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ERROR LOCATING CODES BY USING BLOCKWISE-TENSOR PRODUCT OF BLOCKWISE DETECTING/CORRECTING CODES

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Abstract. In this paper, we obtain lower and upper bounds on the number of parity check digits of a linear code that corrects e or less errors within a sub-block. An example of such a code is provided. We introduce blockwise-tensor product of matrices and using this, we propose classes of error locating codes (or EL-codes) that can detect e or less errors within a sub-block and locate several such corrupted sub-blocks.

1. Introduction

Error detecting codes are used by the receiver merely to detect the presence of errors in a block of received digits, whereas error correcting codes at the receiver are used to correct errors that may have occurred in transmission. To have a good error correcting capability of the codes, long code length is required to be considered, but this results in decreasing the information rate of the system. This problem had been somewhat sorted out by Wolf and Elspas [15]. They introduced the concept of error location by which a compromise between short and long code length can be obtained. The technique in error location concept is that the block of received digits is subdivided into mutually exclusive sub-blocks. The length of the sub-blocks can be chosen relatively smaller requiring less number of parity checks which can improve the information rate of the system. In this concept, errors occurring within sub-block(s) are detected at the receiver, and in addition the receiver is able to specify which particular sub-block(s) contains errors. Of course, the precise location of erroneous digit positions can not be located, but

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the sub-block(s) containing errors can be identified. The codes following this technique is referred to as *error locating codes* (in short *EL-codes*).

Wolf and Elspas [15] devised codes that could locate a single corrupted sub-block containing a given number of random errors. Later on, Wolf in [13] obtained a class of EL-codes capable of identifying corrupted sub-blocks by considering tensor product of parity check matrices of two codes. In EL-codes, the errors are considered to occur within sub-block(s). Such type of errors are found to be in mostly storage and memory systems [6, 9]. Thus, there is a need to study blockwise error detecting/correcting codes. For more study on error locating codes and blockwise error correcting codes, one may refer to [2, 3, 4, 5, 6, 7, 8, 10, 16, 17].

1.1. **Outline:** The paper is organized as follows: In Section 2, we recall basic definition and notations to make the paper self-contained. We present lower and upper bounds on the number of parity check digits for a linear code correcting e or less errors within a sub-block in Section 3. An example is given to support our result. Wolf in [13, 14] studied construction of a class of EL-codes that can locate several corrupted sub-blocks by considering tensor product of parity check matrices of two codes. We introduce the concept of blockwise tensor product and by using this we construct another classes of EL-codes (but of relatively smaller length given in [13, 14]) that can locate several corrupted sub-blocks in Section 4. In Section 5, we present a comparative study between the bounds obtained in Section 3-4, and some existing bounds. We verify that the development of the codes obtained in Section 3-4 improves the efficiency of the communication channel.

2. Basic Tools

In this section, we recall basic definition, notations and results to make the paper self-contained. The largest integer less than or equal to x is denoted by $\lfloor x \rfloor$. An (n, k) linear code will be considered as a subspace of the space of all n-tuples over GF(q). The distance between two vectors shall be considered in the Hamming sense. The minimum distance of a linear code is the minimum of distances between any two code vectors. The information rate of the code is $\frac{k}{n}$. By a burst of length b, we mean a vector with nonzero entries in some b consecutive positions and zeros elsewhere.

Next we consider following conditions for an (n, k) linear codes over $GF(q)$ of which the code length is divided into m mutually exclusive sub-blocks.

(1) The syndrome resulting from the occurrence of e or fewer errors within a sub-block must be distinct from the all zero syndrome.
(2) The syndrome resulting from the occurrence of e or fewer errors within a sub-block must be distinct from the syndrome resulting likewise from e or fewer errors within any *other* sub-block.
(3) The syndrome resulting from the occurrence of e or fewer errors within a single sub-block must be distinct from the syndrome resulting likewise from e or fewer errors within the same sub-block.

(4) The syndrome resulting from the occurrence of such errors within any $l(< m)$ or less sub-blocks must be non-zero and distinct from the syndrome resulting likewise from within any *same* $l(< m)$ or less sub-blocks.
(5) The syndrome resulting from the occurrence of such errors within any $l(< m)$ or less sub-blocks must be distinct from the syndrome resulting likewise from within any *other* $l(< m)$ or less sub-blocks.

The code capable of detecting e or fewer errors within a sub-block must satisfy condition (1). The code capable of detecting and locating such error within a sub-block must satisfy conditions (1) and (2). The code capable of correcting e or fewer errors within a sub-block, the code should satisfy conditions $(1)-(3)$. Further, to locate any $l(< m)$ sub-blocks containing e or fewer errors within a sub-block, conditions (4) and (5) must be satisfied.

To conclude the section, we give the definition of tensor product of matrices which can be found in literature (for example see, [1]).

Definition 2.1. Let $A = [a_{ij}]$ be a m-by-n matrix and let $B = (b_{ij})$ be a p-by-q matrix. The tensor product of A and B is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & . & . & . & . & a_{1n}B \\ a_{21}B & a_{22}B & . & . & . & . & a_{2n}B \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{m1}B & a_{m2}B & . & . & . & . & a_{mn}B \end{bmatrix}.$$

3. Bounds on the number of parity check digits

Peterson and Weldon [11] gave lower and upper bounds for the linear codes that correct e or less errors (see Theorem 4.5 and Theorem 4.7 in [11], also see [12]). By using the technique given in [11], we present lower and upper bounds for the linear codes that correct e or less errors within a sub-block.

Theorem 3.1. [*Lower Bound*] *The number of parity check digits in an (n, k) linear code over $GF(q)$ subdivided into m sub-blocks of length t each, that corrects e or less errors within a sub-block is bounded from below by*

$$n-k \geq \log_q \left\{ 1 + m \sum_{i=1}^{e} \binom{t}{i} (q-1)^i \right\}. \tag{3.1}$$

Proof. We prove the theorem by counting the number of syndromes that are required to be nonzero and distinct by conditions (1), (2) and (3) (in Section 2), and then setting this number less than or equal to q^{n-k}, the number of maximum possible syndromes. By conditions (1), (2) and (3), the syndromes produced by e or less errors within a sub-block are nonzero and distinct, whether in the same sub-block or in different sub-blocks. The total number of such syndromes, including the vector of all zeros, is given by (see Wolf [13])

$$1 + m \sum_{i=1}^{e} \binom{t}{i} (q-1)^i.$$

Therefore, we must have

$$q^{n-k} \geq 1 + m \sum_{i=1}^{e} \binom{t}{i} (q-1)^i$$

or,

$$n-k \geq \log_q \left\{ 1 + m \sum_{i=1}^{e} \binom{t}{i} (q-1)^i \right\}.$$

The result is proved. □

Remark 3.2. For $m = 1$, the inequality (3.1) reduces to

$$n-k \geq \log_q \left\{ 1 + \sum_{i=1}^{e} \binom{t}{i} (q-1)^i \right\},$$

which coincides with the result of Theorem 4.5 of [11].

Theorem 3.3. [*Upper Bound*] *There exists an* (n, k) *linear code over* $GF(q)$ *sub-divided into* m *sub-blocks of length* t *each, that corrects* e *or less errors within a sub-block* $(t \geq 2e)$ *provided that*

$$q^{n-k} > 1 + \sum_{i=1}^{2e-1} \binom{t-1}{i} (q-1)^i + (m-1) \left\{ 1 + \sum_{i=1}^{e-1} \binom{t-1}{i} (q-1)^i \right\} \sum_{i=1}^{e} \binom{t}{i} (q-1)^i. \tag{3.2}$$

Proof. The existence of the code can be ensured by constructing a suitable $(n-k) \times n$ parity check matrix H for the desired code. Without loss of generality, we can assume that the columns of the first $m-1$ sub-blocks of H and the first $j-1$ columns $h_1, h_2, \ldots, h_{j-1}$ of the m^{th} sub-block have been appropriately added. The j^{th} column h_j of the m^{th} sub-block of the matrix H is needed to add according to conditions $(1)-(3)$. This is done in the following two cases:

Case I: First we consider condition (3), according to which the j^{th} column h_j should not be a linear combination of previous any $2e-1$ or less columns of the m^{th} column sub-block. In other words

$$h_j \neq \; u_{j-1}h_{j-1} + u_{j-2}h_{j-2} + \cdots + u_1 h_1, \tag{3.3}$$

where $u_i \in GF(q)$ are any $2e-1$ or less nonzero coefficients.

The number of linear combinations on the R.H.S. of inequality (3.3), including the vector of all zeros, is

$$1 + \sum_{i=1}^{2e-1} \binom{j-1}{i} (q-1)^i. \tag{3.4}$$

(Note that the condition (1) is also taken care of in (3.4).)

Case II: In this case we consider condition (2), accordingly the j^{th} column h_j should not be a linear combination of previous any $e-1$ or less columns of the m^{th} sub-block, together with any e or less columns within any other sub-block of the previous $m-1$ sub-blocks. In other words

$$h_j \neq (u_{j-1}h_{j-1} + u_{j-2}h_{j-2} + \cdots + u_1 h_1) + (v_i h_i + v_{i+1}h_{i+1} + \cdots + v_{i+b-1}h_{i+b-1}), \tag{3.5}$$

where $u_i, v_i \in GF(q)$, u_i are any $e-1$ or less nonzero and h_i are any e or less columns of a sub-block from the previous $m-1$ sub-blocks.

The number of linear combinations on the R.H.S. of the inequality (3.5) is given by

$$(m-1)\left\{1+\sum_{i=1}^{e-1}\binom{j-1}{i}(q-1)^i\right\}\sum_{i=1}^{e}\binom{t}{i}(q-1)^i. \tag{3.6}$$

At worst, all the linear combinations computed in (3.4) and (3.6) might yield distinct sums. Therefore, h_j can be added to the m^{th} sub-block of H provided that

$$q^{n-k} > expr.(3.4) + expr.(3.6)$$

i.e.

$$q^{n-k} > 1+\sum_{i=1}^{2e-1}\binom{j-1}{i}(q-1)^i+(m-1)\left\{1+\sum_{i=1}^{e-1}\binom{j-1}{i}(q-1)^i\right\}\sum_{i=1}^{e}\binom{t}{i}(q-1)^i.$$

Replacing j by t gives the theorem. □

Remark 3.4. For $m=1$, the inequality (3.2) reduces to

$$q^{n-k} > 1+\sum_{i=1}^{2e-1}\binom{t-1}{i}(q-1)^i,$$

which coincides with the result of Theorem 4.7 of [11].

Example 3.5. Consider a $(12,5)$ binary code with the following parity check matrix 7×12 matrix P which has been constructed by the synthesis procedure given in the proof of Theorem 3.3 by taking $m=3, t=4, e=2, q=2$.

$$P=\begin{bmatrix} 1&0&0&1&0&0&0&0&1&0&0&0\\ 0&1&0&1&0&0&0&0&0&1&0&0\\ 0&0&1&1&0&0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0&1&0&0&0&1\\ 0&0&0&0&1&0&0&1&0&0&1&0\\ 0&0&0&0&0&1&0&1&0&1&0&0\\ 0&0&0&0&0&0&1&1&1&0&0&0 \end{bmatrix}$$

The null space of this matrix can be used to correct 2 or less errors within a sub-block of length 4. It may be verified from Error Pattern-Syndrome Table 3.1 that syndromes produced by 2 or less errors within a sub-block, whether in the same sub-block or in different sub-blocks are distinct.

Table 3.1
Error Pattern - Syndrome Table

Error-patterns	Syndromes
Errors within the first sub-block	
1000 0000 0000	1000000
0100 0000 0000	0100000
0010 0000 0000	0010000
0001 0000 0000	1111000
1100 0000 0000	1100000
1010 0000 0000	1010000
1001 0000 0000	0111000
0110 0000 0000	0110000
0101 0000 0000	1011000
0011 0000 0000	1101000
Errors within the second sub-block	
0000 1000 0000	0000100
0000 0100 0000	0000010
0000 0010 0000	0000001
0000 0001 0000	0001111
0000 1100 0000	0000110
0000 1010 0000	0000101
0000 1001 0000	0001011
0000 0110 0000	0000011
0000 0101 0000	0001101
0000 0011 0000	0001110
Errors within the third sub-block	
0000 0000 1000	1000001
0000 0000 0100	0100010
0000 0000 0010	0010100
0000 0000 0001	0001000
0000 0000 1100	1100011
0000 0000 1010	1010101
0000 0000 1001	1001001
0000 0000 0110	0110110
0000 0000 0101	0101010
0000 0000 0011	0011100

4. Blockwise-Tensor Product and EL-Codes

We start this section with the definition of Blockwise-Tensor Product of matrices.

Definition 4.1. If $A = (A_1\ A_2\ ....\ A_m)$, where A_i is a $k_1 \times k_2$ matrix and $B = (B_1\ B_2\ ....\ B_m)$ where B_i is a $s_1 \times s_2$ matrix, the *Blockwise-Tensor(Kronecker) Product* $A \otimes_b B$ is the $(k_1 s_1) \times (m k_2 s_2)$ matrix

$$A \otimes_b B = \left[\begin{array}{cccc} A_1 \otimes B_1 & A_2 \otimes B_2 & .... & A_m \otimes B_m \end{array}\right].$$

By using the blockwise-tensor product and codes discussed in Section 3, we propose new classes of EL-codes that can locate several sub-blocks.

Consider a code of length n consists of m sub-blocks, each sub-block containing t digits, then $n = tm$. Further if $m = m_1 m_2$, then the code of length n can be considered as consisting of m_2 sub-blocks, each sub-block containing tm_1 digits. This new sub-blocks of length tm_1 constitute of m_1 consecutive original sub-blocks of length t. We call the new sub-blocks as *cluster-sub-blocks* of length tm_1. First we present a new class of EL-codes which can locate several sub-blocks within a cluster-sub-block.

Theorem 4.2. *Let $C_1(n_1 = mt, n_1 - k)$ be a linear code over $GF(q)$ with parity check matrix H that detects e_1 or fewer errors within a sub-block of length t and $C_2(n_2 = ms, n_2 - \rho)$ be a linear code over $GF(q)$ with parity check matrix P that corrects e_2 or fewer errors within a sub-block of length s. Then the $(mts, mts - k\rho)$ code C obtained from the parity check matrix $P \otimes_b H$ can detect e_1 or fewer errors within a sub-block of length t and locate any e_2 or fewer such corrupted sub-blocks within a cluster-sub-block of length ts.*

Proof. The parity check matrix of the code C is given by

$$\begin{aligned} P \otimes_b H &= \left[\begin{array}{cccc} P_1 \otimes H_1 & P_2 \otimes H_2 & .... & P_m \otimes H_m \end{array}\right] \\ &= \left[\begin{array}{ccccccccc} p^1_{11}H_1 & p^1_{12}H_1 & .. & p^1_{1s}H_1 & ... & p^m_{11}H_m & p^m_{12}H_m & .. & p^m_{1s}H_m \\ . & . & .. & . & .. & . & .. & . & . \\ . & . & .. & . & .. & . & .. & . & . \\ p^1_{\rho 1}H_1 & p^1_{\rho 2}H_1 & .. & p^1_{\rho s}H_1 & ... & p^m_{\rho 1}H_m & p^m_{\rho 2}H_m & .. & p^m_{\rho s}H_m \end{array}\right] \end{aligned}$$

where P_i and H_j are sub-blocks of length t of P and H, respectively and p^l_{ij} is the $(ij)^{th}$ element of sub matrix P_l $(1 \le l \le m)$.

Consider the situation where e_1 or fewer errors occur only in the λ^{th} sub-block of length t. Let us write $\lambda = (\tau - 1)s + j,\ 1 \le j \le s,\ 1 \le \tau \le m$. Then, the resulting syndrome is $S^\lambda = (S_1, S_2, ...S_\rho)$, where

$$S_i = \sum_{r=1}^{t} u_r p^\tau_{ij} h_{(\tau)_r} = p^\tau_{ij} \sum_{r=1}^{t} u_r h_{(\tau)_r} = p^\tau_{ij} A_\lambda,$$

$u_r \in GF(q)$ are any e_1 or less nonzero coefficients, $h_{(\tau)_r}$ are columns of the sub-block H_τ and $A_\lambda = \sum_{r=1}^{t} u_r h_{(\tau)_r}$. Furthermore, the sum A_λ can not be all zero k-tuple, because the code C_1 detects e_1 or fewer errors within a sub-block of length t. Also since, all p^τ_{ij} are nonzero, the syndrome S cannot be zero (which satisfies condition (1)). This shows that the code obtained from the parity check matrix $P \otimes_b H$ can detect e_1 or fewer errors within a sub-block of length t.

In order to locate any e_2 or fewer sub-blocks of length t containing e_1 or fewer errors within a cluster-sub-block of length s, the syndrome resulting from detectable errors occurring within any e_2 or fewer sub-blocks within a cluster-sub-block is distinct from syndromes resulting from detectable errors occurring within any other e_2 or fewer sub-blocks within the same cluster-sub-block. Let S^{λ_i} be the syndrome corresponding to the λ_i^{th} sub-block of length t within a cluster-sub-block of length s, where $\lambda_i = (\tau_1 - 1)s + j_i,\ i = 1, 2, ..., 2e_2$. Each sub-block contains e_1 or fewer errors.
Assume that

$$\sum_{i=1}^{2e_2} x_i S^{\lambda_i} = 0, \quad x_i \in GF(q), \tag{4.1}$$

i.e.

$$\left(\sum_{i=1}^{2e_2} x_i p_{1j_i}^{\tau_i} A_{\lambda_i}, \sum_{i=1}^{2e_2} x_i p_{2j_i}^{\tau_i} A_{\lambda_i}, ..., \sum_{i=1}^{2e_2} x_i p_{\rho j_i}^{\tau_i} A_{\lambda_i} \right) = 0$$

Since C_2 corrects e_2 or fewer errors within a sub-block of length s, all sets of $2e_2$ or fewer syndromes resulting from detectable errors within a sub-block are linearly independent. So, the equation (4.1) has only trivial solution $x_i = 0$ for $i = 1, 2, \ldots, 2e_2$. This shows that condition (4) is true.
To show that the code C satisfies condition (5), suppose S^{β_i} is the syndrome corresponding to the β_i^{th} sub-block of length t within a cluster-sub-block of length s, where $\beta_i = \tau_2 s + f_i$, $i = 1, 2, ..., e_2$.
Let

$$\sum_{i=1}^{e_2} x_i S^{\lambda_i} + \sum_{i=1}^{e_2} y_i S^{\beta_i} = 0, \quad x_i, y_i \in GF(q) \tag{4.2}$$

i.e.

$$\left(\sum_{i=1}^{e_2} x_i p_{1j_i}^{\tau_1} A_{\lambda_i} + \sum_{i=1}^{e_2} y_i p_{1f_i}^{\tau_2} A_{\beta_i}, \sum_{i=1}^{e_2} x_i p_{2j_i}^{\tau_1} A_{\lambda_i} + \sum_{i=1}^{e_2} y_i p_{2f_i}^{\tau_2} A_{\beta_i}, \ldots \right.$$
$$\left. \ldots, \sum_{i=1}^{e_2} x_i p_{\rho j_i}^{\tau_1} A_{\lambda_i} + \sum_{i=1}^{e_2} y_i p_{\rho f_i}^{\tau_2} A_{\beta_i} \right) = 0.$$

Then

$$\sum_{i=1}^{e_2} x_i p_{rj_i}^{\tau_1} A_{\lambda_i} + \sum_{i=1}^{e_2} y_i p_{rf_i}^{\tau_2} A_{\beta_i} = 0,\ r = 1, 2, \ldots, \rho.$$

The code C_2 corrects e_2 or fewer errors within a sub-block of length s, so the syndromes of such errors within a sub-block are distinct from syndromes resulting from such errors within any other sub-block. Therefore, the equation (4.2) gives only trivial solution $x_i = 0$ and $y_i = 0$ for $i = 1, 2, \ldots, e_2$. Thus, condition (5) is satisfied. Hence the theorem is proved. □

Example 4.3. Consider a $(9 = 3 \times 3, 6)$ binary code that detects 3 or less errors within a sub-block of length 3 whose parity check matrix H is given by

$$H = \begin{bmatrix} 1 & 0 & 0 & & 1 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 1 & 0 & & 0 & 1 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 1 & & 0 & 0 & 1 \end{bmatrix}$$

and the $(12 = 4 \times 3, 5)$ binary code that corrects 2 or less errors within a sub-block of length 4 whose parity check matrix P is given in Example 3.5. Then the blockwise-tensor product matrix $P \otimes_b H$ is given as follows.

$$P \otimes_b H = \begin{bmatrix}
100 & 000 & 000 & 100 & & 000 & 000 & 000 & 000 & & 100 & 000 & 000 & 000 \\
010 & 000 & 000 & 010 & & 000 & 000 & 000 & 000 & & 010 & 000 & 000 & 000 \\
001 & 000 & 000 & 001 & & 000 & 000 & 000 & 000 & & 001 & 000 & 000 & 000 \\
000 & 100 & 000 & 100 & & 000 & 000 & 000 & 000 & & 000 & 100 & 000 & 000 \\
000 & 010 & 000 & 010 & & 000 & 000 & 000 & 000 & & 000 & 010 & 000 & 000 \\
000 & 001 & 000 & 001 & & 000 & 000 & 000 & 000 & & 000 & 001 & 000 & 000 \\
000 & 000 & 100 & 100 & & 000 & 000 & 000 & 000 & & 000 & 000 & 100 & 000 \\
000 & 000 & 010 & 010 & & 000 & 000 & 000 & 000 & & 000 & 000 & 010 & 000 \\
000 & 000 & 001 & 001 & & 000 & 000 & 000 & 000 & & 000 & 000 & 001 & 000 \\
000 & 000 & 000 & 100 & & 000 & 000 & 000 & 100 & & 000 & 000 & 000 & 100 \\
000 & 000 & 000 & 010 & & 000 & 000 & 000 & 010 & & 000 & 000 & 000 & 010 \\
000 & 000 & 000 & 001 & & 000 & 000 & 000 & 001 & & 000 & 000 & 000 & 001 \\
000 & 000 & 000 & 000 & & 100 & 000 & 000 & 100 & & 000 & 000 & 100 & 000 \\
000 & 000 & 000 & 000 & & 010 & 000 & 000 & 010 & & 000 & 000 & 010 & 000 \\
000 & 000 & 000 & 000 & & 001 & 000 & 000 & 001 & & 000 & 000 & 001 & 000 \\
000 & 000 & 000 & 000 & & 000 & 100 & 000 & 100 & & 000 & 100 & 000 & 000 \\
000 & 000 & 000 & 000 & & 000 & 010 & 000 & 010 & & 000 & 010 & 000 & 000 \\
000 & 000 & 000 & 000 & & 000 & 001 & 000 & 001 & & 000 & 001 & 000 & 000 \\
000 & 000 & 000 & 000 & & 000 & 000 & 100 & 100 & & 100 & 000 & 000 & 000 \\
000 & 000 & 000 & 000 & & 000 & 000 & 010 & 010 & & 010 & 000 & 000 & 000 \\
000 & 000 & 000 & 000 & & 000 & 000 & 001 & 001 & & 001 & 000 & 000 & 000
\end{bmatrix}$$

By Theorem 4.2, the null space of this matrix, i.e., the (36, 15) code not only detects 3 or less errors within a sub-block of length 3, but also locates any 2 or less sub-blocks within a cluster-sub-block of length 12. It can be verified from error pattern-syndrome table (as done in Example 3.5) that:
(i) the syndromes of 3 or less errors within a cluster-sub-block of length 3 are nonzero,
(ii) the syndromes of 2 or less such corrupted sub-blocks within a cluster-sub-block of length 12 are all distinct.

The following two classes of EL-codes can be obtained by modifying the nature of the code C_2 in Theorem 4.2.

Theorem 4.4. *Let $C_1(n_1 = mt, n_1 - k)$ be a linear code over $GF(q)$ with parity check matrix H that detects e_1 or fewer errors within a sub-block of length t and $C_2(n_2 = ms, n_2 - \rho)$ be a linear code over $GF(q)$ with parity check matrix P that corrects e_2 or fewer errors. Then the $(mts, mts - k\rho)$ code obtained from the* *parity check matrix $P \otimes_b H$ can detect e_1 or fewer errors within a sub-block of length t and locate any e_2 or fewer such corrupted sub-blocks.*

Proof. Similar to proof of Theorem 4.2. □

Theorem 4.5. *Let $C_1(n_1 = mt, n_1 - k)$ be a linear code over $GF(q)$ with parity check matrix H that detects e_1 or fewer errors within a sub-block of length t and $C_2(n_2 = ms, n_2 - \rho)$ be a linear code over $GF(q)$ with parity check matrix P that corrects any burst of length b or less. Then the $(mts, mts - k\rho)$ code obtained from the parity check matrix $P \otimes_b H$ can detect e_1 or fewer errors within a sub-block of length t and locate any adjacent b or fewer such corrupted sub-blocks.*

Proof. Similar to proof of Theorem 4.2. □

To conclude the section, we observe that distance of a code obtained by blockwise-tensor product of the generator matrices of two blockwise linear codes coincides with Theorem 5.3 given by Peterson and Weldon [11]. This is given in the following theorem.

Theorem 4.6. *The distance of the linear code whose generator matrix is the blockwise-tensor product of generator matrices of the two blockwise linear codes, is the product of the distances of the two codes.*

Proof. Consider the linear code C_1 has distance $D = d_1 + d_2 + ... + d_m$ with generator matrix G_1 and the other linear code C_2 has distance $L = l_1 + l_2 + ... + l_m$ with generator matrix G_2. Then, a codevector of the code whose generator matrix is blockwise-tensor product $G_1 \otimes_b G_2$, must have weight at least

$$d_1(l_1 + l_2 + \cdots + l_m) + d_2(l_1 + l_2 + \cdots + l_m) + \cdots + d_m(l_1 + l_2 + \cdots + l_m) = DL.$$

Since D and L are the distances of the codes C_1 and C_2 respectively, there exists a codevector of weight D in C_1 and a codevector of weight L in C_2. As a result, there exist a codevector of weight DL in the code whose generator matrix is $G_1 \otimes_b G_2$. This completes the proof. □

5. Comparative Study

In the section, we make comparison between bounds on the number of parity check digits for codes obtained in Section 3 and Section 4 with the following bounds which already exist in literature.

Theorem 5.1. [11] *For any (n, k) linear code that corrects e or less errors satisfies the following inequality*

$$q^{n-k} \geq 1 + \sum_{i=1}^{e} \binom{n}{i} (q-1)^i.$$

Theorem 5.2. [11] *There exists an (n, k) linear code that corrects e or less errors provided that*

$$q^{n-k} > 1 + \sum_{i=1}^{2e-1} \binom{n}{i} (q-1)^i.$$

Theorem 5.3. [13] *Let $C_1(n_1 = mt, n_1 - k)$ be a binary linear code with parity check matrix H that detects e_1 or fewer errors within a sub-block of length t and $C_2(n_2 = ms, n_2 - \rho)$ be a nonbinary linear code over $GF(2^k)$ with parity check matrix P that corrects e_2 or less random errors (or any burst of length b or less). Then, the $(n_1n_2, n_1n_2 - k\rho)$ code obtained from the parity check matrix $P \otimes H$ can detect e_1 or fewer errors within a sub-block of length n_1 and locate any e_2 or fewer such corrupted sub-blocks (or any adjacent b or fewer such corrupted sub-blocks).*

Comparison (I): First we compare bounds on the number of parity check digits for the existence of linear codes discussed in Theorem 3.1 (Theorem 3.3) and Theorem 5.1 (Theorem 5.2). Since for $m \neq 1$, we have

$$\sum_{i=1}^{e} \binom{n}{i} (q-1)^i > m \sum_{i=1}^{e} \binom{t}{i} (q-1)^i$$

and

$$\sum_{i=1}^{2e-1} \binom{n}{i} (q-1)^i > \sum_{i=1}^{2e-1} \binom{t-1}{i} (q-1)^i + (m-1)\left\{1 + \sum_{i=1}^{e-1} \binom{t-1}{i} (q-1)^i\right\} \sum_{i=1}^{e} \binom{t}{i} (q-1)^i.$$

So, the necessary (sufficient) number of parity check digits for the existence of linear codes considered in Theorem 3.1 (Theorem 3.3) is less than the necessary (sufficient) number of parity check digits for (n, k) codes considered in Theorem 5.1 (Theorem 5.2).

Outcome: If errors are occurred blockwise, the codes discussed in Theorem 3.1 and Theorem 3.3 will be more efficient than the codes discussed in Theorem 5.1 and Theorem 5.2, respectively.

Comparison (II): We now compare the codes obtained in Theorem 4.4 and Theorem 4.5 with the codes mentioned in Theorem 5.3. In Theorem 4.4 and Theorem 4.5, we obtain linear codes that detect e_1 or fewer errors within a sub-block of length t and locate any e_2 or fewer such corrupted sub-blocks (or any adjacent b or fewer such corrupted sub-blocks) whereas in Theorem 5.3, there exists linear codes that can detect e_1 or fewer errors within a sub-block of length n_1 and locate any e_2 or fewer such corrupted sub-blocks (or any adjacent b or fewer such corrupted sub-blocks). In Theorem 5.3, the identified sub-blocks are of length n_1 whereas in our results (Theorem 4.4 and Theorem 4.5), the identified sub-blocks are of length t which may be made much smaller that n_1.

Outcome: The sender needs to retransmit smaller sub-block(s) instead of bigger sub-block(s) of length n_1. In addition, the rate of transmission in our results is also improved, this is because

$$1 - \frac{k\rho}{m^2ts} > 1 - \frac{k\rho}{mts} \text{ for } m \neq 1,$$

i.e., rate of transmission of the codes obtained in Theorem 4.4 and Theorem 4.5 is better than rate of transmission of the codes given in Theorem 5.3.

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