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ON THE RANKS OF FINITE SIMPLE GROUPS

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Abstract. Let G be a finite group and let X be a conjugacy class of G. The *rank* of X in G, denoted by $rank(G{:}X)$ is defined to be the minimal number of elements of X generating G. In this paper we review the basic results on generation of finite simple groups and we survey the recent developments on computing the ranks of finite simple groups.

1. Introduction

Generation of finite groups by suitable subsets is of great interest and has many applications to groups and their representations. For example, the computations of the genus of simple groups can be reduced to the generations of the relevant simple groups (see Woldar [23] for details). Also Di Martino et al. [16] established a useful connection between generation of groups by conjugates and the existence of elements representable by almost cyclic matrices. Their motivation was to study irreducible projective representations of sporadic simple groups. Recently more attention was given to the generation of finite groups by conjugate elements. In his PhD Thesis [22], Ward considered generation of a simple group by conjugate involutions satisfying certain conditions.

We are interested in generation of finite simple groups by the minimal number of elements from a given conjugacy class of the group. This motivates the following definition.

Definition 1.1. Let G be a finite group and let X be a conjugacy class of G. The *rank* of X in G, denoted by $rank(G{:}X)$ is defined to be the minimal number of elements of X generating G.

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One of the applications of ranks of conjugacy classes of a finite group is that they are involved in the computations of the covering number of the finite group (see Zisser [24]).

In [17, 18, 19] the second author computed the the ranks of the involutary classes of the Fischer sporadic simple groups Fi_{22}. He found that $rank(Fi_{22}{:}2B) = rank(Fi_{22}{:}2C) = 3$, while $rank(Fi_{22}{:}2A) \in \{5, 6\}$. The work of Hall and Soicher [15] implies that $rank(Fi_{22}{:}2A) = 6$. Then in a considerable number of publications (for example but not limited to, see [1, 2, 3, 4, 5] or [19]) J. Moori, F. Ali and M.A.F. Ibrahim explored a large number of ranks of the sporadic simple groups.

2. Preliminaries

Let G be a finite group and $C_1, C_2, \cdots, C_k$ be $k \geq 3$ (not necessarily distinct) conjugacy classes of G with $g_1, g_2, \cdots, g_k$ being representatives for these classes, respectively.

For a fixed representative $g_k \in C_k$ and for $g_i \in C_i,\ 1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ such that $g_1 g_2 \cdots g_{k-1} = g_k$. This number is known as *class algebra constant* or *structure constant*. With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \cdots, \chi_r\}$, the number Δ_G is easily calculated from the character table of G through the formula

$$\Delta_G(C_1, C_2, \cdots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}.$$

Also for a fixed $g_k \in C_k$ we denote by $\Delta_G^*(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1})$ satisfying

$$g_1 g_2 \cdots g_{k-1} = g_k \quad \text{and} \quad G = \langle g_1, g_2, \cdots, g_{k-1} \rangle . \tag{2.1}$$

Definition 2.1. If $\Delta_G^*(C_1, C_2, \cdots, C_k) > 0$, the group G is said to be $(C_1, C_2, \cdots, C_k)$**-generated**.

Furthermore if $H \leq G$ is any subgroup containing a fixed element $g_k \in C_k$, we let $\Sigma_H(C_1, C_2, \cdots, C_k)$ be the total number of distinct tuples $(g_1, g_2, \cdots, g_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$ and $\langle g_1, g_2, \cdots, g_{k-1} \rangle \leq H$. The value of $\Sigma_H(C_1, C_2, \cdots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, c_2, \cdots, c_k)$ of H-conjugacy classes $c_1, c_2, \cdots, c_k$ such that $c_i \subseteq H \cap C_i$.

Theorem 2.2. *Let G be a finite group and $H \leq G$ containing a fixed element g such that $\gcd(o(g), [N_G(H){:}H]) = 1$. Then the number $h(g, H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H. In particular*

$$h(g, H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|},$$

where $x_1, x_2, \cdots, x_m$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of g.

Proof. See Ganief and Moori [12, 13]. □

The above number $h(g, H)$ is useful in giving a lower bound for $\Delta_G^*(C_1, C_2, \cdots, C_k)$, namely $\Delta_G^*(C_1, C_2, \cdots, C_k) \geq \Theta_G(C_1, C_2, \cdots, C_k)$, where

$$\Theta_G(C_1, C_2, \cdots, C_k) = \Delta_G(C_1, C_2, \cdots, C_k) - \sum h(g_k, H)\Sigma_H(C_1, C_2, \cdots, C_k),$$

g_k is a representative of the class C_k and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups of G containing elements of all the classes $C_1, C_2, \cdots, C_k$. Since we have all the maximal subgroups of the sporadic simple groups (except for $G = \mathbb{M}$ the Monster group), it is possible to build a small subroutine in GAP [14] or Magma [8] to compute the values of $\Theta_G = \Theta_G(C_1, C_2, \cdots, C_k)$ for any collection of conjugacy classes and a sporadic simple group.

If $\Theta_G > 0$ then certainly G is $(C_1, C_2, \cdots, C_k)$-generated. In the case $C_1 = C_2 = \cdots = C_{k-1} = C$, G can be generated by $k-1$ elements suitably chosen from C and hence $rank(G{:}C) \leq k-1$.

We now quote some results establishing generation and non-generation of finite simple groups, where these results are important in determining the ranks of finite simple groups.

Lemma 2.3 (See Ali and Moori [5] or Conder et al. [9])**.** *Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is $(\underbrace{lX, lX, \cdots, lX}_{m-times}, (nZ)^m)$-generated.*

Proof. Since G is a (lX, mY, nZ)-generated group, it follows that there exists $x \in lX$ and $y \in mY$ such that $xy \in nZ$ and $\langle x, y\rangle = G$. Let $N := \left\langle x, x^y, x^{y^2}, \cdots, x^{y^{m-1}}\right\rangle$. Then $N \trianglelefteq G$. Since G is a simple group and N is a non-trivial subgroup we obtain that $N = G$. Furthermore, we have

$$\begin{aligned} xx^yx^{y^2}x^{y^{m-1}} &= x(yxy^{-1})(y^2xy^{-2})\cdots(y^{m-1}xy^{1-m}) \\ &= (xy)^m \in (nZ)^m. \end{aligned}$$

Since $x^{y^i} \in lX$ for all i, the result follows. □

Corollary 2.4 (See Ali and Moori [5])**.** *Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then $rank(G{:}lX) \leq m$.*

Proof. Follows immediately by Lemma 2.3. □

Lemma 2.5 (See Ali and Moori [5])**.** *Let G be a finite simple $(2X, mY, nZ)$-generated group. Then G is $(mY, mY, (nZ)^2)$-generated.*

Proof. Since G is a $(2X, mY, nZ)$-generated group, it is also a $(mY, 2X, tK)$-generated group. The result follows immediately by Lemma 2.3. □

Corollary 2.6. *If G is a finite simple $(2X, mY, nZ)$-generated group. Then $rank(G{:}mY) = 2$.*

Proof. By Lemma 2.5 and Corollary 2.4 we have $rank(G{:}mY) \leq 2$. But a non-abelian simple group can not be generated by one element. Thus $rank(G{:}mY) = 2$. □

The following two results are in some cases useful in establishing non-generation for finite groups.

Lemma 2.7 (See Ali and Moori [5] or Conder et al. [9]). *Let G be a finite centerless group. If $\Delta^*_G(C_1, C_2, \cdots, C_k) < |C_G(g_k)|$, $g_k \in C_k$, then $\Delta^*_G(C_1, C_2, \cdots, C_k) = 0$ and therefore G is not $(C_1, C_2, \cdots, C_k)$-generated.*

Proof. We prove the contrapositive of the statement, that is if $\Delta^*_G(C_1, C_2, \cdots, C_k) > 0$ then $\Delta^*_G(C_1, C_2, \cdots, C_k) \geq |C_G(g_k)|$, for a fixed $g_k \in C_k$. So let us assume that $\Delta^*_G(C_1, C_2, \cdots, C_k) > 0$. Thus there exists at least one $(k-1)$-tuple $(g_1, g_2, \cdots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ satisfying Equation (2.1). Let $x \in C_G(g_k)$. Then we obtain

$$x(g_1 g_2 \cdots g_{k-1})x^{-1} = (xg_1x^{-1})(xg_2x^{-1}) \cdots (xg_{k-1}x^{-1}) = (xg_kx^{-1}) = g_k.$$

Thus the $(k-1)$-tuple $(xg_1x^{-1}, xg_2x^{-1}, \cdots, xg_{k-1}x^{-1})$ will generate G. Moreover, if x_1 and x_2 are distinct elements of $C_G(g_k)$, then the $(k-1)$-tuples $(x_1g_1x_1^{-1}, x_1g_2x_1^{-1}, \cdots, x_1g_{k-1}x_1^{-1})$ and $(x_2g_1x_2^{-1}, x_2g_2x_2^{-1}, \cdots, x_2g_{k-1}x_2^{-1})$ are also distinct since G is centerless. Thus we have at least $|C_G(g_k)|, (k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1})$ generating G. Hence $\Delta^*_G(C_1, C_2, \cdots, C_k) \geq |C_G(g_k)|$. □

The following result is due to Ree [20].

Theorem 2.8. *Let G be a transitive permutation group generated by permutations $g_1, g_2, \cdots, g_s$ acting on a set of n elements such that $g_1 g_2 \cdots\ g_s = 1_G$. If the generator g_i has exactly c_i cycles for $1 \leq i \leq s$, then $\sum_{i=1}^{s} c_i \leq (s-2)n + 2$.*

Proof. See for example Ali and Moori [5]. □

The following result is due to Scott ([9] and [21]).

Theorem 2.9 (Scott's Theorem). *Let $g_1, g_2, \cdots, g_s$ be elements generating a group G with $g_1 g_2 \cdots g_s = 1_G$ and $\mathbb{V}$ be an irreducible module for G with $\dim \mathbb{V} = n \geq 2$. Let $C_{\mathbb{V}}(g_i)$ denote the fixed point space of $\langle g_i \rangle$ on $\mathbb{V}$ and let d_i be the codimension of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$. Then $\sum_{i=1}^{s} d_i \geq 2n$.*

With χ being the ordinary irreducible character afforded by the irreducible module $\mathbb{V}$ and $\mathbf{1}_{\langle g_i \rangle}$ being the trivial character of the cyclic group $\langle g_i \rangle$, the codimension d_i of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$ can be computed using the following formula ([11]):

$$\begin{aligned} d_i &= \dim(\mathbb{V}) - \dim(C_{\mathbb{V}}(g_i)) = \dim(\mathbb{V}) - \left\langle \chi\downarrow^G_{\langle g_i \rangle}, \mathbf{1}_{\langle g_i \rangle} \right\rangle \\ &= \chi(1_G) - \frac{1}{|\langle g_i \rangle|} \sum_{j=0}^{o(g_i)-1} \chi(g_i^j). \end{aligned}$$

3. The Ranks of the Simple Groups

The determination of the ranks of the sporadic simple groups is almost completed. With G being a sporadic simple group and nX a non-identity class of G (as listed in the $\mathbb{ATLAS}$ [10]) the results on sporadic simple groups can be summarized as follows:

(1) If $G \neq \mathbb{M}$ and nX is not an involutary class ($n \neq 2$), then $rank(G{:}nX) = 2$ unless:

$$\begin{aligned} (G, nX) \in \{&(J_2, 3A), (HS, 4A), (McL, 3A), (Ly, 3A), \\ &(Co_1, 3A), (Fi_{22}, 3A), (Fi_{22}, 3B), (Fi_{23}, 3A), \\ &(Fi_{23}, 3B), (Fi_{24}^{'}, 3A), (Fi_{24}^{'}, 3B), (Suz, 3A)\}, \end{aligned}$$

where in these cases $rank(G{:}nX) = 3$.

(2) If $G \neq \mathbb{M}$ and nX is an involutary class, then $rank(G{:}2X) = 3$ unless:
- $(G, 2X) \in \{(J_2, 2A), (Co_2, 2A), (\mathbb{B}, 2A)\}$, where in these cases we have $rank(G{:}2A) = 4$
- $(G, 2X) \in \{(Fi_{22}, 2A), (Fi_{23}, 2A)\}$, where in these cases we have $rank(Fi_{22}{:}2A) = 6$ and $rank(Fi_{23}{:}2A) \in \{5, 6\}$.

(3) If $G = \mathbb{M}$ and nX is not an involutary class, then $rank(\mathbb{M}{:}nX) \in \{2, 3\}$.

(4) If $G = \mathbb{M}$ and nX is an involutary class, then $rank(\mathbb{M}{:}2X) \in \{3, 4\}$.

Therefore we can see that only very few cases are remaining as far as sporadic simple groups are concerned. The authors are currently considering some of these cases together with other non-abelian simple groups such as the alternating groups, classical groups and groups of Lie type.

In the following we give an example of some of the results that the authors established (see [6]) on some of the classes of the alternating groups $A_n,\ n \geq 5$.

The alternating group $A_n,\ n \geq 5$ has $\lfloor \frac{n}{3} \rfloor$ conjugacy classes of elements of order 3. The cycle structures of these classes are $3^m 1^{n-3m},\ 1 \leq m \leq \lfloor \frac{n}{3} \rfloor$. For $m = 1$, let $3A$ denote the class of elements of A_n of cycle structure (a, b, c). In [6] we determined the rank of this class in A_n. We use $(A_n)_{[k_1, k_2, \cdots, k_r]}$ to denote the subgroup of A_n fixing the points $k_1, k_2, \cdots, k_r$ and if it fixes a single point k_i, we use $(A_n)_{k_i}$.

Lemma 3.1. $rank(A_5{:}3A) = 2$.

Proof. We claim that $A_5 = \langle (1, 2, 3), (1, 4, 5) \rangle$. The element $(1, 2, 3)(1, 4, 5) = (1, 4, 5, 2, 3)$ has order 5. This implies that $15 \mid |\langle (1, 2, 3), (1, 4, 5) \rangle|$. By looking at the maximal subgroups of A_5 (see the $\mathbb{ATLAS}$ [10] for example) we can see that there is no maximal subgroup of A_5 with order divisible by 15. It follows that $\langle (1, 2, 3), (1, 4, 5) \rangle = A_5$ and hence $rank(A_5{:}3A) = 2$. □

Lemma 3.2. $rank(A_n{:}3A) \neq 2,\ \forall\ n \geq 6$.

Proof. Suppose that $x, y \in 3A$ of $A_n,\ n \geq 6$ and let $x = (a, b, c)$ and $y = (d, e, f)$. If $xy = yx$ then $\langle x, y \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. If $xy \neq yx$, then x and y are not disjoint cycles and have some common points, i.e, $\{a, b, c\} \cap \{d, e, f\} \neq \phi$. Thus the number of moved points by $\langle x, y \rangle$ is at most 5 and it follows that $\langle x, y \rangle \leq A_5$. Hence $rank(A_n{:}3A) \neq 2$ for $n \geq 6$. □

Lemma 3.3. $rank(A_6{:}3A) = 3$.

Proof. We show that $A_6 = \langle (1, 2, 3), (1, 4, 5), (1, 5, 6) \rangle$. Let $H = \langle (1, 2, 3), (1, 4, 5) \rangle$. Then $H \cong A_5$ and $H = (A_6)_6$, which is a maximal subgroup of A_6. Since $(1, 5, 6) \notin H$, we have $\langle H, (1, 5, 6) \rangle = A_6$. Now since $rank(A_6{:}3A) \neq 2$ by Lemma 3.2, it follows that $rank(A_6{:}3A) = 3$. □

The next theorem states an important result on the rank of the class $3A$ of $A_n,\ n \geq 5$.

Theorem 3.4. *For the alternating group $A_n,\ n \geq 5$ we have*

$$rank(A_n{:}3A) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

Proof. We use the mathematical induction on n. The result is true for $n = 5$ and $n = 6$ by Lemmas 3.1 and 3.3, respectively. We will show that

$$A_n = \begin{cases} \langle (1, 2, 3), (1, 4, 5), (1, 6, 7), \cdots, (1, n-3, n-2), (1, n-1, n) \rangle & \text{if } n \text{ is odd,} \\ \langle (1, 2, 3), (1, 4, 5), (1, 6, 7), \cdots, (1, n-2, n-1), (1, n-1, n) \rangle & \text{if } n \text{ is even.} \end{cases} \tag{3.1}$$

Suppose that the result is true for n odd, then we will show that the result will be true for $n+1$ and $n+2$. So assume that Equation (3.1) is true for n odd. Note that if $H = \langle (1,2,3),(1,4,5),(1,6,7),\cdots,\ (1,n-1,n)\rangle$, then Equation (3.1) implies that $A_n \cong (A_{n+1})_{n+1} = H$. Since $(1,n,n+1) \in A_{n+1} \setminus A_n$ and H is a maximal subgroup of A_{n+1}, we have

$$K = \langle H, (1,n,n+1)\rangle = A_{n+1}.$$

Since $n+1$ is even, we have proven the result for the even case. Now since K is a maximal subgroup of A_{n+2} ($K = (A_{n+2})_{n+2}$), then $\langle K, (1,n+1,n+2)\rangle = A_{n+2}$ as $(1,n+1,n+2) \in A_{n+2} \setminus A_{n+1}$. Thus

$$\langle H, (1,n,n+1), (1,n+1,n+2)\rangle = A_{n+2}. \tag{3.2}$$

We now show that we do not need the element $(1,n,n+1)$ in Equation (3.2) to generate A_{n+2}; that is $(1,n,n+1)$ is redundant. Let $\alpha = (1,n-1,n) \in H$ and $\beta = (1,n+1,n+2)$. Then $\alpha^\beta = (n-1,n,n+1) := \gamma$ and $\gamma^\alpha = (1,n+1,n) = (1,n,n+1)^{-1}$. This shows that $\langle H, (1,n,n+1), (1,n+1,n+2)\rangle$ can actually be reduced to $\langle H, (1,n+1,n+2)\rangle$. That is $\langle H, (1,n+1,n+2)\rangle = A_{n+2}$, i.e

$$\langle (1,2,3),(1,4,5),(1,6,7),\cdots,(1,n-1,n),(1,n+1,n+2)\rangle = A_{n+2},$$

completing the proof. □

In Section 4 of [6] we completely determined the ranks of all the classes of A_8 and A_9. Also in [7] we supplied further general results on some of the classes of A_n, $n \geq 5$ and we completely determined the ranks of all the classes of A_{10}.

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