(\mathbb{F}_\ell)$ (resp. $W(\overline{\mathbb{F}}_\ell)$) the ring of Witt vectors of $\mathbb{F}_\ell$ (resp. $\overline{\mathbb{F}}_\ell$), $\mathbb{Q}_\ell = \mathrm{Frac}(\mathbb{Z}_\ell)$) the field of fractions of $\mathbb{Z}_\ell$. Let $\varphi : W(\overline{\mathbb{F}}_\ell) \to W(\overline{\mathbb{F}}_\ell)$ be the Frobenius automorphism of $W(\overline{\mathbb{F}}_\ell)$.

For a ring R, let $R^\times$ denote the group of units in R. For a positive integer $n \in \mathbb{Z}_{>0}$, let $\boldsymbol{\mu}_n(R)$ denote the group of n-th roots of unity in R, $\boldsymbol{\mu}_{n^\infty}(R)$ denotes the union $\cup_i \boldsymbol{\mu}_{n^i}(R)$.

For a finite extension L/K of fields, let $[L:K]$ denote the degree of L over K. For a subgroup H of a group G of finite index, its index is denoted by $[G:H]$.

For a finite field k of characteristic $\neq 2$, let $\left(\frac{}{k}\right) : k^\times \to \{\pm 1\}$ denote the unique surjective homomorphism.

Throughout this paper, we fix once and for all a prime number p. We consider a complete discrete valuation field K whose residue field is perfect of characteristic p. We say such a field K is a *p-CDVF.* We sometimes consider a p-CDVF whose residue field is finite. We say such a field is a *p-local field.*

For a p-CDVF K, let $\mathcal{O}_K$ denote its ring of integers, $\mathfrak{m}_K$ the maximal ideal of $\mathcal{O}_K$, $k_K = \mathcal{O}_K/\mathfrak{m}_K$ the residue field of $\mathcal{O}_K$, and $v_K : K^\times \to \mathbb{Z}$ the normalized valuation. If K is a p-local field, we also denote by $(\ ,\ )_K : K^\times \times K^\times \to \{\pm 1\}$ the Hilbert symbol, by W_K the Weil group of K, by $\mathrm{rec} = \mathrm{rec}_K : K^\times \xrightarrow{\cong} W_K^{\mathrm{ab}}$ the reciprocity map of local class field theory, which sends a prime element of K to a lift of geometric Frobenius of k. If

L/K is a finite separable extension of p-CDVFs, let $e_{L/K} \in \mathbb{Z}$, $f_{L/K} \in \mathbb{Z}$, $D_{L/K} \in \mathcal{O}_L/\mathcal{O}_L^\times$, $d_{L/K} \in \mathcal{O}_K/\mathcal{O}_K^{\times 2}$ denotes the ramification index of L/K, the residual degree of L/K, the different of L/K, the discriminant of L/K respectively.

For a topological group (or more generally for a topological monoid) G and a commutative topological ring R, let $\mathrm{Rep}(G, R)$ denote the category whose objects are pairs (ρ, V) of a finitely generated free R-module V and a continuous group homomorphism $\rho : G \to GL_R(V)$ (we endow $GL_R(V)$ with the topology induced from the direct product topology of $\mathrm{End}_R(V)$), and whose morphisms are R-linear maps compatible with actions of G.

A sequence
$$0 \to (\rho', V') \to (\rho, V) \to (\rho'', V'') \to 0$$
of morphisms in $\mathrm{Rep}(G, R)$ is called a *short exact sequence* in $\mathrm{Rep}(G, R)$ if $0 \to V' \to V \to V'' \to 0$ is the short exact sequence of R-modules.

In this paper, a noetherian local ring with residue field of characteristic $\neq p$ is called a *p'-coefficient ring.* Any p'-coefficient ring $(R, \mathfrak{m}_R)$ is considered as a topological ring with the $\mathfrak{m}_R$-preadic topology. A *strict p'-coefficient ring* is a p'-coefficient ring R with an algebraically closed residue field such that $(R^\times)^p = R^\times$.

3. Review of basic facts

3.1. Ramification subgroups. Let K be a p-CDVF with a residue field k, and $\overline{K}$ (resp. $\overline{k}$) a separable closure of K (resp. k). Let k_0 be the algebraic closure of $\mathbb{F}_p$ in k. If k_0 is finite, define the *Weil group* $W_k \subset \mathrm{Gal}(\overline{k}/k)$ of k as the inverse image of $\mathbb{Z}$ under the canonical map
$$\mathrm{Gal}(\overline{k}/k) \to \mathrm{Gal}(\overline{k}_0/k_0) \xrightarrow{\cong} \widehat{\mathbb{Z}}.$$
If k_0 is infinite, we put $W_k = \mathrm{Gal}(\overline{k}/k)$. Define the *Weil group* $W_K \subset \mathrm{Gal}(\overline{K}/K)$ of K as the inverse image of W_k under the canonical map $\mathrm{Gal}(\overline{K}/K)$
$\to \mathrm{Gal}(\overline{k}/k)$. Let $G = W_K$ denote the Weil group of K. Put $G^v = G \cap \mathrm{Gal}(\overline{K}/K)^v$ and $G^{v+} = G \cap \mathrm{Gal}(\overline{K}/K)^{v+}$, where $\mathrm{Gal}(\overline{K}/K)^v$ and $\mathrm{Gal}(\overline{K}/K)^{v+}$ are the upper numbering ramification subgroups (see [9, IV, §3] for definition) of $\mathrm{Gal}(\overline{K}/K)$. The groups G^v, G^{v+} are called the *upper numbering ramification subgroups* of G. They have the following properties:

- G^v and G^{v+} are closed normal subgroups of G.
- $G^v \supset G^{v+} \supset G^w$ for every $v, w \in \mathbb{Q}_{\geq 0}$ with $w > v$.
- G^{v+} is equal to the closure of $\bigcup_{w>v} G^w$.
- $G^0 = I_K$, the inertia subgroup of W_K. $G^{0+} = P_K$, the wild inertia subgroup of W_K. In particular, G^w for $w > 0$ and G^{w+} for $w \geq 0$ are pro p-groups.
- For $w \in \mathbb{Q}$, $w > 0$, G^w/G^{w+} is an abelian group which is killed by p.

3.2. Herbrand's function $\psi_{L/K}$. Let L/K be a finite separable extension of a p-CDVF. Let $\psi_{L/K} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the Herbrand fuction (see [9, IV, §3] for definition) of L/K. The function $\psi_{L/K}$ has the following properties:

- $\psi_{L/K}$ is continuous, strictly increasing, piecewise linear, and convex function on $\mathbb{R}_{\geq 0}$.
- For sufficiently large w, $\psi_{L/K}(w)$ is linear with slope $e_{L/K}$.
- We have $\psi_{L/K}(0) = 0$.
- We have $\psi_{L/K}(\mathbb{Z}_{\geq 0}) \subset \mathbb{Z}_{\geq 0}$ and $\psi_{L/K}(\mathbb{Q}_{\geq 0}) = \mathbb{Q}_{\geq 0}$.
- Let $G = W_K$, $H = W_L$. Then for $w \in \mathbb{Q}_{\geq 0}$, we have $G^w \cap H = H^{\psi_{L/K}(w)}$ and $G^{w+} \cap H = H^{\psi_{L/K}(w)+}$.

3.3. Slope decomposition and refined slope decomposition. Let K be a p-CDVF, $G = W_K$ the Weil group of K. Let $(R, \mathfrak{m}_R)$ be a p'-coefficient ring.

Let V be an $R[G]$-module. We say that V is *tamely ramified* or *pure of slope* 0 if G^{0+} acts trivially on V. V is called *totally wild* if the G^{0+}-fixed part $V^{G^{0+}}$ is 0. For $v \in \mathbb{Q}_{>0}$, we say that V is *pure* of slope v if V^{G^v} is 0 and if G^{v+} acts trivially on V.

Let K^{tm} be the maximal tamely ramified extension of K in a fixed separable closure $\overline{K}$ of K. Let (ρ, V) be an object in $\mathrm{Rep}(G, R)$. Since G^{0+} is a pro-p group, there exists a finite Galois extension L of K^{tm} in $\overline{K}$ such that ρ factors through the quotient $W(L/K) = W_K/\mathrm{Gal}(\overline{K}/L)$ of W_K. Let $G(L/K)^v$ (resp. $G(L/K)^{v+}$) denotes the image of G^v (resp. G^{v+}) in $W(L/K)$.

Lemma 3.1. *There exists a finite number of rational numbers $v_1, \cdots, v_n \in \mathbb{Q}_{\geq 0}$ with $0 = v_1 < \cdots < v_n$ such that $G(L/K)^{v_i+} = G(L/K)^{v_{i+1}}$ for $1 \leq i \leq n-1$ and that $G(L/K)^{v_n} = \{1\}$.*

Proof. There exists a finite Galois extension L' of K contained in L such that the composite map $G(L/K)^{0+} \subset W(L/K) \twoheadrightarrow \mathrm{Gal}(L'/K)$ is injective. Then the image of $G(L/K)^v$, $G(L/K)^{v+}$ in $\mathrm{Gal}(L'/K)$ is equal to the upper numbering ramification subgroups $\mathrm{Gal}(L'/K)^v$, $\mathrm{Gal}(L'/K)^{v+}$ of $\mathrm{Gal}(L'/K)$ respectively. Hence the lemma follows. □

Corollary 3.2. *Let (ρ, V) be an object in $\mathrm{Rep}(G, R)$. Then for any $v \in \mathbb{Q}_{\geq 0}$, there exists a unique maximal sub $R[G]$-module V^v of V which is pure of slope v. $V^v = \{0\}$ except for a finite number of v and we have*

$$V = \bigoplus_{v \in \mathbb{Q}_{\geq 0}} V^v.$$

□

For $v \in \mathbb{Q}_{\geq 0}$, the object in $\mathrm{Rep}(G, R)$ defined by V^v is called the *slope v-part* of (ρ, V).

$V \mapsto V^v$ define a functor from $\mathrm{Rep}(G, R)$ to itself which preserves short exact sequences. These functors commute with base changes by $R \to R'$.

Definition 3.3. Let (ρ, V) be an object in $\mathrm{Rep}(G, R)$, $V = \bigoplus_{v \in \mathbb{Q}_{\geq 0}} V^v$ its slope decomposition. We define the *Swan conductor* $sw(V)$ of V by

$$\mathrm{sw}(V) = \sum_{v \in \mathbb{Q}_{\geq 0}} v \cdot \mathrm{rank}\, V^v.$$

Lemma 3.4. $\mathrm{sw}(V) \in \mathbb{Z}$.

Proof. Since $\mathrm{sw}(V) = \mathrm{sw}(V \otimes_R R/\mathfrak{m}_R)$, we may assume that R is a field. Then the lemma is classical. $\square$

Assume further that R contains a primitive p-th root of unity. Let (ρ, V) be an object in $\mathrm{Rep}(G, R)$. Let $v \in \mathbb{Q}_{>0}$ and let V^v denote the slope v part of (ρ, V). We have a decomposition

$$V^v = \bigoplus_{\chi \in \mathrm{Hom}(G^v/G^{v+}, R^\times)} V_\chi$$

of V^v by the sub $R[G^v/G^{v+}]$-modules V_χ on which G^v/G^{v+} acts by χ. The group G acts on the set $\mathrm{Hom}(G^v/G^{v+}, R^\times)$ by conjugation : $(g.\chi)(h) = \chi(g^{-1}hg)$. The action of $g \in G$ on V induces an R-linear isomorphism $V_\chi \xrightarrow{\cong} V_{g.\chi}$. Let X^v denote the set of G-orbits in the G-set $\mathrm{Hom}(G^v/G^{v+}, R^\times)$. Then for any $\Sigma \in X^v$,

$$V^\Sigma = \bigoplus_{\chi \in \Sigma} V_\chi$$

is a sub $R[G]$-module of V and we have

$$V = \bigoplus_{\Sigma \in X^v} V^\Sigma.$$

The object in $\mathrm{Rep}(G, R)$ defined by V^Σ is called the *refined slope Σ-part* of (ρ, V). (ρ, V) is called *pure of refined slope* Σ if $V = V^\Sigma$. $V \mapsto V^\Sigma$ defines a functor from $\mathrm{Rep}(G, R)$ to itself which preserves short exact sequences. These functors commute with base changes by $R \to R'$.

Lemma 3.5. *Let (ρ, V) be a non-zero object in* $\mathrm{Rep}(G, R)$ *which is pure of refined slope $\Sigma \in X^v$, $\chi \in \Sigma$, and $V_\chi \subset \mathrm{Res}^G_{G^v} V$ be the χ-part of* $\mathrm{Res}^G_{G^v} V$*. Let $H_\chi \subset G$ be the stabilizing subgroup of χ.*

(1) *H_χ is a subgroup of G of finite index.*
(2) *V_χ is stable under the action of H_χ on V.*
(3) *V is, as an object in* $\mathrm{Rep}(G, R)$*, isomorphic to* $\mathrm{Ind}^G_{H_\chi} V_\chi$.

Proof. Obvious. □

Remark 3.6. The claim $[G : H_\chi] < \infty$ also follows from the explicit description of the group $\mathrm{Hom}(G^v/G^{v+}, R^\times)$ by Saito [5, p. 3, Thm. 1].

3.4. Character sheaves. Let S be a scheme of characteristic p, $(R, \mathfrak{m}_R)$ a complete p'-coefficient ring, and G a commutative group scheme over S. An *invertible character R-sheaf* on G is a smooth invertible étale R-sheaf (that is, a pro-system of smooth invertible $R/\mathfrak{m}_R^n$-sheaves in the étale topology) $\mathcal{L}$ on G such that $\mathcal{L} \boxtimes \mathcal{L} \cong \mu^*\mathcal{L}$, where $\mu : G \times_S G \to G$ is the group law. We have $i^*\mathcal{L} \cong \mathcal{L}$, where $i : G \to G$ is the inverse morphism. If $\mathcal{L}_1$, $\mathcal{L}_2$ are two invertible character R-sheaf on G, then so is $\mathcal{L}_1 \otimes_R \mathcal{L}_2$.

Lemma 3.7 (Orthogonality relation)**.** *Suppose that S is quasi-compact and quasi-separated, and that the structure morphism $\pi : G \to S$ is compactifiable. Let $\mathcal{L}$ be an invertible character R-sheaf on G such that $\mathcal{L} \otimes_R R/\mathfrak{m}_R$ is non-trivial. Then we have $R\pi_!\mathcal{L} = 0$.*

Proof. We may assume that R is a field. Since $R\mathrm{pr}_1(\mathcal{L}\boxtimes\mathcal{L}) \cong (\pi^*R\pi_!\mathcal{L})\otimes\mathcal{L}$ and $R\mathrm{pr}_1(\mu^*\mathcal{L}) \cong \pi^*R\pi_!\mathcal{L}$, we have $(\pi^*R^i\pi_!\mathcal{L}) \otimes \mathcal{L} \cong \pi^*R^i\pi_!\mathcal{L}$ for all i. Hence $R^i\pi_!\mathcal{L} = 0$ for all i. □

Lemma 3.8. *Suppose further that S and G are noetherian and connected, and that R is a finite ring. Let $\mathcal{L}$ be a smooth invertible R-sheaf on G. Then $\mathcal{L}$ is an invertible character R-sheaf if and only if there is a finite etale homomorphism $G' \to G$ of commutative S-group schemes with a constant kernel H_S and a homomorphism $\chi : H \to R^\times$ of groups such that $\mathcal{L}$ is the sheaf defined by G' and χ.*

Proof. This is [11, Lem. 3.2]. □

4. $\widetilde{\varepsilon}_0$-characters

Throughout this section, let K be a p-CDVF with residue field k and $(R, \mathfrak{m}_R)$ a complete strict p'-coefficient ring with a positive residue characteristic. In this section, we generalize the theory of local ε_0-constants to that for objects in $\mathrm{Rep}(W_K, R)$.

We use the following notation: for any k-algebra A, let R_A denote A (*resp.* $W(A)$) when K is of equal characteristic (resp. mixed characteristic). Then $\mathcal{O}_K$ has a natural structure of R_k-algebra.

4.1. Additive character sheaves. For two integers $m, n \in \mathbb{Z}$ with $m \leq n$, let $K^{[m,n]}$ denote $\mathfrak{m}_K^m/\mathfrak{m}_K^{n+1}$ regarded as an affine commutative k-group. More precisely, take a prime element π_K of K. If $\mathrm{char}\, K = p$, then $K^{[m,n]}$ is canonically isomorphic to the affine k-group which associates every k-algebra A the group $\bigoplus_{i=m}^n A$. If $\mathrm{char}\, K = 0$, let $e = [K : \mathrm{Frac}\, W(k)]$ be the absolute ramification index of K. Then $K^{[m,n]}$ is canonically isomorphic to the affine k-group which associates every k-algebra A the group $\bigoplus_{i=0}^{e-1} W_{1+\lfloor \frac{n-m-i}{e} \rfloor}(A)$.

Let R_0 be a pro-finite local ring on which p is invertible. Let $\mathrm{ACh}(K^{[m,n]}, R_0)$ (resp. $\mathrm{ACh}^0(K^{[m,n]}, R_0)$) denote the set of all isomorphism classes of invertible character R_0-sheaves (resp. non-trivial invertible character R_0-sheaves) on $K^{[m,n]}$. For a p'-coefficient ring $(R, \mathfrak{m}_R)$, let $\mathrm{ACh}(K^{[m,n]}, R)$ denote the set $\varinjlim_{R_0} \mathrm{ACh}(K^{[m,n]}, R_0)$, where R_0 runs over all isomorphism classes of injective local ring homomorphisms $R_0 \hookrightarrow R$ from pro-finite local rings R_0 to R.

For four integers m_1, m_2, n_1, and $n_2 \in \mathbb{Z}$ with $m_1 \leq m_2 \leq n_2$ and $m_1 \leq n_1 \leq n_2$, the canonical morphism $K^{[m_2,n_2]} \to K^{[m_1,n_1]}$ induces a map $\mathrm{ACh}(K^{[m_1,n_1]}, R) \to \mathrm{ACh}(K^{[m_2,n_2]}, R)$.

Definition 4.1. A *non-trivial additive character sheaf of K with coefficients in R* is an element $\widetilde{\psi}$ in

$$\coprod_{n\in\mathbb{Z}} \varprojlim_{m\leq -n-1} \mathrm{ACh}^0(K^{[m,-n-1]}, R).$$

When $\widetilde{\psi} \in \varprojlim_{m\leq -n-1} \mathrm{ACh}^0(K^{[m,-n-1]}, R)$, the integer n is called the *conductor* of $\widetilde{\psi}$ and is denoted by $\mathrm{ord}\, \widetilde{\psi}$.

Let $a \in K$ with $v_K(a) = v$. The multiplication-by-a map

$$a_{[m,n]} : K^{[m-v,n-v]} \to K^{[m,n]}$$

induces a canonical isomorphism

$$a^*_{[m,n]} : \mathrm{ACh}^0(K^{[m,n]}) \xrightarrow{\cong} \mathrm{ACh}^0(K^{[m-v,n-v]})$$

and hence an isomorphism

$$\varprojlim_{m\leq -n-1} \mathrm{ACh}^0(K^{[m,-n-1]}, R) \xrightarrow{\cong} \varprojlim_{m\leq -n-v-1} \mathrm{ACh}^0(K^{[m,-n-v-1]}, R).$$

We denote by $\widetilde{\psi}_a$ the image of $\widetilde{\psi}$ by this isomorphism.

Let L be a finite separable extension of K. The trace map $\mathrm{Tr}_{L/K} : L \to K$ induces the map

$$\mathrm{Tr}^*_{L/K} : \mathrm{ACh}(K^{[m,-n-1]}, R) \to \mathrm{ACh}(L^{[-e_{L/K}m - v_L(D_{L/K}), -e_{L/K}n - v_L(D_{L/K})-1]}, R).$$

We denote by $\widetilde{\psi} \circ \mathrm{Tr}_{L/K}$ the image of $\widetilde{\psi}$ by this map. We have $\mathrm{ord}\,(\widetilde{\psi} \circ \mathrm{Tr}_{L/K}) = e_{L/K} \mathrm{ord}\, \widetilde{\psi} + v_L(D_{L/K})$.

Lemma 4.2. *Let k be a perfect field of characteristic p, and $G = \mathbb{G}_{a,k}$ be the additive group scheme over k, $\phi_0 : \mathbb{F}_p \to R_0^\times$ a non-trivial additive character, and $\mathcal{L}_{\phi_0}$ the Artin-Schreier sheaf on $\mathbb{G}_{a,\mathbb{F}_p}$ associated to ϕ_0. Then for any additive character sheaf $\mathcal{L}$ on G, there exists a unique element $a \in k$ such that $\mathcal{L}$ is isomorphic to the pull-back of $\mathcal{L}_{\phi_0}|_G$ by the multiplication-by-a map $G \to G$.*

Proof. This follows from Lemma 3.8 and [6, 8.3, Prop. 3]. □

Corollary 4.3. *Let K be a p-local field and R a complete strict p'-coefficient ring with a positive residue characteristic. Then for any non-trivial continuous additive character $\psi : K \to R^\times$ of conductor n. Then there exists a unique non-trivial additive character R-sheaf $\widetilde{\psi}$ of conductor n such that for any $a \in K$ with $v_K(a) < -n-1$, we have*

$$\psi(a) = \mathrm{Tr}(\mathrm{Fr}_{\overline{a}}\,;\, \widetilde{\psi}|_{K^{[v_K(a),-n-1]}}),$$

where $\overline{a}$ is the k-rational point of $K^{[v_K(a),-n-1]}$ corresponding to a. Furthermore, $\psi \mapsto \widetilde{\psi}$ gives a one-to-one correspondence between the non-trivial continuous R-valued additive characters of K of conductor n and the non-trivial additive character R-sheaves of conductor n.

Proof. The only non-trivial part is the existence of the sheaf $\widetilde{\psi}$. When $\mathrm{char}\, K = p$, take a non-trivial additive character $\phi_0 : \mathbb{F}_p \to R_0^\times$ with values in a pro-finite local subring R_0. Then there exists a unique continuous 1-differential ω on K over k such that $\psi(x) = \phi_0(\mathrm{Tr}_{k/\mathbb{F}_p}\mathrm{Res}(x\omega))$ for all $x \in K$ (Here Res denotes the residue at the closed point of $\mathrm{Spec}\, \mathcal{O}_K$). Then for all $m < -n-1$, the map $x \mapsto \mathrm{Res}(x\omega)$ defines a morphism $f : K^{[m,-n-1]} \to \mathbb{G}_{a,k}$ of k-groups. The sheaf $\widetilde{\psi}|_{K^{[m,-n-1]}}$ is realized as the pull-back of the Artin-Schreier sheaf on $\mathbb{G}_{a,k}$ associated to ϕ_0.

When $\mathrm{char}\, K = 0$, fix a non-trivial continuous additive character $\psi_0 : \mathbb{Q}_p \to R^\times$ with $\mathrm{ord}\, \psi_0 = 0$. For each integer $n \geq 1$, let $\mathbb{Q}_p^{[-n,-1]}$ is canonically isomorphic to the group of Witt covectors $CW_{n,\mathbb{F}_p}$ of length n. Then the morphism $1 - F : CW_{n,\mathbb{F}_p} \to CW_{n,\mathbb{F}_p}$ and the character ϕ_0 defines a non-trivial additive character R-sheaf $\widetilde{\psi}_0$ of conductor 0. There exists a unique element $a \in K^\times$ such that $\psi(x) = \psi_0(\mathrm{Tr}_{K/\mathbb{Q}_p}(ax))$ for all x. Then the sheaf $\widetilde{\psi}$ is realized as $(\widetilde{\psi}_0 \circ \mathrm{Tr}_{K/\mathbb{Q}_p})_a$. □

Corollary 4.4. *Let K be a p-CDVF with a residue field k.*

1. *Suppose that $\mathrm{char}\, K = 0$. Let $K_0 = \mathrm{Frac}\, W(k)$ the maximal absolutely unramified subfield of K and let $K_{00} = \mathrm{Frac}\, W(\mathbb{F}_p)$. Fix a non-trivial additive character sheaf $\widetilde{\psi}_0$ on K_{00}. Then for any non-trivial additive character sheaf $\widetilde{\psi}$ on K, there exists a unique element $a \in K^\times$ with*

$$v_K(a) = \mathrm{ord}\, \widetilde{\psi} - v_L(D_{K/K_0}) - e_{K/K_0} \cdot \mathrm{ord}\, \widetilde{\psi}_0$$

such that for all $m \in \mathbb{Z}$ with $m \le \operatorname{ord} \widetilde{\psi}_0 - 1$, the sheaf

$$\widetilde{\psi}|_{K^{[me_{K/K_0}+e_{K/K_0}\cdot \operatorname{ord} \widetilde{\psi}_0 - \operatorname{ord} \widetilde{\psi}, -\operatorname{ord} \widetilde{\psi}-1]}}$$

is the pull-back of $\widetilde{\psi}_0$ by the morphism

$$\begin{aligned} & K^{[me_{K/K_0}+e_{K/K_0}\cdot \operatorname{ord} \widetilde{\psi}_0 - \operatorname{ord} \widetilde{\psi}, -\operatorname{ord} \widetilde{\psi}-1]} \\ \xrightarrow{a} \; & K^{[-v_L(D_{K/K_0})+me_{K/K_0}, -v_L(D_{K/K_0})-e_{K/K_0}\cdot \operatorname{ord} \widetilde{\psi}_0-1]} \\ \xrightarrow{\operatorname{Tr}_{K/K_0}} \; & K_0^{[m,-\operatorname{ord} \widetilde{\psi}_0-1]} \to K_{00}^{[m,-\operatorname{ord} \widetilde{\psi}_0-1]}. \end{aligned}$$

(2) *If* $\operatorname{char} K = p > 0$*, take a prime element π_K in K and set $K_{00} = \mathbb{F}_p((\pi_K))$. Fix a non-trivial additive character sheaf $\widetilde{\psi}_0$ on K_{00}. Then for any non-trivial additive character sheaf $\widetilde{\psi}$ on K, there exists a unique element $a \in K^\times$ with $v_K(a) = \operatorname{ord} \psi - \operatorname{ord} \psi_0$ such that for all $m \in \mathbb{Z}$ with $m \le -\operatorname{ord} \widetilde{\psi}_0 - 1$, the sheaf*

$$\widetilde{\psi}|_{K^{[m+\operatorname{ord} \widetilde{\psi}_0 - \operatorname{ord} \psi, -\operatorname{ord} \widetilde{\psi}-1]}}$$

is the pull-back of $\widetilde{\psi}_0$ by the morphism

$$K^{[m+\operatorname{ord} \widetilde{\psi}_0 - \operatorname{ord} \psi, -\operatorname{ord} \widetilde{\psi}-1]} \xrightarrow{a} K^{[m,-\operatorname{ord} \widetilde{\psi}_0-1]} \to K_{00}^{[m,-\operatorname{ord} \widetilde{\psi}_0-1]}.$$

□

4.2. A map from the Brauer group. There is a canonical map $\partial : Br(K) \to H^1(k, \mathbb{Q}/\mathbb{Z})$. Let us recall its definition: we have

$$Br(K) = \bigcup_L Br(L/K),$$

where L runs over all unramified finite Galois extension of K in a fixed separable closure of K and $Br(L/K) := \operatorname{Ker}(Br(K) \to Br(L))$. We define ∂ to be the composition

$$\begin{aligned} Br(K) &\xrightarrow{\cong} \varinjlim_{L,\mathrm{Inf}} H^2(\mathrm{Gal}(L/K), L^\times) \to \varinjlim_{L,\mathrm{Inf}} H^2(\mathrm{Gal}(L/K), \mathbb{Z}) \\ &\xrightarrow{\cong} \varinjlim_{L,\mathrm{Inf}} H^1(\mathrm{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} H^1(k, \mathbb{Q}/\mathbb{Z}). \end{aligned}$$

By local class field theory, the following lemma holds.

Lemma 4.5. *Suppose that k is finite. Then the invariant map* $\operatorname{inv} :$ $Br(K) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}$ *of local class field theory is equal to the composition*

$$Br(K) \xrightarrow{\partial} H^1(k, \mathbb{Q}/\mathbb{Z}) \xrightarrow{ev} \mathbb{Q}/\mathbb{Z}$$

where $ev : H^1(k, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}$ denotes the evaluation map at Fr_k. □

Let $\chi : W_K \to R^\times$ be a character of W_K of finite order n, and let $a \in K^\times$. Take a generator $\zeta \in R^\times$ of $\operatorname{Im}\chi$. Let L be the finite cyclic extension of K corresponding to $\operatorname{Ker}\chi$, Let $\sigma \in \operatorname{Gal}(L/K)$ be the generator of $\operatorname{Gal}(L/K)$ such that $\chi(\sigma) = \zeta$. Then the cyclic algebra $(a, L/K, \sigma)$ defines an element $[(a, L/K, \sigma)]$ in ${}_nBr(K)$. We identify $H^1(k, \mathbb{Z}/n\mathbb{Z})$ with $\operatorname{Hom}(G_k, \boldsymbol{\mu}_n(\overline{\mathbb{Q}}_\ell))$ by the isomorphism $\mathbb{Z}/n\mathbb{Z} \to \boldsymbol{\mu}_n(R)$, $1 \mapsto \zeta$, and regard $\partial_n([(a, L/K, \sigma)])$ as a character of G_k of finite order. This character does not depend on the choice of ζ, and is denoted by $\chi_{[a]} : G_k \to R^\times$. It is well-known that $(\chi, a) \mapsto \chi_{[a]}$ is biadditive with respect to χ and a. If R' is another complete strict p'-coefficient ring and if $h : R \to R'$ is a local homomorphism, then we have $\chi_{[a]} \otimes_R R' \cong (h \circ \chi)_{[a]}$.

Corollary 4.6. *Suppose that k is finite. Let χ be a character of G_K of finite order, and let $a \in K^\times$. Then we have*

$$\chi_{[a]}(\mathrm{Fr}) = \chi(\mathrm{rec}(a)).$$

□

The following lemma is easily proved:

Lemma 4.7. *Let $\chi : W_K \to R^\times$ be an unramified character of W_K of finite order. Then, for $a \in K^\times$, we have $\chi_{[a]} = \chi^{\otimes v_K(a)}$.* □

Let $\chi : W_K \to R^\times$ be an arbitrary character of W_K. Then there exists an unramified character χ_1 and a character χ_2 such that $\chi_2 \mod \mathfrak{m}_R^n$ is of finite order for all $n \in \mathbb{Z}_{>0}$ and that $\chi = \chi_1 \otimes_R \chi_2$. For $a \in K^\times$ define $\chi_{[a]} : W_k \to R^\times$ by

$$\chi_{[a]} := \chi_1^{\otimes v_K(a)} \otimes_R (\varprojlim_n (\chi_2 \mod \mathfrak{m}_R^n)_{[a]}).$$

This does not depend on the choice of χ_1 and χ_2. By definition, we have $\chi_{[aa']} \cong \chi_{[a']} \otimes_R \chi_{[a']}$ and $(\chi \otimes_R \chi')[a] \cong \chi_{[a]} \otimes_R \chi'_{[a]}$.

Lemma 4.8. *If $a \in 1 + \mathfrak{m}_K$, then the character $\chi_{[a]}$ is finite of a p-power order.*

Proof. We may assume that $\chi \mod \mathfrak{m}_R^n$ is of finite order for all $n \in \mathbb{Z}_{>0}$. Let L_n be the finite cyclic extension of K corresponding to $\operatorname{Ker}(\chi \bmod \mathfrak{m}_R^n)$. There exists a p-power N such that $a^N \in 1 + \mathfrak{m}_K^{\mathrm{sw}(\chi)}$. Since $1 + \mathfrak{m}_K^{\mathrm{sw}(\chi)}$ is contained in $\mathrm{N}_{L_n/K}(L_n^\times)$, the character $\chi_{[a^N]}$ is trivial. This completes the proof. □

4.3. Serre-Hazewinkel's geometric class field theory. For any finite separable extension L of K, let U_L, $U_{L,n}$, and $U_L^{(n)}$ denote the affine commutative k-group schemes which represent the functors which associate to each

k-algebra A the multiplicative group $(R_A\widehat{\otimes}_{R_k}\mathcal{O}_L)^\times$, $(R_A\otimes_{R_k}\mathcal{O}_L/\mathfrak{m}_L^n)^\times$, and $\mathrm{Ker}[(R_A\widehat{\otimes}_{R_k}\mathcal{O}_L)^\times \to (R_A\otimes_{R_k}\mathcal{O}_L/\mathfrak{m}_L^n)^\times]$, respectively.

Let L be a totally ramified finite abelian extension of K. Then the homomorphism $U_L \to U_K$ of affine k-groups induced by the norm is surjective, and if we denote by B the neutral component of is its kernel, then by [3, p. 659, 4.2] the kernel of the induced homomorphism

$$U_L/B \to U_K$$

is canonically isomorphic to the constant k-group $\mathrm{Gal}(L/K)$.

The isomorphism is realized as follows: take a prime element π of L, then for $\sigma \in \mathrm{Gal}(L/K)$, $\sigma(\pi)/\pi$ defines an element of $U_L(k)$ of norm 1. Since the image of $\sigma - 1 : U_L \to U_L$ is connected, the class of $\sigma(\pi)/\pi$ in U_L/B does not depend on the choice of π, which we denote by $\mathrm{class}(\sigma)$. For $\sigma_1, \sigma_2 \in \mathrm{Gal}(L/K)$, it is easily checked that $\mathrm{class}(\sigma_1)\cdot\mathrm{class}(\sigma_2) = \mathrm{class}(\sigma_1\sigma_2)$. Hence $\sigma \mapsto \mathrm{class}(\sigma)$ defines a group homomorphism $\mathrm{Gal}(L/K) \to U_L/B$.

Suppose further that L/K is cyclic. Let σ be a generator of $\mathrm{Gal}(L/K)$. Then, by Hilbert 90, B is equal to the image of

$$1 - \sigma : U_L \to U_L.$$

4.4. Local $\widetilde{\varepsilon}_0$-character for rank one objects. Let $\widetilde{\psi}$ be an additive character sheaf of K. Let $(R_0, \mathfrak{m}_{R_0})$ be a pro-finite local subring of R such that $R_0 \hookrightarrow R$ is a local homomorphism and that $\widetilde{\psi}$ is defined over R_0.

In this subsection we attach, for every rank one object (χ, V) in $\mathrm{Rep}(W_K, R_0)$, a rank one object $\widetilde{\varepsilon}_{0,R}(V, \widetilde{\psi})$ in $\mathrm{Rep}(W_k, R)$, which we call the *local ε_0-character* of V.

For each integer $m \in \mathbb{Z}$, let $K^{[m,\infty]}$ denote the affine k-scheme $\varprojlim_n K^{[m,n]}$. This represents the functor associating for any k-algebra A the set $R_A\widehat{\otimes}_{R_k}\mathfrak{m}_K^m$. Take a prime element π_K of K and let $\pi_K^{-m} : K^{[m,\infty]} \to K^{[0,\infty]}$ be the morphism defined by the multiplication by π_K^{-m}. The inverse image of $U_K \subset K^{[0,\infty]}$ by π_K^{-m} is an open subscheme of $K^{[m,\infty]}$ which we denote by $K^{v=m}$. This does not depend on the choice of π_K.

For $m, n \in \mathbb{Z}$, the multiplication map defines a morphism $K^{v=n} \times K^{v=m} \to K^{v=m+n}$ of k-schemes. This defines a structure of commutative k-group scheme on the disjoint union $\coprod_m K^{v=m}$. There is a canonical exact sequence

$$1 \to U_K \to \coprod_m K^{v=m} \to \mathbb{Z} \to 0,$$

where $\mathbb{Z}$ is a constant k-group scheme.

Now we shall define, for every rank one object χ in $\mathrm{Rep}(W_K, R_0)$, a character sheaf $\mathcal{L}_\chi$ on $\coprod_m K^{v=m}$.

(1) First assume that χ is unramified, let $\mathcal{L}'_\chi$ be the invertible R_0-sheaf on $\mathrm{Spec}\,(k)$ corresponding to χ. Define an invertible R_0-sheaf $\mathcal{L}_\chi$ on $\coprod_m K^{v=m}$

by

$$\mathcal{L}_\chi|_{K^{v=m}} = \pi^{m,*}(\mathcal{L}'_\chi)^{\otimes m},$$

where $\pi^m : K^{v=m} \to \mathrm{Spec}\,(k)$ is the structure morphism. It is easily checked that $\mathcal{L}_\chi$ is a character sheaf.

(2) Next assume that χ is a character of the Galois group of a finite separable totally ramified abelian extension L of K. Consider the norm map $\coprod_m L^{v=m} \to \coprod_m K^{v=m}$. It is surjective and the group of the connected components of its kernel is canonically isomorphic to $\mathrm{Gal}(L/K)$. Hence we have a canonical group extension of $\coprod_m K^{v=m}$ by $\mathrm{Gal}(L/K)$. Define $\mathcal{L}_\chi$ to be the character sheaf on $\coprod_m K^{v=m}$ defined by this extension and χ.

(3) Assume that χ is of finite order. Then χ is a tensor product $\chi = \chi_1 \otimes_{R_0} \chi_2$, where χ_1 is unramified and χ_2 is of the form in (2). Define $\mathcal{L}_\chi$ to be $\mathcal{L}_{\chi_1} \otimes_{R_0} \mathcal{L}_{\chi_2}$.

Let L/K be a finite abelian extension such that χ factors through $\mathrm{Gal}(L/K)$. Let L_0 be the maximal unramified subextension of L/K. From the norm map $L^{v=1} \to L_0^{v=1}$ and the canonical morphism $L_0^{v=1} \cong K^{v=1} \otimes_k k_L \to K^{v=1}$, we obtain a canonical etale $\mathrm{Gal}(L/K)$-torsor T on $K^{v=1}$. The following lemma is easily proved.

Lemma 4.9. *$\mathcal{L}_\chi|_{K^{v=1}}$ is isomorphic to the smooth R_0-sheaf defined by T and χ.* $\square$

Corollary 4.10. *The sheaf $\mathcal{L}_\chi$ does not depend on the choice of χ_1 and χ_2.* $\square$

(4) General case. For each $n \in \mathbb{Z}_{>0}$, $\chi_n := \chi \mod \mathfrak{m}_{R_0}^n$ is a character of finite order. Define $\mathcal{L}_\chi$ to be $(\mathcal{L}_{\chi_n})_n$.

Corollary 4.11. *Let χ_1, χ_2 be two rank one objects in $\mathrm{Rep}(W_K, R_0)$. Then we have an isomorphism $\mathcal{L}_{\chi_1 \otimes_{R_0} \chi_2} \cong \mathcal{L}_{\chi_1} \otimes_{R_0} \mathcal{L}_{\chi_2}$.* $\square$

Lemma 4.12. *Let $s = \mathrm{sw}(\chi)$ be the Swan conductor of χ. Then the restriction of $\mathcal{L}_\chi$ to U_K is the pull-back of a character sheaf $\overline{\mathcal{L}}_\chi$ on $U_{K,s+1}$. Furthermore, if $s \geq 1$, the restriction of $\overline{\mathcal{L}}_\chi \otimes R_0/\mathfrak{m}_{R_0}$ to $U_K^{(s)}/U_K^{(s+1)}$ is non-trivial.*

Proof. We may assume that χ is of the form of (2). Let L be the finite extension of K corresponding to $\mathrm{Ker}\,\chi$, π_L a prime element in L. For the first (resp. the second) assertion, it suffices to prove that there does not exist (resp. there exists) $\sigma \in \mathrm{Gal}(L/K)$ with $\sigma \neq 1$ such that $\sigma(\pi_L)/\pi_L$ lies in the neutral component of the kernel of the map $U_L \xrightarrow{\mathrm{N}_{L/K}} U_K \to U_{K,s+1}$, which is easy to prove. $\square$

Lemma 4.13. *For $a \in K^\times$, let $\overline{a} : \mathrm{Spec}\,(k) \to U_K$ be the k-rational point of U_K defined by a. Then $\overline{a}^*\mathcal{L}_\chi$ is isomorphic to $\chi_{[a]}$.*

Proof. We may assume that χ is of the form of case (1) or (2). In case (1), The assertion follows from Lemma 4.7. In case (2), let L be the finite extension of K corresponding to $\operatorname{Ker}\chi$. Take a generator $\sigma \in \operatorname{Gal}(L/K)$ and let us consider the cyclic algebra $(a, L/K, \sigma)$. This is isomorphic to a matrix algebra over a central division algebra $D = D_{(a,L/K,\sigma)}$ over K. The valuation of K is canonically extended to a valuation of D. Let $\mathcal{O}_D$ denote the valuation ring of D, k_D the residue field of D. There is a maximal commutative subfield of D which is isomorphic (as a k-algebra) to a subextension of L/K. Since L/K is totally ramified, k_D is a commutative field. Let π_D be a prime element of D. The conjugation by π_D; $x \mapsto \pi_D^{-1} x \pi_D$ defines an automorphism τ of k_D over k. It is checked that the fixed field of τ is equal to k. Hence k_D/k is a cyclic extension whose Galois group is generated by τ. Let K_D is the unramified extension of K corresponding to k_D/k. Then $D \otimes_K K_D$ is split. Hence there exists an element $b \in (\mathcal{O}_{LK_D})^\times$ such that $a = \mathrm{N}_{LK_D/K_D}(b)$.

Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & (K_D)^\times & \longrightarrow & GL_n(K_D) & \longrightarrow & PGL_n(K_D) & \longrightarrow & 1 \\
 & & {\scriptstyle v_{K_D}}\big\downarrow & & {\scriptstyle \frac{1}{n}v_{K_D}\circ\det}\big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0.
\end{array}$$

By [9, X, § 5], $(a, L/K, \sigma)$ gives a canonical element in $H^1(\operatorname{Gal}(K_D/K), PGL_n(K_D))$ whose image by the canonical map

$$H^1(\operatorname{Gal}(K_D/K), PGL_n(K_D)) \to H^1(\operatorname{Gal}(K_D/K), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{Inf}} H^1(k, \mathbb{Q}/\mathbb{Z})$$

is equal to $\partial([(a, L/K, \sigma)])$.

By definition, $(a, L/K, \sigma) = \bigoplus_{i=0}^{n-1} L \cdot \alpha^i$ with $\alpha^n = a$, $\alpha x = \sigma(x)\alpha$ for $x \in K$. Let $\iota : LK_D \hookrightarrow \operatorname{End}_{K_D}(LK_D)$ be the canonical homomorphism. Let $\varphi : (a, L/K, \sigma) \otimes_K K_D \cong \operatorname{End}_{K_D}(LK_D)$ be the K_D-algebra isomorphism defined by $\varphi(x) = \iota(x)$ for $x \in LK_D$ and by $\varphi(\alpha) = \iota(b) \cdot \sigma$. It is easily checked that the composition

$$\begin{aligned}
\tau \circ \varphi \circ \tau^{-1} \circ \phi^{-1} : \operatorname{End}_{K_D}(LK_D) \quad & \xrightarrow{\varphi^{-1}} (a, L/K, \sigma) \otimes_K K_D \\
& \xrightarrow{\tau^{-1}} (a, L/K, \sigma) \otimes_K K_D \\
& \xrightarrow{\varphi} \operatorname{End}_{K_D}(LK_D) \cong \operatorname{End}_K(L) \otimes_K K_D \\
& \xrightarrow{\tau} \operatorname{End}_K(L) \otimes_K K_D \cong \operatorname{End}_{K_D}(LK_D)
\end{aligned}$$

is a K_D-algebra automorphism which is identity on $\iota(LK_D)$ and which sends σ to $\frac{\tau(b)}{b}\sigma$. By Skolem-Noether theorem, there exists an element $c \in LK_D^\times$ such that $\frac{\tau(b)}{b} = \frac{\sigma(c)}{c}$ and that $\tau \circ \varphi \circ \tau^{-1} \circ \phi^{-1}$ is the conjugation by $\iota(c)$. Then $\chi_{[a]}$ is the inflation of the character of $\operatorname{Gal}(k_D/k)$ which sends τ to $\zeta^{v_{LK_D}(c)}$. Hence the assertion follows. $\square$

Let $s = \mathrm{sw}(\chi)$ be the Swan conductor of χ and set $m = -s - \mathrm{ord}\,\widetilde{\psi} - 1$. Then the character $\widetilde{\psi}$ defines an invertible character R_0-sheaf $\widetilde{\psi}^{[m,-\mathrm{ord}\,\widetilde{\psi}-1]}$ on $K^{[m,-\mathrm{ord}\,\psi-1]}$. The sheaf $\mathcal{L}_\chi$ is a pull-back of a character sheaf $\overline{\mathcal{L}}_\chi$ on $\coprod_{m'} K^{v=m'}/U_K^{(s+1)}$. Let

$$i : K^{v=m}/U_K^{(s+1)} \hookrightarrow K^{[m,-\mathrm{ord}\,\widetilde{\psi}-1]}$$

be the canonical inclusion and let $f : K^{v=m}/U_K^{(s+1)} \to \mathrm{Spec}\,(k)$ be the structure morphism. Define the $\widetilde{\varepsilon}_0$-character $\widetilde{\varepsilon}_{0,R}(\chi,\widetilde{\psi})$ to be the rank one object in $\mathrm{Rep}(W_k, R)$ corresponding to the invertible R_0-sheaf

$$\widetilde{\varepsilon}_{0,R}(\chi,\widetilde{\psi}) = \\ \det{}_{R_0}(Rf_!((\overline{\mathcal{L}}_\chi|_{K^{v=m}/U_K^{(s+1)}})^{\otimes -1} \otimes_{R_0} i^*\widetilde{\psi}^{[m,-\mathrm{ord}\,\widetilde{\psi}-1]})[s+1](\mathrm{ord}\,\widetilde{\psi})).$$

Here [ ] denotes a shift in the derived category and ( ) is a Tate twist.

Proposition 4.14. *Let*

$$\mathcal{F} := (\overline{\mathcal{L}}_\chi|_{K^{v=m}/U_K^{(s+1)}})^{\otimes -1} \otimes_{R_0} i^*\widetilde{\psi}^{[m,-\mathrm{ord}\,\widetilde{\psi}-1]}).$$

(1) *Suppose that $s = 0$. Then $R^i f_! \mathcal{F} = 0$ for $i \neq 1$ and $R^i f_! \mathcal{F}$ is an invertible R-sheaf on* $\mathrm{Spec}\,(k)$.

(2) *Suppose that $s = 2b - 1$ is odd and ≥ 1. Let $f' : K^{v=m}/U_K^{(s+1)} \to K^{v=m}/U_K^{(b)}$ be the canonical morphism. Then $R^i f'_! \mathcal{F} = 0$ for $i \neq 2b$ and there exists a k-rational point P in $K^{v=m}/U_K^{(b)}$ such that $R^{2b} f_! \mathcal{F}$ is supported on P whose fiber is free of rank one.*

(3) *Suppose that $s = 2b$ is even and ≥ 2. Let $f' : K^{v=m}/U_K^{(s+1)} \to K^{v=m}/U_K^{(b+1)}$ be the canonical morphism. Then $R^i f'_! \mathcal{F} = 0$ for $i \neq 2b - 2$ and there exists a k-rational point P in $K^{v=m}/U_{K,b}$ such that $R^{2b-2} f_! \mathcal{F}$ is supported on the fiber $A \cong \mathbb{A}^1_k$ at P by the canonical morphism*

$$K^{v=m}/U_K^{(b+1)} \to K^{v=m}/U_K^{(b)}$$

and that $R^{2b-2} f_! \mathcal{F}|_A$ is a smooth invertible R-sheaf on A, whose swan conductor at infinity is equal to 2.

Proof. The assertions (1) and (2) are easy and their proofs are left to the reader. We will prove (3). We may assume that k is algebraically closed.

For any closed point Q in $K^{v=m}/U_K^{(s+1)}$, the pull-back of $\mathcal{F}$ by the multiplication-by-Q map

$$U_K^{(b+1)}/U_K^{(s+1)} \hookrightarrow K^{v=m}/U_K^{(s+1)}$$

is an invertible character R_0-sheaf, which we denote by $\mathcal{L}_Q$.

There exists a unique k-rational point P in $K^{v=m}/U_K^{(b)}$ such that $\mathcal{L}_Q$ is trivial if and only if Q lies in the fiber $A \cong \mathbb{A}_k^1$ at P by the canonical morphism $K^{v=m}/U_K^{(b+1)} \to K^{v=m}/U_K^{(b)}$. By the orthogonality relation of character sheaves, $R^i f_!'\mathcal{F} = 0$ for $i \neq 2b-2$, $R^{2b-2}f_!\mathcal{F}$ is supported on A and $\mathcal{G} = R^{2b-2}f_!\mathcal{F}|_A$ is a smooth invertible R-sheaf on A. Take a closed point P_0 in $A \subset K^{v=m}/U_K^{(b+1)}$ and identify A with $U_K^{(b)}/U_K^{(b+1)} \cong \mathbb{G}_{a,k}$ by P_0. The sheaf $\mathcal{G}$ is has the following property: there exists a non-trivial invertible character sheaf $\mathcal{L}_1$ on $\mathbb{G}_{a,k}$ such that $\mathcal{G} \boxtimes \mathcal{G} \cong \alpha^*\mathcal{G} \otimes \mu^*\mathcal{L}_1$, where $\alpha, \mu : \mathbb{G}_{a,k} \times \mathbb{G}_{a,k} \to \mathbb{G}_{a,k}$ denote the addition map and the multiplication map respectively.

If $p \neq 2$, then let $f : \mathbb{G}_{a,k} \to \mathbb{G}_{a,k}$ denote the map defined by $x \mapsto \frac{x^2}{2}$. Then $\mathcal{G} \otimes f^*\mathcal{L}_1$ is an invertible character sheaf on $\mathbb{G}_{a,k}$. Hence the swan conductor of $\mathcal{G}$ at infinity is equal to 2.

It remains to consider the case $p = 2$. Let $W_{2,k}$ be the k-group of Witt vectors of length two. Let $\mathcal{G}'$ be the invertible sheaf on $W_{2,k}$ defined by $\mathcal{G}' = a_0^*\mathcal{G} \otimes a_1^*\mathcal{L}$, where $a_i : W_{2,k} \to \mathbb{G}_{a,k}$ are k-morphisms defined by $(x_0, x_1) \mapsto x_i$.

Then the sheaf $\mathcal{G}'$ is an invertible character sheaf on $W_{2,k}$. There exists an element $a \in W_2(k)^\times$ such that the pull-back $a^*\mathcal{G}'$ of $\mathcal{G}'$ by the multiplication-by-a map is trivial on the finite etale covering $1-F : W_{2,k} \to W_{2,k}$ of $W_{2,k}$. Since $\mathcal{G}$ is isomorphic to the pull-back of $\mathcal{G}'$ by the Teichmüller map $\mathbb{G}_{a,k} \to W_{2,k}$, the assertion of the lemma follows from direct computation. $\square$

Lemma 4.15. *For $a \in K^\times$, we have*

$$\widetilde{\varepsilon}_{0,R}(\chi, \widetilde{\psi}_a) = \chi_{[a]} \otimes_R \widetilde{\varepsilon}_{0,R}(\chi, \widetilde{\psi}_a) \otimes_R R(v_K(a)).$$

Proof. It follows from Lemma 4.13. $\square$

4.5. $\widetilde{\boldsymbol{\lambda}}_{\boldsymbol{R}}$-characters. Let L be a finite separable extension, and $\widetilde{\psi}$ an additive character sheaf on K.

Let $V = \mathrm{Ind}_{W_L}^{W_K} 1 \in \mathrm{Rep}(W_K, R)$, and $V_{\mathbb{C}} = \mathrm{Ind}_{W_L}^{W_K} 1 \in \mathrm{Rep}(W_K, \mathbb{C})$. Since $V_{\mathbb{C}}^0 + \det V_{\mathbb{C}} - ([L : K] + 1)1_{\mathbb{C}}$ is an real virtual representation of W_K of virtual rank 0, we can define a canonical element

$$\mathrm{sw}_2(V_{\mathbb{C}}^0) \in {}_2Br(K)$$

as in [2, 1.4.1]. Let $\mathrm{sw}_{2,R}(V_{\mathbb{C}}^0)$ be the rank one object in $\mathrm{Rep}(W_k, R)$ induced by $\partial_2(\mathrm{sw}_2(V_{\mathbb{C}}^0)) \in H^1(k, \mathbb{Z}/2\mathbb{Z})$ and the map $\mathbb{Z}/2\mathbb{Z} \to R^\times$, $n \mapsto (-1)^n$.

Next we define $\widetilde{\varepsilon}_R(\det V, \widetilde{\psi})$. When $\det V$ is unramified, we denote by the same symbol $\det V$ the rank one object in $\mathrm{Rep}(W_k, R)$ corresponding to $\det V$ and set $\widetilde{\varepsilon}_R(\det V, \widetilde{\psi}) := (\det V)^{\otimes \mathrm{ord}\,\widetilde{\psi}} \otimes_R R(-\mathrm{ord}\,\widetilde{\psi})$. When $\det V$ is not unramified, we set $\widetilde{\varepsilon}_R(\det V, \widetilde{\psi}) := \widetilde{\varepsilon}_{0,R}(\det V, \widetilde{\psi})$.

Definition 4.16. Define the rank one object $\widetilde{\lambda}_R(L/K,\widetilde{\psi})$ in $\mathrm{Rep}(W_k,R)$ by

$$\begin{aligned}\widetilde{\lambda}_R(L/K,\widetilde{\psi}) := \quad & \mathrm{sw}_{2,R}(V^0_{\mathbb{C}}) \otimes_R \widetilde{\varepsilon}_R(\det V,\widetilde{\psi})^{\otimes -1} \otimes_R \det(\mathrm{Ind}^{W_k}_{W_{k_L}} 1) \\ & \otimes_R R\left(\tfrac{1}{2}(v_{L/K}(d_{L/K}) - a(\det V_{\mathbb{C}})) - \mathrm{ord}\,\widetilde{\psi}\right),\end{aligned}$$

where $a(\det V_{\mathbb{C}})$ is the Artin conductor of $\det V_{\mathbb{C}}$.

The following lemma is easily checked:

Lemma 4.17. *Suppose that k is finite. Let $\psi : K \to R^\times$ be the additive character corresponding to $\widetilde{\psi}$. Then we have*

$$\widetilde{\lambda}_R(L/K,\widetilde{\psi})(\mathrm{Fr}_k) = (-1)^{v_K(d_{L/K})+f_{L/K}+1}\lambda_R(L/K,\psi).$$

$\square$

4.6. Local $\widetilde{\varepsilon}_0$-characters of representations of G_K whose images are finite. Let R be an algebraically closed field of positive characteristic $\neq p$. In this subsection we shall define, for an object (ρ, V) in $\mathrm{Rep}(W_K,R)$ such that $\mathrm{Im}\,\rho$ is finite, a rank one object $\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi})$ in $\mathrm{Rep}(W_k,R)$, which is called the *local $\widetilde{\varepsilon}_0$-character* of V.

Let L be the finite Galois extension of K corresponding to the kernel of ρ and let $G = \mathrm{Gal}(L/K)$.

By Brauer's theorem for modular representations (cf. [8]), there exist subgroups $H_1,\cdots,H_m$ of G, characters $\chi_1,\cdots,\chi_m$ of $H_1,\cdots,H_m$ and integers $n_1,\cdots,n_m \in \mathbb{Z}$ such that $\rho = \sum_i n_i \mathrm{Ind}^G_{H_i}\chi_i$ as a virtual representation of G over R. Let K_i be the subextension of L/K corresponding to H_i.

Define $\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi})$ by

$$\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi}) = \bigotimes_i (\widetilde{\varepsilon}_{0,R}(\chi_i,\widetilde{\psi}\circ \mathrm{Tr}_{K_i/K}) \otimes \widetilde{\lambda}_R(K_i/K,\widetilde{\psi}))^{\otimes n_i}.$$

Lemma 4.18. *The sheaf $\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi})$ does not depend on the choice of H_i, χ_i and n_i.*

A proof of this lemma is given in the next two subsections.

The following two lemmas are easily proved.

Lemma 4.19. *Let χ be an unramified rank one object in $\mathrm{Rep}(W_K,R)$ of finite order. Then*

$$\widetilde{\varepsilon}_{0,R}(V\otimes\chi,\widetilde{\psi}) \cong \widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi}) \otimes_R \chi^{\otimes \mathrm{sw}(V)+\mathrm{rank}\,V\cdot(\mathrm{ord}\,\widetilde{\psi}+1)}.$$

$\square$

Lemma 4.20. *Let*

$$0 \to V' \to V \to V'' \to 0$$

be a short exact sequence of objects in $\mathrm{Rep}(W_K, R)$ *with finite images. Then we have*

$$\widetilde{\varepsilon}_{0,R}(V, \widetilde{\psi}) \cong \widetilde{\varepsilon}_{0,R}(V', \widetilde{\psi}) \otimes \widetilde{\varepsilon}_{0,R}(V'', \widetilde{\psi}).$$

$\square$

4.7. *A*-structures. For any ring A of characteristic p, let R_A denote $W(A)$ (resp. A) if K is of mixed characteristic (resp. of equal characteristic). When A is a subring of k, we regard R_A as a subalgebra of $\mathcal{O}_K$ in canonical way.

If $\mathrm{char}\, K = 0$, a prime element π_K of K is called *A-admissible* if the minimal polynomial of π_K over $\mathrm{Frac}\, W(k)$ has coefficients in R_A and has the constant term in $pR_A^\times$.

If $\mathrm{char}\, K = p$, any prime element π_K of K is called *A-admissible*.

Lemma 4.21. *For any prime element π'_K of K and for any positive integer $N \in \mathbb{Z}_{>0}$, there exists a finitely generated $\mathbb{F}_p$-subalgebra $A \subset k$ and an A-admissible prime element π_K of K congruent to π'_K modulo $\mathfrak{m}_K^N$.*

Before proving this lemma, we prove the following lemma:

Lemma 4.22. *Let K be a p-CDVF, and $f(T) \in \mathcal{O}_K[T]$ a polynomial. Suppose that $x_0 \in \mathcal{O}_K$ satisfies $f'(x_0) \neq 0$ and $f(x_0) \in \mathfrak{m}_K^{2v_K(f'(x_0))+1}$. Then there exists a unique element x in $\mathcal{O}_K$ such that $x \equiv x_0 \mod \mathfrak{m}_K^{v_K(f'(x))+1}$ and that $f(x) = 0$. Moreover we have $v_K(f'(x)) = v_K(f'(x_0))$.*

Proof. We prove the lemma using induction. Put $v = v_K(f'(x_0))$. It suffices to prove the following statement:

> If $n > 2v$ and if an element $y_0 \in \mathcal{O}_K$ satisfies $y_0 \equiv x_0 \mod \mathfrak{m}_K^{v+1}$. and $f(y_0) \in \mathfrak{m}_K^n$, then there exists an element $y \in \mathcal{O}_K$ satisfying $y \equiv y_0 \mod \mathfrak{m}_K^{n-v}$ and $f(y) \in \mathfrak{m}_K^{n+1}$. Furthermore, the class of such y modulo $\mathfrak{m}_K^{n-v+1}$ is unique.

Since $y_0 \equiv x_0 \mod \mathfrak{m}_K^{v+1}$, we have

$$f'(y_0) \equiv f'(x_0) \not\equiv 0 \mod \mathfrak{m}_K^{v+1}.$$

Hence $\frac{f(y_0)}{f'(y_0)}$ is an element in $\mathfrak{m}_K^{n-v}$. Then the polynomial $f(y_0 + \frac{f(y_0)}{f'(y_0)}T)$ is congruent to $f(x) + f(x)T$ modulo $\mathfrak{m}_K^{n+1}$. Hence the assertion follows. $\square$

Proof of Lemma 4.21. We may assume that $\mathrm{char}\, K = 0$. Take an arbitrary prime element π'_K of K. Let $f(T) = T^e + \sum_{i=0}^{e-1} a_i T^i$ be the minimal polynomial of π_K over $K_0 = \mathrm{Frac}\, W(k)$. Set $M = \max\{N, v_K(D_{K/K_0})+1\}$. For each i, write $a_i \in W(k)$ as the form of a Witt vector $a_i = (0, a_{i,1}, a_{i,2}, \ldots)$. Define $a'_i \in W(k)$ by $a'_i = (0, a_{i,1}, \ldots, a_{i,M+v_K(D_{K/K_0})}, 0, \ldots)$ and let $g(T) =$

$T^e + \sum_{i=0}^{e-1} a'_i T^i$. By Lemma 4.22, there exists a root π_K of $g(T)$ such that $\pi_K \equiv \pi'_K \mod \mathfrak{m}_K^M$.

Let A be the $\mathbb{F}_p$-subalgebra of k generated by

$$\{a_{i,j}\,;\, 0 \le i \le e-1,\, 1 \le j \le M + v_K(D_{K/K_0})\} \cup \{a_{0,1}^{-1}\}.$$

Then π_K is A-admissible. $\square$

For an A-admissible prime element π_K of K, let $\widetilde{R}_{A,\pi_K}$ denote the subalgebra $R_A[[\pi_K]]$ of $\mathcal{O}_K$.

There exists a canonical morphism $\widetilde{R}_{A,\pi_K} \to A$. The following lemma is easily checked:

Lemma 4.23. (1) *Let any $n \in \mathbb{Z}_{>0}$, $x \in \widetilde{R}_{A,\pi_K} \cap \mathfrak{m}_K^n$, and $y \in k$ the class of $\frac{x}{\pi_K^n}$. Then there exists a positive integer m satisfying $y^{p^m} \in A$.*

(2) *An element in $\widetilde{R}_{A,\pi_K}$ is invertible if and only if its canonical image in A is invertible.*

(3) *If A is perfect, then $\widetilde{R}_{A,\pi_K} \cap \mathfrak{m}_K^n = \pi_K^n \cdot \widetilde{R}_{A,\pi_K}$.*

$\square$

In a manner similar to that in the proof of Lemma 4.22, we have:

Lemma 4.24. *Let $f(T) \in \widetilde{R}_{A,\pi_K}[T]$ be a polynomial. Let v be a positive integer. Suppose that $x_0 \in \widetilde{R}_{A,\pi_K}$ satisfies $f'(x_0) \in \pi_K^v(\widetilde{R}_{A,\pi_K})^\times$ and $f(x_0) \in \pi_K^{2v+1}\widetilde{R}_{A,\pi_K}$. Then there exists a unique element x in $\widetilde{R}_{A,\pi_K}$ such that $x \equiv x_0 \mod \pi_K^{v+1}\widetilde{R}_{A,\pi_K}$ and that $f(x) = 0$. Furthermore we have $f'(x) \in \pi_K^v(\widetilde{R}_{A,\pi_K})^\times$.* $\square$

Let $m, n \in \mathbb{Z}$ be two integers with $m \le n$. When $\operatorname{char} K = p$, let $\widetilde{R}_{A,\pi_K}^{[m,n]}$ denote the affine A-group scheme which associates to any A-algebra A' the group

$$\{\sum_{i=m}^{n} a_i \pi_K^i\,;\, a_i \in A'\}.$$

There exists a canonical isomorphism of k-groups $\widetilde{R}_{A,\pi_K}^{[m,n]} \otimes_A k \cong K^{[m,n]}$. When $\operatorname{char} K = 0$, let $K_0 = \operatorname{Frac} W(k)$ and $e = [K : K_0]$. Let $\widetilde{R}_{A,\pi_K}^{[m,n]}$ denote the affine A-group scheme which associates to any A-algebra A' the group

$$\bigoplus_{i=0}^{e-1} W_{1+\lfloor \frac{n-m-i}{e} \rfloor}(A').$$

Then the multiplication by π_K^{-m} induces a canonical isomorphism of k-groups $\widetilde{R}_{A,\pi_K}^{[m,n]} \otimes_A k \cong K^{[m,n]}$.

Definition 4.25. Let L be a finite separable totally ramified extension of K of degree d. Let A be a subring of k, π_K an A-admissible prime element. A prime element π_L of L is called (A, π_K)-*admissible* over K if the minimal monic polynomial $f(T) \in \mathcal{O}_K[T]$ of π_L satisfies the following two conditions:

- $f(T) \in T^d + \pi_K \widetilde{R}_{A,\pi_K}[T]$ and $f(0) \in \pi_K(\widetilde{R}_{A,\pi_K})^\times$.
- Set $f(\pi_L + T) = T^d + \sum_{i=1}^{d-1} a_i T^i$. For each i, let $b_i = N_{L/K}(a_i)$ and $v_i = v_K(b_i)$. Then for any i such that $(i, v_K(a_i))$ is a vertex of the Newton polygon of $f(\pi_L + T)$, $b_i \in \pi_K^{v_i}(\widetilde{R}_{A,\pi_K})^\times$.

Definition 4.26. Let L/K be a finite Galois extension. Let L_0 be the maximal unramified subextension of L/K. A *pre A-structure of L/K* consists of the following data (π_K, B, π_L):

- π_K is an A-admissible prime element in K.
- B is a finite etale A-subalgebra of k_L such that $B \otimes_A k \cong k_L$.
- π_L is a prime element of L which is (B, π_K)-admissible over L_0 such that all $\mathrm{Gal}(L/K)$-conjugates of π_L belong to $\pi_L(\widetilde{R}_{B,\pi_L})^\times$.

Definition 4.27. An $\mathbb{F}_p$-algebra A is *good perfect* if A is isomorphic to the perfection of a smooth $\mathbb{F}_p$-algebra.

Lemma 4.28. *For any finite Galois extension L/K as above, there exists a good perfect $\mathbb{F}_p$-subalgebra A of k and a pre A-structure (π_K, B, π_L) of L/K.*

Proof. By Lemma 4.21, there exists a subring A_1 of k which is finitely generated over $\mathbb{F}_p$ and an A_1-admissible prime element π_K of K. Since k_L is a finite separable extension of k, there exists a monic polynomial $g(T) \in k[T]$ such that $k_L \cong k[T]/(g(T))$ as a k-algebra. Let A_2 be the subring of k obtained by adjoining all coefficients of $g(T)$ and by inverting the discriminant of $g(T)$. Then there exists a finite etale A_2-subalgebra B_2 of k_L such that $B_2 \otimes_{A_2} k \cong k_L$.

By Lemma 4.22, there exist a finitely generated B_2-subalgebra B_3 of k_L and a prime element π_L of L such that the minimal polynomial $f(T) \in L[T]$ of π_L over L_0 has coefficients in $\widetilde{R}_{B_3,\pi_K}$. There exists a finitely generated B_3-subalgebra B_4 of k_L such that π_L is (B_4, π_K)-admissible. By Lemma 4.23 (2) and Lemma 4.24, there exists a finitely generated B_4-subalgebra B_5 of k such that $(\pi_L, B_5^{\mathrm{perf}}, \pi_K)$ satisfies the third condition of Definition 4.26.

Take a finite set of generators $y_1, \ldots, y_n \in B_5$ of the B_2-algebra B_5. Let A_5 be the A_2-subalgebra of k obtained by adjoining all coefficients of the minimal polynomials of all y_i over k. There exists a non-empty affine open subscheme $\mathrm{Spec}\,(A_6)$ of $\mathrm{Spec}\,(A_5)$ which is smooth over $\mathbb{F}_p$. Let B_6 be the subring of k_L generated by A and B_2. Since B_6 is etale over $\mathbb{F}_p$, B is regular.

In particular B_6 is normal. Hence B_6 contains all y_i and π_L is $(B_6^{\mathrm{perf}}, \pi_K)$-admissible. We put $A = A_6^{\mathrm{perf}}$ and $B = B_6^{\mathrm{perf}}$. Then (π_K, B, π_L) is a pre A-structure of L/K. □

Lemma 4.29. *Let (π_K, B, π_L) be a pre A-structure of L/K. Then for any ring homomorphism from A to a perfect field k' of characteristic p, $K_{k'} := (\widetilde{R}_{A,\pi_K} \widehat{\otimes}_{R_A} R_{k'})[\frac{1}{\pi_K}]$ is a p-CDVF with residue field k'. Decompose $B \otimes_A k'$ into a direct product $\prod_i k'_i$ of a finite number of finite separable extensions of k'. Then $L_{k'} = (\widetilde{R}_{B,\pi_K}[\pi_L] \widehat{\otimes}_{R_A} R_{k'})[\frac{1}{\pi_K}]$ is a direct product $\prod_i L_{k'_i}$, where $L_{k'_i}$ is a finite separable extension of $K_{k'}$ with residue field k_i. The scheme* $\mathrm{Spec}\,(L_{k'})$ *is an etale* $\mathrm{Gal}(L/K)$*-torsor on* $\mathrm{Spec}\,(K_{k'})$*. Furthermore, for each i and for each $n \geq 0$, the lower numbering ramification subgroup* $\mathrm{Gal}(L_{k'_i}/K_{k'})_n$ *is canonically identified with* $\mathrm{Gal}(L/K)_n$.

Proof. All assertions are clear except the last one.

Let L_0 be the maximal unramified subextension of L/K and $f(T) \in L_0[T]$ the minimal polynomial of π_L over L_0. Then last assertion holds because the Herbrand function of L/L_0 is completely determined by the Newton polygon of $f(T + \pi_L) \in L[T]$. □

Let L/K be a finite totally ramified abelian extension of p-CDVFs. Let $n \in \mathbb{Z}_{\geq 0}$ be a non-negative integer such that the Herbrand function $\psi_{L/K}(v)$ is linear for $v > n$. Let $\mathrm{N}_{L/K} : U_{L,\psi_{L/K}(n)+1} \to U_{K,n+1}$ be the morphism of affine k-group schemes induced by norms, B_n the neutral component of the kernel of $\mathrm{N}_{L/K}$, and $V_{L/K,n}$ the quotient group $U_{L,\psi_{L/K}(n)+1}/B_n$. Then the kernel of the canonical morphism $\beta_{L/K,n} : V_{L/K,n} \to U_{K,n+1}$ is canonically isomorphic to the constant group scheme $\mathrm{Gal}(L/K)$. In particular the morphism $\beta_{L/K,n}$ is finite etale.

Let (π_K, A, π_L) be a pre A-structure of L/K. Let $U_{K,A}$ (resp. $U_{L,A}$) be the A-group scheme $(\widetilde{R}_{A,\pi_K})^\times$ (resp. $(\widetilde{R}_{A,\pi_L})^\times$). We define $U_{K,n+1,A}$ and $U_{L,\psi_{L/K}(n)+1,A}$ for $n \in \mathbb{Z}_{\geq 0}$ in a similar way. There exists norm maps $U_{L,A} \to U_{K,A}$ and $U_{L,\psi_{L/K}(n)+1,A} \to U_{K,n+1,A}$ which are homomorphisms of affine A-group schemes.

Definition 4.30. Let notation be as above. A pre A-structure (π_K, A, π_L) of is called *good* if there exists an affine A-group scheme $V_{L/K,n,A}$ and homomorphisms

$$\gamma_{L/K,n,A} : U_{L,\psi_{L/K}(n)+1,A} \to V_{L/K,n,A}, \ \beta_{L/K,n,A} : V_{L/K,n,A} \to U_{K,n+1,A}$$

of A-group schemes which satisfy the following four conditions:

- The composition $\beta_{L/K,n,A} \circ \gamma_{L/K,n,A}$ is equal to the norm map.
- The morphism $\beta_{L/K,n,A}$ is finite etale.

- The homomorphisms

$$U_{L,\psi_{L/K}(n)+1,A} \otimes_A k \xrightarrow{\gamma_{L/K,n,A}} V_{L/K,n,A} \otimes_A k \xrightarrow{\beta_{L/K,n,A}} U_{K,n+1,A} \otimes_A k$$

are canonically identified with the homomorphisms

$$U_{L,\psi_{L/K}(n)+1} \to V_{L/K,n} \xrightarrow{\beta_{L/K,n}} U_{K,n+1}.$$

- For any ring homomorphism $A \to B$, the homomorphism

$$\Gamma(V_{L/K,n,A}, \mathcal{O}_{V_{L/K,n,A}}) \otimes_A B \to \Gamma(U_{L,\psi_{L/K}(n),A}), \mathcal{O}_{U_{L,\psi_{L/K}(n),A}}) \otimes_A B$$

induced by $\gamma_{L/K,n,A}$ is injective.

Lemma 4.31. *There exists a good perfect subring A of k and a good pre A-structure (π_K, A, π_L) of L/K.*

Proof. By Lemma 4.28, there exists a good perfect subring A_1 of k and a pre A_1-structure (π_K, A_1, π_L) of L/K. Let us denote the coordinate rings of $U_{K,n+1,A_1}$, $U_{L,\psi_{L/K}(n)+1,A_1}$ and $V_{L/K,n}$ by C_{K,A_1}, C_{L,A_1} and C_V, respectively. There exist canonical injective ring homomorphisms

$$C_{K,A_1} \otimes_{A_1} k \to C_V \to C_{L,A_1} \otimes_{A_1} k.$$

We regard C_V as a subring of $C_{L,A_1} \otimes_{A_1} k$ by the latter homomorphism. The rings C_{K,A_1} and C_{L,A_1} are each isomorphic to polynomial rings over A_1 with finitely many variables. So we write $C_{K,A_1} = A_1[x_1, x_2, \cdots, x_m]$, and $C_{L,A_1} = A_1[y_1, y_2, \cdots, y_{m'}]$. The ring C_V is finite free as a $C_{K,A_1} \otimes_{A_1} k$-module. Take a $C_{K,A_1} \otimes_{A_1} k$-basis $b_1, \ldots, b_{n'}$ of C_V. There exist n' monomials $s_1, \ldots, s_{n'}$ of $y_1, \ldots, y_{m'}$ such that the matrix (c_{ij}) of coefficients of s_i in $b_j \in A_1[y_1, y_2, \cdots, y_{m'}]$ is invertible. Let I be the kernel of the homomorphism $\varphi : C_{K,A_1} \otimes_{A_1} k[z_1, \cdots, z_{n'}] \to C_V$ which sends z_i to b_i. Take a generator $f_1, \ldots, f_{n'}$ of I. Then the image of the determinant of the matrix $(\frac{\partial f_i}{\partial z_j})$ by φ belongs to $B^\times$.

There exists a finitely generated A_1-algebra A which satisfies the following seven properties:

- For all i, $b_i \in C_{L,A_1} \otimes_{A_1} A$.
- The determinant of the matrix (c_{ij}) belongs to $A^\times$.
- For all i_1, i_2, $b_{i_1} b_{i_2} \in \sum_j C_{K,A_1} \otimes_{A_1} A \cdot b_j$.
- For all i, $f_i \in C_{K,A_1} \otimes_{A_1} A[z_1, \cdots, z_{n'}]$.
- The image of the determinant of the matrix $(\frac{\partial f_i}{\partial z_j})$ by φ belongs to $(\bigoplus_i C_{K,A_1} \otimes_{A_1} A \cdot b_i)^\times$.
- Let

$$\Delta : C_V \to C_V \otimes_k C_V \cong \bigoplus_{i_1,i_2} (C_{K,A_1} \otimes_{A_1} C_{K,A_1}) \otimes_{A_1} k \cdot b_{i_1} \otimes b_{i_2}$$

be the morphism induced by the group law of $V_{L/K,n}$. Then $\Delta(b_j) \in (C_{K,A_1} \otimes_{A_1} C_{K,A_1}) \otimes_{A_1} A \cdot b_{i_1} \otimes b_{i_2}$.
- Let $S : C_V \to C_V$ be the morphism induced by the inverse morphism of $V_{L/K,n}$. Then $S(b_i) \in \bigoplus_j C_{K,A_1} \otimes_{A_1} A \cdot b_j$.

Put

$$C_{V,A} = \bigoplus_i C_{K,A_1} \otimes_{A_1} A \cdot b_i.$$

This is a finite etale $C_{K,A_1} \otimes_{A_1} A$-algebra. There exists a canonical structure of A-group scheme on $V_{L/K,n,A} = \operatorname{Spec}(C_{V,A})$. It is easily checked that this A and $V_{L/K,n,A}$ satisfy the desired properties. □

Definition 4.32. Let L/K be a finite separable extension, An A-*structure* of L/K is a pair $(B, (\pi_M)_M)$ which satisfies the following conditions:

- B is a finite etale A-subalgebra of k_L such that $B \otimes_A k \cong k_L$.
- $(\pi_M)_M$ is a system of a prime element π_M of M, where π_M runs over all subextensions of L/K.
- For any two subextensions M_1, M_2 of L/K with $M_2 \supset M_1$, let $B_{k_{M_i}}$ $(i = 1, 2)$ be the finite etale A-subalgebra of B corresponding to the residue field k_{M_i} of M_i. Then $(\pi_{M_1}, B_{k_{M_2}}, \pi_{M_2})$ is a pre $B_{k_{M_1}}$-structure of M_2/M_1.
- For any two subextensions M_1, M_2 of L/K with $M_2 \supset M_1$ such that M_2/M_1 is a totally ramified abelian extension, the pre $B_{k_{M_1}}$-structure $(\pi_{M_1}, B_{k_{M_1}}, \pi_{M_2})$ of M_2/M_1 is a good pre $B_{k_{M_1}}$-structure of M_2/M_1.

By Lemma 4.28 and Lemma 4.31, we have:

Lemma 4.33. *For any finite separable extension L/K as above, there exists a good perfect $\mathbb{F}_p$-subalgebra A of k and an A-structure $(B, (\pi_M)_M)$ of L/K.* □

Definition 4.34. Let L/K be a finite separable extension of p-CDVFs, R_0 a pro-finite p'-coefficient ring, and $\widetilde{\psi}$ a non-trivial additive character R_0-sheaf on K. Let $N > \operatorname{ord} \widetilde{\psi}$ be an integer. A pre A-structure (π_K, B, π_L) of L/K is called $(N, \widetilde{\psi})$-*admissible* if the following two conditions are satisfied:

- The sheaf $\widetilde{\psi}|_{K^{[-N,-\operatorname{ord}\widetilde{\psi}-1]}}$ is the pull back of an invertible character sheaf on the A-group scheme $\widetilde{R}_{A,\pi_K}^{[-N,-\operatorname{ord}\widetilde{\psi}-1]}$.
- There exists a unit $b \in A^\times$ such that when $K^{[-\operatorname{ord}\widetilde{\psi}-1,-\operatorname{ord}\widetilde{\psi}-1]}$ is identified with $\mathbb{G}_{a,k}$ by using π_K, the sheaf $\widetilde{\psi}|_{K^{[-\operatorname{ord}\widetilde{\psi}-1,-\operatorname{ord}\widetilde{\psi}-1]}}$ is the pull-back of a non-trivial Artin-Schreier sheaf on $\mathbb{G}_{a,\mathbb{F}_p}$ by the multiplication-by-b map $\mathbb{G}_{a,k} \to \mathbb{G}_{a,k} \to \mathbb{G}_{a,\mathbb{F}_p}$.

Definition 4.35. Let L/K, R_0 and $\widetilde{\psi}$ be as above. Let N be a sufficiently large integer. An A-structure $(B, (\pi_M)_M)$ of L/K is called $(N, \widetilde{\psi})$-*admissible* if for any two intermediate extensions M_1, M_2 of L/K with $M_2 \supset M_1$, the pre $B_{k_{M_1}}$-structure $(\pi_{M_1}, B_{M_2}, \pi_{M_2})$ of M_2/M_1 is $(N, \widetilde{\psi} \circ \mathrm{Tr})$-admissible.

Proposition 4.36. *Let L be a finite separable extension of K, R_0 a finite p'-coefficient ring, $\widetilde{\psi}$ a non-trivial additive character R_0-sheaf on K, and $N \in \mathbb{Z}$ an integer larger than* $\operatorname{ord} \widetilde{\psi}$. *Then there exists a good perfect $\mathbb{F}_p$-subalgebra $A \subset k$ and a $(N, \widetilde{\psi})$-admissible pre A-structure of L/K.*

Proof. By Lemma 4.28, there exists a good perfect $\mathbb{F}_p$-subalgebra A_1 of k and a pre A_1-structure (π_K, B_1, π_L) of L/K. It is easily checked that there exists a good perfect $\mathbb{F}_p$-subalgebra A_2 of k which satisfies the two conditions in the definition of $(N, \widetilde{\psi})$-admissibility. We may take as A an arbitrary $\mathbb{F}_p$-subalgebra of k which is the perfection of a smooth $\mathbb{F}_p$-algebra and which contains both A_1 and A_2. $\square$

Corollary 4.37. *Let L, R_0, and $\widetilde{\psi}$ be as the above proposition. Let $N \in \mathbb{Z}$ be a sufficiently large integer. Then there exists a good perfect $\mathbb{F}_p$-subalgebra $A \subset k$ and a $(N, \widetilde{\psi})$-admissible A-structure of L/K.* $\square$

Proposition 4.38. *Let L be a finite Galois extension of K, R_0 a finite subring of R, $\widetilde{\psi}$ a non-trivial additive character R_0-sheaf on K, and χ a rank one object in* $\operatorname{Rep}(W_K, R_0)$ *which comes from an object in* $\operatorname{Rep}(\operatorname{Gal}(L/K), R_0)$. *Let N be an integer larger than* $\operatorname{ord} \widetilde{\psi} + \operatorname{sw}(\chi) + 1$. *Let A be a smooth $\mathbb{F}_p$-algebra, and let $(B, (\pi_M)_M)$ be a $(N, \widetilde{\psi})$-admissible A-structure of L/K. Let $\widetilde{\psi}_A$ be an invertible character sheaf on the A-group scheme $\widetilde{R}_{A,\pi_K}^{[-N, -\operatorname{ord} \psi - 1]}$ whose pull-back on $K^{[-N, -\operatorname{ord} \psi - 1]}$ is isomorphic to $\widetilde{\psi}|_{K^{[-N, -\operatorname{ord} \widetilde{\psi} - 1]}}$.*

Then there exists a smooth invertible R_0-sheaf $\widetilde{\varepsilon}_{0,R_0,A}(\chi, \widetilde{\psi})$ on $\operatorname{Spec}(A)$ *which satisfies the following property:*

> *For any ring homomorphism from A to a perfect field k' of characteristic p, let $K_{k'}$ be as in Lemma 4.29. Let $\chi_{k'}$ be the rank one object in* $\operatorname{Rep}(W_{K_{k'}}, R_0)$ *induced from the canonical homomorphism $W_{K_{k'}} \to \operatorname{Gal}(L/K)$ and χ. Let $\widetilde{\psi}_{k'}$ be the pull-back of $\widetilde{\psi}_A$ by the canonical morphism*
>
> $$K_{k'}^{[-N, -\operatorname{ord} \widetilde{\psi} - 1]} \cong \widetilde{R}_{A,\pi_K}^{[-N, -\operatorname{ord} \widetilde{\psi} - 1]} \otimes_A k' \to \widetilde{R}_{A,\pi_K}^{[-N, -\operatorname{ord} \widetilde{\psi} - 1]}.$$
>
> *Then the pull-back of $\widetilde{\varepsilon}_{0,R_0,A}(\chi, \widetilde{\psi})$ on* $\operatorname{Spec}(k')$ *is the smooth invertible R_0-sheaf on* $\operatorname{Spec}(k')$ *corresponding to $\widetilde{\varepsilon}_{0,R}(\chi_{k'}, \widetilde{\psi}_{k'})$.*

Proof. Decompose χ into the tensor product $\chi = \chi_1 \otimes \chi_2$ of two rank one objects χ_1, χ_2 in $\mathrm{Rep}(W_K, R_0)$, both of which come from objects in $\mathrm{Rep}(\mathrm{Gal}(L/K), R_0)$, such that χ_1 is an unramified and that the extension K'/K corresponding to $\mathrm{Ker}\,\chi_2$ is a totally ramified cyclic abelian extension. The etale A-algebra B and χ_1 induces a smooth invertible sheaf $\chi_{1,A}$ on $\mathrm{Spec}\,(A)$.

Let $s = \mathrm{sw}(\chi_2)$. For $m \in \mathbb{Z}$, let $K_{s,A}^{v=m}$ (resp. $(K'_{s,A})^{v=m}$) denote the A-scheme $U_{K,s+1,A}$ (resp. $U_{K',\psi_{K'/K}(s)+1,A}$). The scheme $\coprod_{m\in\mathbb{Z}} K_{s,A}^{v=m} = U_{K,s+1,A} \times \mathbb{Z}$ (resp. $\coprod_{m\in\mathbb{Z}} (K'_{s,A})^{v=m} = U_{K',\psi_{K'/K}(s)+1,A} \times \mathbb{Z}$) has a canonical structure of an A-group scheme. The multiplication by π_K^m (resp. $\pi_{K'}^m$) induces a canonical isomorphism $K_{s,A}^{v=m} \otimes_A k \cong K^{v=m}/U_K^{(s+1)}$ (resp. $(K'_{s,A})^{v=m} \otimes_A k \cong (K')^{v=m}/U_{K'}^{(\psi_{K'/K}(s)+1)}$).

The norm map $\mathrm{N}_{K'/K} : K' \to K$ induces a homomorphism $\mathrm{N}_{K'/K,A} : \coprod_{m\in\mathbb{Z}} (K'_{s,A})^{v=m} \to \coprod_{m\in\mathbb{Z}} K_{s,A}^{v=m}$ of A-group schemes. By the definition of good pre A-structure, there exists an affine A-group scheme $\coprod_m V_{K'/K,s,A}^{v=m}$ and homomorphisms

$$\gamma_{K'/K,s,A} : \coprod_{m\in\mathbb{Z}} (K'_{s,A})^{v=m} \to \coprod_m V_{K'/K,s,A}^{v=m}$$

and

$$\beta_{K'/K,s,A} : \coprod_m V_{K'/K,s,A}^{v=m} \to \coprod_{m\in\mathbb{Z}} K_{s,A}^{v=m}$$

of A-group schemes such that $\beta_{K'/K,s,A} \circ \gamma_{K'/K,s,A} = \mathrm{N}_{K'/K,A}$, that $\beta_{K'/K,s,A}$ is finite etale and the kernel of $\beta_{K'/K,s,A}$ is isomorphic to the constant group scheme $\mathrm{Gal}(K'/K)$, and that for any A-algebra B, the homomorphism

$$\Gamma(V_{K'/K,s,A}^{v=m}, \mathcal{O}_{V_{K'/K,s,A}^{v=m}}) \otimes_A B \to \Gamma((K'_{s,A})^{v=m}, \mathcal{O}_{(K'_{s,A})^{v=m}}) \otimes_A B$$

is injective. Let $\mathcal{L}_{\chi_2,A}$ be the invertible character sheaf on $\coprod_{m\in\mathbb{Z}} K_{s,A}^{v=m}$ defined by $\beta_{K'/K,s,A}$ and χ_2.

Let $m_0 = -s - \mathrm{ord}\,\widetilde{\psi} - 1$ and let $\mathcal{L}_{\chi,A}$ be the invertible sheaf $\mathcal{L}_{\chi_2,A}|_{K_{s,A}^{v=m_0}} \otimes \pi^* \chi_{1,A}^{\otimes m_0}$ on $K_{s,A}^{v=m_0}$, where $\pi : K_{s,A}^{v=m_0} \to \mathrm{Spec}\,(A)$ is the structure morphism.

Let $\widetilde{\varepsilon}_{0,R_0,A}(\chi, \widetilde{\psi})$ be the invertible sheaf

$$\widetilde{\varepsilon}_{0,R_0,A}(\chi, \widetilde{\psi}) = \det{}_{R_0}(Rf_!((\overline{\mathcal{L}}_{\chi A}^{\otimes -1} \otimes_{R_0} i^* \widetilde{\psi}_A^{[m, -\mathrm{ord}\,\widetilde{\psi} - 1]})[s+1](\mathrm{ord}\,\widetilde{\psi})).$$

It is easy to prove that this $\widetilde{\varepsilon}_{0,R_0,A}(\chi, \widetilde{\psi})$ has desired properties. □

Corollary 4.39. *Let L, R_0, and $\widetilde{\psi}$ be as the above proposition. Let N be a sufficiently large integer. Let A be a good perfect $\mathbb{F}_p$-algebra, and* *let $(B,(\pi_M)_M)$ be a $(N,\widetilde{\psi})$-admissible A-structure of L/K. Let $\widetilde{\psi}_A$ be an invertible character sheaf on the A-group scheme $\widetilde{R}_{A,\pi_K}^{[-N,-\mathrm{ord}\,\psi-1]}$ whose pull-back on $K^{[-N,-\mathrm{ord}\,\psi-1]}$ is isomorphic to $\widetilde{\psi}|_{K^{[-N,-\mathrm{ord}\,\widetilde{\psi}-1]}}$.*

Then there exists a smooth invertible R_0-sheaf $\widetilde{\lambda}_{R_0,A}(L/K,\widetilde{\psi})$ on Spec (A) *which satisfies the following property:*

> *For any ring homomorphism from A to a perfect field k' of characteristic p, let $K_{k'}$ be as in Lemma 4.29. Let $\widetilde{\psi}_{k'}$ be the pull-back of $\widetilde{\psi}_A$ by the canonical morphism*
> $$K_{k'}^{[-N,-\mathrm{ord}\,\widetilde{\psi}-1]} \cong \widetilde{R}_{A,\pi_K}^{[-N,-\mathrm{ord}\,\widetilde{\psi}-1]} \otimes_A k' \to \widetilde{R}_{A,\pi_K}^{[-N,-\mathrm{ord}\,\widetilde{\psi}-1]}.$$
> *Then the pull-back of $\widetilde{\lambda}_{R_0,A}(L/K,\widetilde{\psi})$ on* Spec (k') *is the smooth invertible R_0-sheaf on* Spec (k') *corresponding to $\widetilde{\lambda}_R(L_{k'}/K_{k'},\widetilde{\psi}_{k'})$.*

□

4.8. Proof of Lemma 4.18. Suppose that
$$\rho = \sum_i n_i \mathrm{Ind}_{H_i}^G \chi_i = \sum_j n'_j \mathrm{Ind}_{H'_j}^G \chi'_j.$$

Take a sufficiently large integer $N \in \mathbb{Z}$. Take a good perfect $\mathbb{F}_p$-subalgebra A of k and a $(\widetilde{\psi},N)$-admissible A-structure $(B,(\pi_M)_M)$ of L/K. Take an invertible character sheaf $\widetilde{\psi}_A$ on the A-group scheme $\widetilde{R}_{A,\pi_K}^{[-N,-\mathrm{ord}\,\psi-1]}$ whose pull-back on $K^{[-N,-\mathrm{ord}\,\psi-1]}$ is isomorphic to $\widetilde{\psi}|_{K^{[-N,-\mathrm{ord}\,\widetilde{\psi}-1]}}$. We define, in a canonical way, smooth invertible R-sheaves $\widetilde{\varepsilon}_{0,R,A}(\chi_i,\widetilde{\psi}\circ \mathrm{Tr}_{K_i/K})$, $\widetilde{\varepsilon}_{0,R,A}(\chi'_j,\widetilde{\psi}\circ \mathrm{Tr}_{K'_j/K})$, $\widetilde{\lambda}_{R,A}(K'_i/K,\widetilde{\psi})$, and $\widetilde{\lambda}_{R,A}(K'_j/K,\widetilde{\psi})$ on Spec (A) whose pull-backs on Spec (k) correspond to $\widetilde{\varepsilon}_{0,R}(\chi_i,\widetilde{\psi}\circ \mathrm{Tr}_{K_i/K})$, $\widetilde{\varepsilon}_{0,R}(\chi'_j,\widetilde{\psi}\circ \mathrm{Tr}_{K'_j/K})$, $\widetilde{\lambda}_R(K'_i/K,\widetilde{\psi})$, and $\widetilde{\lambda}_R(K'_j/K,\widetilde{\psi})$ respectively.

Let $\widetilde{\varepsilon}_{0,R,A}(V,\widetilde{\psi})$ and $\widetilde{\varepsilon}'_{0,R,A}(V,\widetilde{\psi})$ be two invertible sheaves on Spec (A) defined by
$$\widetilde{\varepsilon}_{0,R,A}(V,\widetilde{\psi}) := \bigotimes_i (\widetilde{\varepsilon}_{0,R,A}(\chi_i,\widetilde{\psi}\circ \mathrm{Tr}_{K_i/K}) \otimes \widetilde{\lambda}_R(K_i/K,\widetilde{\psi}))^{\otimes n_i}.$$

and
$$\widetilde{\varepsilon}'_{0,R,A}(V,\widetilde{\psi}) := \bigotimes_j (\widetilde{\varepsilon}_{0,R,A}(\chi'_j,\widetilde{\psi}\circ \mathrm{Tr}_{K'_j/K}) \otimes \widetilde{\lambda}_R(K'_j/K,\widetilde{\psi}))^{\otimes n'_j}.$$

Let $x \in$ Spec (A) be a closed point. We denote by $\kappa(x)$ (resp. i_x) the residue field at x (resp. the canonical inclusion) $i_x : x \hookrightarrow$ Spec (A). Let K_x denote the field $K_{\kappa(x)}$ introduced in Lemma 4.29 for the ring homomorphism $A \to \kappa(x)$. The canonical homomorphism $W_{K_x} \to \mathrm{Gal}(L/K)$ and V defines an object in $\mathrm{Rep}(W_{K_x},R)$ which we denote by V_x. Then we have an isomorphism $i_x^*\widetilde{\varepsilon}_{0,R,A}(V,\widetilde{\psi}) \cong i_x^*\widetilde{\varepsilon}'_{0,R,A}(V,\widetilde{\psi})$, since the main result of [10] implies that both sides, as characters of $W_{\kappa(x)}$, send the geometric Frobenius at x to $(-1)^{\operatorname{rank} V+\operatorname{sw}(V)}\varepsilon_{0,R}(V_x,\psi_x)$, where ψ_x denotes the additive character of K_x corresponding to the specialization of $\widetilde{\psi}_A$ at x. Let $A_0 \subset A$ be a smooth $\mathbb{F}_p$-subalgebra of A whose perfection equals A. By the standard argument using Chebotarev's theorem which states that the geometric Frobenii at the closed points are dense in $\pi_1(\operatorname{Spec}(A_0))$ ([7, Theorem 7]), we conclude that $\widetilde{\varepsilon}_{0,R,A}(V,\widetilde{\psi}) \cong \widetilde{\varepsilon}'_{0,R,A}(V,\widetilde{\psi})$. This completes the proof. □

4.9. Local $\widetilde{\varepsilon}_0$-characters of representations of G_K (field coefficients). Let R be an algebraically closed field of positive characteristic $\neq p$. In this subsection we shall define, for an object (ρ,V) in $\operatorname{Rep}(W_K,R)$, a rank one object $\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi})$ in $\operatorname{Rep}(W_k,R)$, which is called the local $\widetilde{\varepsilon}_0$-character of V.

First assume that V is irreducible. Then V is of the form $V = V' \otimes_R \chi$, where V' is an object in $\operatorname{Rep}(W_K,R)$ such that the image of W_K in $GL_R(V')$ is finite, and χ is an unramified rank one object in $\operatorname{Rep}(W_K,R)$. Define $\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi})$ to be $\widetilde{\varepsilon}_{0,R}(V',\widetilde{\psi}) \otimes_R \chi^{\otimes \operatorname{sw}(V)+\operatorname{rank} V\cdot(\operatorname{ord}\widetilde{\psi}+1)}$. By Lemma 4.19, $\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi})$ is independent of the choice of V' and χ.

For a general V, let $V_1,\dots,V_n$ denote the Jordan-Hölder constituents of V counted with multiplicity. Define $\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi})$ to be $\bigotimes_i \widetilde{\varepsilon}_{0,R}(V_i,\widetilde{\psi})$.

4.10. Local $\widetilde{\varepsilon}_0$-characters for torsion coefficients (totally wild case). Let $(R,\mathfrak{m}_R)$ be a complete strict p'-coefficient ring with a positive residue characteristic and $\widetilde{\psi}$ a non-trivial additive character R-sheaf on K.

Assume that $p \neq 2$. For $x \in K^\times$ with $v_K(x) + \operatorname{ord}\widetilde{\psi} = 2b+1$ is odd, define a quadratic Gauss sum sheaf $\widetilde{\tau}_{K,\widetilde{\psi}}(x)$ by

$$\widetilde{\tau}_{K,\widetilde{\psi}}(x) = \widetilde{\varepsilon}_{0,R}(\chi_{-\frac{x}{2}},\widetilde{\psi}) \otimes_R R(\operatorname{ord}\widetilde{\psi}),$$

where $\chi_{-\frac{x}{2}} : W_K \to R^\times$ is the composition of the quadratic character $W_K \to \{\pm 1\}$ corresponding to the quadratic extension $K(\sqrt{-\frac{x}{2}})$ of K and the canonical map $\{\pm 1\} \to R^\times$. The sheaf $\widetilde{\tau}_{K,\widetilde{\psi}}(x)$ does not depend on the choice of y and depends only on the class of $x \in \{x \in K^\times \mid v_K(x)+\operatorname{ord}\widetilde{\psi} \equiv 1 \bmod 2\}$ in $(K^\times/1+\mathfrak{m}_K) \otimes \mathbb{Z}/2\mathbb{Z}$. Thus we can define $\widetilde{\tau}_{K,\widetilde{\psi}}(x)$ for $x \in \{x \in (K^\times/1+\mathfrak{m}_K) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}] \mid v_K(x) + \operatorname{ord}\widetilde{\psi} \in 1 + 2\mathbb{Z}[\frac{1}{p}]\}$.

Remark 4.40. Suppose that k is finite. Let $\psi : K \to R^\times$ be the additive character of K corresponding to $\widetilde{\psi}$. Then we have

$$\widetilde{\tau}_{K,\widetilde{\psi}}(x)(\mathrm{Fr}_k) = -\tau_{K,\psi}(x),$$

where $\tau_{K,\psi}(x)$ is the quadratic Gauss sum defined in [10, § 7.3].

Definition 4.41. Let $v \in \mathbb{Q}_{>0}$, $G = W_K$, and let $\chi \in \mathrm{Hom}(G^v/G^{v+}, R^\times)$. Let K_χ be the extension of K corresponding to the stabilizing subgroup of χ, and k_χ the residue field of K_χ. Let $r \in \mathbb{Z}$ be an integer such that $rv \in \mathbb{Z}$. Define the *Gauss sum part sheaf* $g_R(\chi, \widetilde{\psi})^{\otimes r}$ *of* $\overline{\overline{\varepsilon}}_{0,R}$*-constant* to be the object in $\mathrm{Rep}(W_k, R)$ defined by

$$g_R(\chi, \widetilde{\psi})^{\otimes r} = (\widetilde{\lambda}_R(K_\chi/K, \widetilde{\psi}))^{\otimes r} \otimes_R R(-[k_\chi : k]\mathrm{ord}\,(\widetilde{\psi} \circ \mathrm{Tr}_{K_\chi/K}))$$
$$\otimes_R \begin{cases} R(-[k_\chi : k]r \cdot \frac{1+w}{2}) \\ \quad \text{if } p = 2 \text{ or } p \neq 2 \text{ and } \mathrm{ord}_2(v) \leq 0, \\ R(-[k_\chi : k]r \cdot \frac{w}{2}) \otimes_R \widetilde{\tau}_{K_\chi, \widetilde{\psi} \circ \mathrm{Tr}_{K_\chi/K}}(\sigma_{\widetilde{\psi}}(\chi))^{\otimes r} \\ \quad \text{if } p \neq 2 \text{ and } \mathrm{ord}_2(v) > 0, \end{cases}$$

where $w = e_{K_\chi/K} v$.

Let (ρ, V) be an object in $\mathrm{Rep}(W_K, R)$ which is pure of slope v and of refined slope Σ. As in [10, § 7], for $w \in \mathbb{Q}$, let N_K^w denote the set

$$N_K^w := \{x \in \overline{K} | v_K(x) \geq w\} / \{x \in \overline{K} | v_K(x) > w\},$$

endowed with the canonical W_K-action. By [5, p. 3, Thm. 1] there is a canonical isomorphism from $\mathrm{Hom}(G^v/G^{v+}, R)$ to the set of all isomorphism classes of character sheaves on N_K^v considered as a $\overline{k}$-group scheme. This isomorphism and the additive character sheaf $\widetilde{\psi}$ induce a canonical isomorphism $\sigma_{\widetilde{\psi}} : \mathrm{Hom}(G^v/G^{v+}, \mathbb{Z}/p\mathbb{Z}) \to N_K^{-v-\mathrm{ord}\,\widetilde{\psi}-1}$ in a similar way as in [10, § 7].

Take a $\chi \in \Sigma$, and let V' be the χ-part of V. Then V' is an object in $\mathrm{Rep}(W_{K_\chi}, R)$.

Let K_χ be the extension of K corresponding to the stabilizing subgroup of χ, and k_χ the residue field of K_χ. We consider the element $\sigma_{\widetilde{\psi}}(\chi) \in N_K^{-v-\mathrm{ord}\,\widetilde{\psi}-1}$ as an element in $(K_\chi^\times/1 + \mathfrak{m}_{K\chi}) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$.

Definition 4.42. Let R be a p'-coefficient ring, and V an object in $\mathrm{Rep}(G, R)$ which is pure of refined slope Σ. We define the *refined* $\widetilde{\psi}$*-Swan conductor* $\mathrm{rsw}_{\widetilde{\psi}}(V) \in K^\times/1 + \mathfrak{m}_K$ by

$$\mathrm{rsw}_{\widetilde{\psi}}(V) = \mathrm{N}_{K_\chi/K}(\sigma_{\widetilde{\psi}}(\chi))^{-\frac{\mathrm{rank}\,V}{e_{K_\chi/K}}}.$$

For an arbitrary object W in $\mathrm{Rep}(G, R)$, define $\mathrm{rsw}_{\widetilde{\psi}}(W) \in K^\times/1 + \mathfrak{m}_K$ by

$$\mathrm{rsw}_{\widetilde{\psi}}(W) = \prod_{\Sigma'} \mathrm{rsw}_{\widetilde{\psi}}(W^{\Sigma'}),$$

where

$$W = W^0 \oplus \bigoplus_{\Sigma'} W^{\Sigma'}$$

is the refined slope decomposition of W.

Lemma 4.43. *The p-power map* $\mathrm{Hom}_{\mathrm{cont}}(W_k, R^\times) \to \mathrm{Hom}_{\mathrm{cont}}(W_k, R^\times)$ *is surjective.*

Proof. It suffices to prove that the p-power map $H^1(k, R^\times) \to H^1(k, R^\times)$ is surjective. Since $H^2(k, \mathbb{Z}/p\mathbb{Z}) = \{0\}$ by Artin-Schreier theory, the map $H^1(k, (R/\mathfrak{m}_R)^\times) \to H^1(k, (R/\mathfrak{m}_R)^\times)$ induced by the p-power map $(R/\mathfrak{m}_R)^\times \to (R/\mathfrak{m}_R)^\times$ is surjective. Since the p-power map $1 + \mathfrak{m}_R \to 1 + \mathfrak{m}_R$ is a homeomorphism, it suffices to prove that the natural map $H^1(k, R^\times) \to H^1(k, (R/\mathfrak{m}_R)^\times)$ is surjective. Let ℓ be the residue characteristic of R. Then the composition

$$H^1(k, (R/\mathfrak{m}_R)^\times) \cong H^1(k, \overline{\mathbb{F}}_\ell^\times) \to H^1(k, W(\overline{\mathbb{F}}_\ell)^\times) \to H^1(k, R^\times)$$

gives the right inverse of the last map. This complete the proof. $\square$

Let $\overline{\mathrm{Hom}}(W_k, R^\times)$ be the quotient of $\mathrm{Hom}_{\mathrm{cont}}(W_k, R^\times)$ by the subgroup of the characters of p-power orders. For an object V in $\mathrm{Rep}(G, R)$ which is pure of refined slope Σ, define the $\overline{\widetilde{\varepsilon}}_0$-character of V to be an element in $\overline{\mathrm{Hom}}(W_k, R^\times)$ defined by

$$\overline{\widetilde{\varepsilon}}_{0,R}(V, \widetilde{\psi}) := ((\det V')_{[\sigma_{\widetilde{\psi}}(\chi)]} \circ \mathrm{Ver}_{W_{k_\chi}}^{W_k})^{\otimes -1} \otimes g_R(\chi, \widetilde{\psi})^{\otimes \mathrm{rank}\, V'}.$$

Here $\mathrm{Ver}_{W_{k_\chi}}^{W_k}$ is the transfer map.

Lemma 4.44. *Suppose that R is a field. Then $\overline{\widetilde{\varepsilon}}_{0,R}(V, \widetilde{\psi})$ is equal to the class of $\widetilde{\varepsilon}_{0,R}(V, \widetilde{\psi})$ in* $\overline{\mathrm{Hom}}(W_k, R^\times)$.

Proof. We may assume that (ρ, V) is irreducible. Twisting V by an unramified character, we may assume that the image of ρ is finite. Take a representative $\widetilde{\widetilde{\varepsilon}}_{0,R}(V, \widetilde{\psi}) \in \mathrm{Hom}_{\mathrm{cont}}(W_k, R^\times)$ of $\overline{\widetilde{\varepsilon}}_{0,R}(V, \widetilde{\psi})$. Take a finite Galois extension L of K such that ρ factors through the finite quotient $G = \mathrm{Gal}(L/K)$ of W_K. By Brauer's theorem of modular representations, we may assume that V is of the form $V = \mathrm{Ind}_H^G \chi$ for a subgroup $H \subset G$ and a character χ of H. Let K' be the subextension of L/K corresponding to H. There exist a sufficiently large integer $N \in \mathbb{Z}$ and a good perfect $\mathbb{F}_p$-subalgebra A of k and a $(N, \widetilde{\psi})$-admissible A-structure $(B, (\pi_M)_M)$ of L/K such that we can define smooth invertible R-sheaves $\widetilde{\varepsilon}_{0,R,A}(V, \widetilde{\psi})$ and $\widetilde{\widetilde{\varepsilon}}_{0,R,A}(V, \widetilde{\psi})$ on $\mathrm{Spec}\,(A)$ whose pull-backs on $\mathrm{Spec}\,(k)$ correspond to $\widetilde{\varepsilon}_{0,R,A}(V, \widetilde{\psi})$ and $\widetilde{\widetilde{\varepsilon}}_{0,R,A}(V, \widetilde{\psi})$, respectively. Then for any closed point $x \in \mathrm{Spec}\,(A)$, there exists an integer n_x such that $i_x^* \widetilde{\varepsilon}_{0,R,A}(V, \widetilde{\psi})^{\otimes p^{n_x}} \cong i_x^* \widetilde{\varepsilon}'_{0,R,A}(V, \widetilde{\psi})^{\otimes p^{n_x}}$, where $i_x : x \hookrightarrow \mathrm{Spec}\,(A)$ is the canonical inclusion. By the construction of $\widetilde{\varepsilon}_{0,R,A}(V, \widetilde{\psi})$ and $\widetilde{\widetilde{\varepsilon}}_{0,R,A}(V, \widetilde{\psi})$, there exists a positive integer $n \in \mathbb{Z}_{>0}$ such that $n_x \leq n$ for every closed point $x \in \operatorname{Spec}(A)$. Since $(\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi}) \otimes_R \widetilde{\overline{\varepsilon}}_{0,R}(V,\widetilde{\psi})^{\otimes -1})^{\otimes p^n}$ is trivial by Chebotarev's theorem, this completes the proof. $\square$

Definition 4.45. Define the *local $\widetilde{\varepsilon}_0$-character* $\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi})$ to be the unique continuous character $W_k \to R^\times$ satisfying

$$\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi}) = \widetilde{\overline{\varepsilon}}_{0,R}(V,\widetilde{\psi}) \text{ in } \overline{\operatorname{Hom}}(W_k, R^\times)$$

and

$$\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi}) \mod \mathfrak{m}_R = \widetilde{\varepsilon}_0(V \otimes_R R/\mathfrak{m}_R, \widetilde{\psi}).$$

Proposition 4.46 (cf. [10, Prop. 8.3])**.** *Let R be a strict p'-coefficient ring, R_0 a pro-finite subring of R, and $V \neq \{0\}$ a totally wild object in* $\operatorname{Rep}(G, R_0)$*. Then for every tamely ramified object W in* $\operatorname{Rep}(G,R_0)$*, we have*

$$\widetilde{\varepsilon}_{0,R}(V \otimes_R W, \widetilde{\psi}) = (\det W)_{[\operatorname{rsw}_{\widetilde{\psi}}(V)]} \otimes \widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi})^{\operatorname{rank} W}.$$

Proof. We may assume that R_0 is finite and that V is pure of refined slope Σ. Take $\chi \in \Sigma$ and let K_χ be the finite separable extension of K corresponding to the stabilizing subgroup of χ.

Take a sufficiently large Galois extension L of K containing K_χ such that V and W come from objects in $\operatorname{Rep}(\operatorname{Gal}(L/K), R_0)$.

Take a sufficiently large integer $N \in \mathbb{Z}$. Take a good perfect $\mathbb{F}_p$-subalgebra A of k and a $(N,\widetilde{\psi})$-admissible A-structure $(B, (\pi_M)_M)$ of L/K. Then we can define, in a canonical way, smooth invertible R-sheaves $\widetilde{\varepsilon}_{0,R,A}(V,\widetilde{\psi})$ and $\widetilde{\varepsilon}_{0,R,A}(V \otimes W, \widetilde{\psi})$ on $\operatorname{Spec}(A)$ whose pull-backs on $\operatorname{Spec}(k)$ are identified with $\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi})$ and $\widetilde{\varepsilon}_{0,R}(V \otimes W, \widetilde{\psi})$ respectively.

Let $v \in \mathbb{Q}_{>0}$ be the slope of V. Let K' and L' be the subextensions of L/K_χ corresponding to $\operatorname{Gal}(L/K_\chi)^{e_{K_\chi/K} v}$ and $\operatorname{Gal}(L/K_\chi)^{e_{K_\chi/K} v+}$, respectively. Let $v_{K'} = \psi_{K'/K}(v)$ and $v_{L'} = \psi_{L'/K}(v)$. Consider the homomorphism

$$\alpha : L'/K', v_{K'} : \mathfrak{m}_{L'}^{v_{L'}}/\mathfrak{m}_{L'}^{v_{L'}+1} \to \mathfrak{m}_{K'}^{v_{K'}}/\mathfrak{m}_{K'}^{v_{K'}+1}$$

defined by $\mathrm{N}_{L'/K'}(1+x) = 1 + \alpha : L'/K', v_{K'}$. It is easily checked that the homomorphism $\alpha : L'/K', v_{K'}$ is induced from a morphism

$$\widetilde{\alpha} : \widetilde{R}^{[v_{L'}, v_{L'}]}_{B_{k'_K}, \pi_{L'}} \to \widetilde{R}^{[v_{K'}, v_{K'}]}_{B_{k'_K}, \pi_{K'}}$$

of $B_{k'_K}$-group schemes. By localizing A if necessary, we may assume that $\widetilde{\alpha}$ is finite etale and $\operatorname{Ker}\widetilde{\alpha}$ is constant. Put $A_\chi = A_{k_{K_\chi}}$. Using the character sheaf on $\widetilde{R}^{[v_{K'}, v_{K'}]}_{B_{k'_K}, \pi_{K'}}$ and the sheaf $\widetilde{\psi}_{A_\chi}$, we can define the A_χ-valued point of the group scheme $\coprod_{m \in \mathbb{Z}[\frac{1}{p}]} \mathbb{G}_{m, A_\chi}$ which induces $\sigma_{\widetilde{\psi}}(\chi)$. Using this, we can define the refined swan conductor $\operatorname{rsw}_{\widetilde{\psi},A}(V)$ of V as a A-valued point of the group scheme $\coprod_{m \in \mathbb{Z}[\frac{1}{p}]} \mathbb{G}_{m, A_\chi}$.

On the other hand, we have an invertible character sheaf $(\det W)_A$ on $\coprod_{m\in\mathbb{Z}} \mathbb{G}_{m,A}$ corresponding to $\det W$. Thus we obtain an invertible sheaf $(\det W)_{A,[\mathrm{rsw}_{\widetilde{\psi},A}(V)]}$ on $\mathrm{Spec}\,(A)$ which induces $(\det W)_{[\mathrm{rsw}_{\widetilde{\psi}}(V)]}$. Then we have

$$i_z^*\widetilde{\varepsilon}_{0,R,A}(V\otimes_R W,\widetilde{\psi}) = i_z^*\left((\det W)_{A,[\mathrm{rsw}_{\widetilde{\psi},A}(V)]}\otimes\widetilde{\varepsilon}_{0,R,A}(V,\widetilde{\psi})^{\otimes \mathrm{rank}\, W}\right)$$

for all closed point $i_z : z \hookrightarrow \mathrm{Spec}\,(A)$. Hence the assertion follows. □

4.11. Local $\widetilde{\varepsilon}_0$-characters for torsion coefficients (tame case). Let R be a complete strict p'-coefficient ring with a positive residue characteristic.

Define the k-algebra $\mathrm{Gr}^\bullet K$, $\mathrm{Gr}^{\geq 0}K$ and $\widehat{\mathrm{Gr}}^\bullet K$ in a similar way to that in [10, § 10.1]. Let ℓ be the residue characteristic of R, and set $R_0 := W(\mathbb{F}_\ell(\boldsymbol{\mu}_p))$. Let ϕ_0 be a non-trivial additive character R_0-sheaf on $K^{[-1,-1]}$. Let $\mathcal{L}_{\phi_0}$ be the invertible R_0-sheaf on $\mathrm{Spec}\,(\mathrm{Gr}^{\geq 0}K)$ corresponding to ϕ_0 by the canonical isomorphism $K^{[-1,-1]}\cong \mathrm{Spec}\,(\mathrm{Gr}^{\geq 0}K)$.

Define schemes X_0, X, and X_m, groups G, I, and I_m and a smooth R_0-sheaf $\widetilde{\mathcal{L}}'_{\phi_0}$ on X_0 in a similar way to that in [10, § 10.1]. Put $W_m := H^1_c(X_m,\widetilde{\mathcal{L}}'_{\phi_0})$. W_m is a free $R_0[I_m]$-module of rank one with a semi-linear action of $\mathrm{Gal}(X_m/X_0)$.

Definition 4.47. Let (ρ,V) is a tamely ramified object in $\mathrm{Rep}(W_K,R)$. Let $\widetilde{\psi}_0$ be a non-trivial additive character R-sheaf on $K^{[0,0]}$.

For each $n\in\mathbb{Z}_{>0}$, take a sufficiently divisible m such that W_K^0 acts on $V\otimes_R R/\mathfrak{m}_R^n$ via the quotient I_m.

Define the $R/\mathfrak{m}_R^n[W_k]$-module $\widetilde{\varepsilon}_{0,R/\mathfrak{m}_R^n}(V\otimes_R R/\mathfrak{m}_R^n,\widetilde{\psi}_0,\phi_0)$ by

$$\begin{aligned}\widetilde{\varepsilon}_{0,R/\mathfrak{m}_R^n}(V\otimes_R R/\mathfrak{m}_R^n,\widetilde{\psi}_0,\phi_0) := {}& R/\mathfrak{m}_R^n(\mathrm{rank}\, V)\\ &\otimes_{R/\mathfrak{m}_R^n}(V\otimes_R R/\mathfrak{m}_R^n\otimes_{R_0} W_m)_{I_m}\\ &\otimes_{R/\mathfrak{m}_R^n}\widetilde{\varepsilon}_{0,R/\mathfrak{m}_R^n}((\rho,V)_{\widehat{\mathrm{Gr}}}\otimes_{R_0}\widehat{\mathcal{L}}_{\phi_0},\widetilde{\psi}')^{\otimes -1}\end{aligned}$$

where $\widetilde{\psi}'$ is an additive character R_0-sheaf on $\widehat{\mathrm{Gr}}^\bullet K$ of conductor -1 which is the pull-back of $\widetilde{\psi}$ by the canonical morphism

$$\widehat{\mathrm{Gr}}^\bullet K^{[-n,0]}\cong K^{[0,0]}\times_k\cdots K^{[1,1]}\times_k K^{[n,n]}\xrightarrow{\mathrm{pr}_1} K^{[0,0]}\xrightarrow{-1} K^{[0,0]}.$$

Define $\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi}_0,\phi_0)$ by the projective limit

$$\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi}_0,\phi_0) := \varprojlim_m \widetilde{\varepsilon}_{0,R/\mathfrak{m}_R^n}(V\otimes_R R/\mathfrak{m}_R^n,\widetilde{\psi}_0,\phi_0).$$

Lemma 4.48. *The character $\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi}_0,\phi_0)$ does not depend on the choice of ϕ_0.*

Proof. We may assume that V, $\widetilde{\psi}_0$ and ϕ_0 are defined over a finite p'-coefficient ring R_0.

Let $\phi'_0 : k \to R_0^\times$ be another additive character. Define the representation W'_m by

$$W'_m := H^1_c(X, (\pi_{m*}R_0) \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi'_0}) \in \mathrm{Rep}(\mathrm{Gal}(X_m/X_0), R_0).$$

Then we have a canonical isomorphism

$$H^1_c(X, \widetilde{V} \otimes_{R_0} \widetilde{\mathcal{L}}'_{\phi'_0}) \cong (V \otimes_{R_0} W'_m)_{I_m}.$$

There exists a unique element $a \in k^\times$ such that $\phi'(x) = \phi(ax)$ for all $x \in k$. Take an element $\alpha \in \overline{k}$ satisfying $\alpha^m = a$. Then the map $X_m \to X_m$ induced by the multiplication-by-α map $\mathfrak{m}_K/\mathfrak{m}_K^2 \to \mathfrak{m}_K/\mathfrak{m}_K^2$ induces an isomorphism $\varphi : W_m \cong W'_m$ of $R_0[I_m]$-modules. Let $\sigma \in W_k$. Let $[\frac{\sigma^{-1}(\alpha)}{\alpha}] \in I_m$ be the element corresponding to $\frac{\sigma^{-1}(\alpha)}{\alpha} \in \boldsymbol{\mu}_m(\overline{k})$ by the canonical isomorphism $I_m \cong \boldsymbol{\mu}_m(\overline{k})$. It is easily checked that the action of σ on W_m is identified with the action of $\sigma \cdot [\frac{\sigma^{-1}(\alpha)}{\alpha}]$. Hence the proposition follows from Proposition 4.46. □

4.12. Local $\widetilde{\varepsilon}_0$-characters for torsion coefficients (general case). Let R be a complete strict p'-coefficient ring with positive residue characteristic and $\widetilde{\psi}$ a non-trivial additive character R-sheaf on K. For every object V in $\mathrm{Rep}(W_K, R)$, let $V = V^0 \oplus \bigoplus_\Sigma V^\Sigma$ be the refined slope decomposition of V and set

$$\widetilde{\varepsilon}_{0,R}(V, \widetilde{\psi}) := \widetilde{\varepsilon}_{0,R}(V^0, \widetilde{\psi}) \otimes \bigotimes_{\Sigma} \widetilde{\varepsilon}_{0,R}(V^\Sigma, \widetilde{\psi}).$$

Then $\widetilde{\varepsilon}_{0,R}(V, \widetilde{\psi})$ has the following properties:

(1) The isomorphism class of $\widetilde{\varepsilon}_{0,R}(V, \widetilde{\psi})$ depends only on the isomorphism class of (ρ, V).

(2) Let R' be another strict p'-coefficient ring, and $h : R \to R'$ a local ring homomorphism. Then we have

$$\widetilde{\varepsilon}_{0,R}(V, \widetilde{\psi}) \otimes_R R' \cong \widetilde{\varepsilon}_{0,R'}(V \otimes_R R', \widetilde{\psi} \otimes_R R').$$

(3) Suppose that there exists an exact sequence

$$0 \to V' \to V \to V'' \to 0$$

in $\mathrm{Rep}(W_K, R)$. Then we have

$$\widetilde{\varepsilon}_{0,R}(V, \widetilde{\psi}) \cong \widetilde{\varepsilon}_{0,R}(V', \widetilde{\psi}) \otimes_R \widetilde{\varepsilon}_{0,R}(V'', \widetilde{\psi}).$$

(4) Suppose that the residue field k of K is finite. Let $\psi : K \to R^\times$ be the additive character canonically corresponding to $\widetilde{\psi}$. Then

$$\widetilde{\varepsilon}_{0,R}(V, \widetilde{\psi})(\mathrm{Fr}_k) = (-1)^{\mathrm{rank}\, V + \mathrm{sw}(V)} \cdot \varepsilon_{0,R}(V, \psi).$$

(5) Let $a \in K^\times$. Then we have

$$\widetilde{\varepsilon}_{0,R}(V, \widetilde{\psi}_a) = \det(V)_{[a]} \otimes_R R(-v_K(a) \cdot \operatorname{rank} V) \otimes_R \widetilde{\varepsilon}_{0,R}(V, \widetilde{\psi}).$$

(6) Let W be an object in $\operatorname{Rep}(W_K, R)$ on which W_K acts via $W_K/W_K^0 \cong W_k$. Then we have

$$\widetilde{\varepsilon}_{0,R}(V \otimes W, \widetilde{\psi}) = (\det W)^{\otimes \operatorname{sw}(V)+\operatorname{rank} V \cdot (\operatorname{ord} \widetilde{\psi}+1)} \otimes_R \widetilde{\varepsilon}_{0,R}(V, \widetilde{\psi})^{\otimes \operatorname{rank} W}.$$

(7) Suppose that the coinvariant $(V)_{W_K^0}$ is zero. Let V^* be the R-linear dual of V. Then we have

$$\widetilde{\varepsilon}_{0,R}(V, \widetilde{\psi}) \otimes \widetilde{\varepsilon}_{0,R}(V^*, \widetilde{\psi}) = (\det V)_{[-1]} \otimes_R R(-\operatorname{sw}(V) - \operatorname{rank} V \cdot (2\operatorname{ord} \widetilde{\psi} + 1)).$$

Theorem 4.49. *Let L/K be a finite separable extension of p-CDVFs, R a complete strict p'-coefficient ring with a positive residue characteristic, $\widetilde{\psi}$ a non-trivial additive character R-sheaf. Then for every object (ρ, V) in $\operatorname{Rep}(W_L, R)$, we have*

$$\widetilde{\varepsilon}_{0,R}(\operatorname{Ind}_{W_L}^{W_K} V, \widetilde{\psi}) = \widetilde{\lambda}_R(L/K, \widetilde{\psi})^{\otimes \operatorname{rank} V} \otimes_R \widetilde{\varepsilon}_{0,R}(V, \widetilde{\psi} \circ \operatorname{Tr}_{L/K}).$$

Proof. Take a sufficiently large finite Galois extension L' of K containing L such that ρ factors through $W_K/W_{L'}$. Take a sufficiently large integer $N \in \mathbb{Z}$. Take a good perfect $\mathbb{F}_p$-subalgebra A of k and a $(N, \widetilde{\psi})$-admissible A-structure $(B, (\pi_M)_M)$ of L'/K. Then we can define, in a canonical way, smooth invertible R-sheaves $\widetilde{\varepsilon}_{0,R,A}(\operatorname{Ind}_{W_K}^{W_L} V, \widetilde{\psi})$, $\widetilde{\varepsilon}_{0,R,A}(V, \widetilde{\psi} \circ \operatorname{Tr}_{L/K})$, and $\widetilde{\lambda}_{R,A}(L/K, \widetilde{\psi})$ on $\operatorname{Spec}(A)$ whose pull-backs on $\operatorname{Spec}(k)$ are identified with $\widetilde{\varepsilon}_{0,R}(\operatorname{Ind}_{W_K}^{W_L} V, \widetilde{\psi})$, $\widetilde{\varepsilon}_{0,R}(V, \widetilde{\psi} \circ \operatorname{Tr}_{L/K})$, and $\widetilde{\lambda}_R(L/K, \widetilde{\psi})$ respectively.

Then for any closed point $x \in \operatorname{Spec}(A)$, we have

$$i_x^* \widetilde{\varepsilon}_{0,R,A}(\operatorname{Ind}_{W_K}^{W_L} V, \widetilde{\psi}) \cong i_x^* \widetilde{\varepsilon}'_{0,R,A}(V, \widetilde{\psi} \circ \operatorname{Tr}_{L/K}) \otimes i_x^* \widetilde{\lambda}_R(L/K, \widetilde{\psi})^{\otimes \operatorname{rank} V},$$

where $i_x : x \hookrightarrow \operatorname{Spec}(A)$ be the canonical inclusion. Hence we have $\widetilde{\varepsilon}_{0,R,A}(\operatorname{Ind}_{W_K}^{W_L} V, \widetilde{\psi}) \cong \widetilde{\varepsilon}_{0,R,A}(V, \widetilde{\psi} \circ \operatorname{Tr}_{L/K}) \otimes \widetilde{\lambda}_R(L/K, \widetilde{\psi})^{\otimes \operatorname{rank} V}$. This completes the proof. □

Let k be a perfect field of characteristic p, X_0 a proper smooth connected curve over k, $U_0 \subset X_0$ a non-empty open subscheme of X_0, $j_0 : U_0 \hookrightarrow X_0$ the inclusion, $X = X_0 \otimes_k \overline{k}$, $U = U_0 \otimes_k \overline{k}$, $j = j_0 \times id : U \hookrightarrow X$, R a strict p'-coefficient ring, $R_0 \subset R$ a finite subring, and $\mathcal{F}$ a smooth R_0-flat R_0-sheaf on U_0. Define the global $\widetilde{\varepsilon}$-character $\widetilde{\varepsilon}_{R_0}(U_0, \mathcal{F})$ by

$$\varepsilon_{R_0}(U_0, \mathcal{F}) := \det(R\Gamma_c(U, \mathcal{F}))^{\otimes -1} = \det(R\Gamma_c(X, j_{0,!}\mathcal{F}))^{\otimes -1}.$$

Let $\omega \in \Gamma(U_0, \Omega^1_{U_0/k})$ be a non-zero differential on U. Fix a non-trivial additive character R_0-sheaf $\widetilde{\psi}$ on $\mathbb{G}_{a,k}$. For a closed point $x \in X$, let $\kappa(x)$ be the residue field at x, K_x the completion of the function field of X at x, and $\mathcal{F}_x$ the isomorphism class in $\operatorname{Rep}(W_{K_x}, R)$ corresponding to the pull-back of $\mathcal{F}$ by the canonical morphism $\mathrm{Spec}\,(K_x) \to U$. Define the additive character R-sheaf $\widetilde{\psi}_{\omega,x}$ on K_x to be the pull-back of $\widetilde{\psi}|_{\mathbb{G}_{a,\kappa(x)}}$ by the morphism $K_x^{[m,-\mathrm{ord}\,_x(\omega)-1]} \to \mathbb{G}_{a,\kappa(x)}$ defined by $a \mapsto \mathrm{Res}(a\omega)$.

Theorem 4.50. *With the above notations, we have*

$$\widetilde{\varepsilon}_{R_0}(U_0,\mathcal{F}) = R(-\frac{1}{2}\chi(X)\mathrm{rank}\,(\mathcal{F})) \otimes_R \bigotimes_{x\in X_0-U_0} \widetilde{\varepsilon}_{0,R}(\mathcal{F}_x,\widetilde{\psi}_{\omega,x}),$$

where $\chi(X)$ is the Euler number of X.

Proof. We may assume that $X_0 = \mathbb{P}^1_k$.

Let K_0 denote the function field of $\mathbb{P}_1$. For a closed point x on $\mathbb{P}^1_k$, let K_x be the completion of K at x. Take a sufficiently large integer N. Take a finite etale Galois covering V_0 of U_0 such that the sheaf $\mathcal{F}$ and the sheaves $\widetilde{\psi}_{\omega,x}|_{K_x^{[-N,-\mathrm{ord}\,_x(\omega)-1]}}$ is constant on V_0.

Let $\overline{V}_0$ be the smooth completion of V_0. The morphism f is canonically extension to the morphism $\overline{f}: \overline{V}_0 \to \mathbb{P}^1_k$. Let L_0 denote the function field of V_0. For a closed point y on $\overline{V}_0$, let L_y denote the completion of L_0 at y.

There exists a datum

$$(A,(i_{A,x})_x,U_A,\omega_A,V_A,(B_y,(\pi_{M_y})_{M_y})_y)$$

which satisfies the following conditions:

- A is a good perfect $\mathbb{F}_p$-subalgebra of k,
- In $(i_{A,x})_x$, x runs over all points in $\mathbb{P}^1_k - U_0$. For each such x, $i_{A,x}$ is a closed A-immersion $\mathrm{Spec}\,(A_x) \hookrightarrow \mathbb{P}^1_A$ from a finite etale A-subalgebra of $\kappa(x)$ to $\mathbb{P}^1_A$ which is equal to $i_x : x \hookrightarrow \mathbb{P}^1_k$ after tensored with k over A.
- $U_A := \mathbb{P}^1_A - \bigcup_x i_x(\mathrm{Spec}\,(A_x))$.
- $\omega_A \in \Gamma(U_A,\Omega^1_{U_A/A})$ is a 1-differential on U_A such that $\omega_A|_{U_0} = \omega$.
- $V_A \to U_A$ is a finite etale morphism such that $V_A \otimes_A k \cong V_0$ as U_0-schemes.
- In $(B_y,(\pi_{M_y})_{M_y})_y$, y runs over all pairs of closed point $y \in \overline{Y}_0$ with $x = \overline{f}(y) \notin U_0$. For each such y, $(B_y,(\pi_{M_y})_{M_y})$ is an $(N,\widetilde{\psi}_{\omega,x})$-admissible A_x-structure of L_y/K_x.

Let $\mathcal{F}_A$ be the smooth etale R_0-sheaf on U_A corresponding to $\mathcal{F}$. Let $\mathcal{F}_{A,x}$ be the object in

$$\mathrm{Rep}(\pi_1(\mathrm{Spec}\,(A_x((\pi_{K_x}))),R_0)$$

corresponding to $\mathcal{F}_x$.

By using $\mathcal{F}_{A,x}$, we define a smooth invertible R_0-sheaf $\widetilde{\varepsilon}_{0,R_0,A_x}(\mathcal{F}_x,\widetilde{\psi}_{\omega,x})$ on $\mathrm{Spec}\,(A_x)$ whose pull-back on $\mathrm{Spec}\,(k)$ is the sheaf corresponding to $\widetilde{\varepsilon}_{0,R}(\mathcal{F}_x,\widetilde{\psi}_{\omega,x})$. We also define the smooth invertible R_0-sheaf $\widetilde{\varepsilon}_{R_0,A}(U,\mathcal{F})$

on $\mathrm{Spec}\,(A)$ to be $\det_{R_0} Rf_!(U_A, \mathcal{F}_A)$, where $f : U_A \to \mathrm{Spec}\,(A)$ is the structure morphism.

Let $z \in \mathrm{Spec}\,(A)$ be a closed point, and $i_z : \mathrm{Spec}\,(\kappa(z)) \hookrightarrow \mathrm{Spec}\,(A)$ the canonical inclusion. Set $U_z := U_A \otimes_A \kappa(z)$. Then U_z is an open subscheme of $\mathbb{P}^1_k$.

Let $i_{U,z} : U_z \hookrightarrow U_A$ be the canonical inclusion and set $\mathcal{F}_z = i^*_{U,z}\mathcal{F}$ and $\omega_z = i^*_{U,z}\omega_A$. Then we have

$$\mathrm{Tr}(-\mathrm{Fr}_z\,;\, i_z^*\widetilde{\varepsilon}_{R_0,A}(U,\mathcal{F})) = \varepsilon_{R_0}(U_z, \mathcal{F}_z).$$

For $x \in \mathbb{P}^1_k - U_0$, put $z_x = z \times_{\mathrm{Spec}\,(A)} \mathrm{Spec}\,(A_x)$. For all point $y \in z_x$, let $i_{z,y} : y \hookrightarrow z_x \hookrightarrow \mathrm{Spec}\,(A_x)$ be the canonical inclusion. Then we have

$$\mathrm{Tr}(\mathrm{Fr}_y\,;\, i^*_{z,y}\widetilde{\varepsilon}_{0,R,A_x}(\mathcal{F}_x, \widetilde{\psi}_{\omega,x})) = (-1)^{\mathrm{rank}\,\mathcal{F}+\mathrm{sw}_y(\mathcal{F}_z)}\varepsilon_{0,R}(\mathcal{F}_{z,y}, \psi_{\omega_z,y}).$$

By [11, Thm. 4.1], we have

$$\varepsilon_{R_0}(U_z, \mathcal{F}_z) = (\sharp\kappa(z))^{\mathrm{rank}\,(\mathcal{F}_z)} \prod_{x\in\mathbb{P}^1_{\kappa(z)}-U_z} \varepsilon_{0,R}(\mathcal{F}_{z,x}, \psi_{\omega_z,x}).$$

From this we have

$$i_z^*\widetilde{\varepsilon}_{R_0,A}(U,\mathcal{F}) \cong R_0(-\frac{1}{2}\chi(X)\mathrm{rank}\,(\mathcal{F})) \otimes_{R_0} \bigotimes_x i_z^*\pi_{x,*}\widetilde{\varepsilon}_{0,R,A_x}(\mathcal{F}_x, \widetilde{\psi}_{\omega,x})),$$

where $\pi_x : \mathrm{Spec}\,(A_x) \to \mathrm{Spec}\,(A)$ is the structure morphism.

Hence the theorem follows from the standard argument using Chebotarev's theorem (cf. proof of Lemma 4.18). □

Acknowledgements. The author would like to thank his supervisor Prof. Takeshi Saito for many fruitful discussions and helpful comments.

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