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ABEL-SCHUR MULTIPLIERS ON BANACH SPACES OF INFINITE MATRICES

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Abstract. We consider a more general class than the class of Schur multipliers namely the *Abel-Schur multipliers,* which in turn coincide with the bounded linear operators on ℓ_2 preserving the diagonals. We extend to the matrix framework Theorem 2.4 (a) of a paper of Anderson, Clunie, and Pommerenke published in 1974, and as an application of this theorem we obtain a new proof of the necessity of an old theorem of Hardy and Littlewood in 1941.

1. Introduction and preliminaries

In [1] the authors characterized the topological dual of the space

$$\mathcal{I} = \{f : D \to \mathbb{C};\ f(z) = \sum_{n=0}^{\infty} a_n z^n,\ z \in D,\ \text{an analytic function such that}$$

$$||f||_{\mathcal{I}} := |f(0)| + \frac{1}{2\pi}\int_0^1 \int_0^{2\pi} |f'(re^{i\theta})| d\theta r dr < \infty\}$$

as being given by the Bloch space of functions

$$\mathcal{B} = \{f : D \to \mathbb{C}; f \text{ is an analytic function such that } ||f||_{\mathcal{B}} := |f(0)| + \sup_{|z|<1}(1 - |z|)||f'(z)||_\infty < \infty\}$$

in the following manner (cf. [1, Theorem2.3]):
Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$ *and* $g(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{I}$. *Then*

$$h(z) = \sum_{n=0}^{\infty} a_n b_n z^n$$

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is continuous on $|z| \le 1$ *and further,*

$$|h(z)| \le 2||f||_{\mathcal{B}}||g||_{\mathcal{I}} \qquad (|z| \le 1).$$

In particular it follows that

$$\langle f, g\rangle := \lim_{\rho\to 1^-} \sum_{n=0}^{\infty} a_n b_n \rho^n$$

exists for $f \in \mathcal{B},\ g \in \mathcal{I}$. It is showed in [1, Theorem 2.5] that this sum need not converge for $\rho = 1$.

It was also proved in [1, Theorem 2.4] that

For every $\psi \in \mathcal{I}^*$ *there exists a unique* $f \in \mathcal{B}$ *such that*

$$\psi(g) = \langle f, g\rangle := \lim_{\rho\to 1^-} \sum_{n=0}^{\infty} a_n b_n \rho^n \quad (g \in \mathcal{I}).$$

Conversely this defines a bounded linear functional on $\mathcal{I}$ *for each* $f \in \mathcal{B}$. *Also*

$$\frac{1}{3}||f||_{\mathcal{B}} \le ||\psi||_{\mathcal{I}^*} \le 2||f||_{\mathcal{B}}.$$

In 1976 the previous theorem was rephrased in [2] as follows

$$\mathcal{I}^a = \mathcal{B}, \tag{1.1}$$

where $\mathcal{I}^a$ denotes the *Abel dual* of $\mathcal{I}$, that is the space of all analytic functions $g = \sum_{n=0}^{\infty} b_n z^n$, for $|z| < 1$, such that

$$\lim_{\rho\to 1^-} \sum_{0}^{\infty} a_n b_n \rho^n$$

exists for all functions $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{I}$.

In this paper we would like to introduce for some spaces of upper triangular infinite matrices a similar notion to that of the Abel dual, and to extend in an appropriate manner the identity (1.1) .

To this end we introduce a new class of bounded linear operators acting on Banach spaces of infinite matrices. The Schur multipliers belong to this class, and also to another more general class called the class of Abel-Schur multipliers.

In what follows we focus our attention on the upper triangular infinite matrices, and particularly on Toeplitz matrices of this kind. Such a Toeplitz matrix may be identified in a natural manner with an analytic function.
To the function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in X$ we associate the matrix

$$A_f := \begin{pmatrix} a_0 & a_1 & a_2 & \dots \\ 0 & a_0 & a_1 & \dots \\ 0 & 0 & a_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with the equality of the norms: $||A_f||_X = ||f||_X$.

Next, let X, Y be two Banach spaces of infinite matrices (the matrices are always meant to be upper triangular). We denote by $S(X,Y)$ the set of all upper triangular matrices A, such that

$$A * B \in Y \text{ for all } B \in X.$$

Here $A * B$ is the matrix $C = (c_{ij})$ given by

$$c_{ij} = a_{ij} b_{ij} \quad (i, j \geq 0),$$

where $A = (a_{ij})$, $B = (b_{ij})$.
We call $A * B$ *the Schur (Hadamard) product of matrices* A *and* B, and A will be called *the Schur multiplier* for the pair X and Y.

On the space $S(X,Y)$ we consider the natural quasi-norm

$$||A||_{S(X,Y)} := \sup_{||B||_X \leq 1} ||A * B||_Y.$$

In the sequel we consider a class of bounded linear operators between X and Y, extending the class of Schur multipliers.

First of all we make some assumptions on the spaces X and Y.
If $A = (a_{ij})_{j \geq i \geq 1} \in X$, the matrix $A_k = (a'_{ij})$, where $k \geq 0$, is called the kth *diagonal of* A if

$$a'_{ij} := \begin{cases} a_{ij} & \text{whenever} \quad j = i + k, \ i \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $P_k^X : X \to X$, $k \geq 0$ be the natural projection on the kth diagonal, that is

$$P_k^X(A) := A_k, \quad (k \geq 0).$$

A Banach space X of upper triangular infinite matrices has property (i) if

$$\sup_{k \geq 0} ||P_k^X|| \leq C < \infty,$$

where the norm is meant to be the usual norm of the space of all bounded linear operators $B(X,Y)$, and $C > 0$ is a constant $C = C(X)$.

The space $B(\ell_2)$ of all bounded linear operators $T : \ell_2 \to \ell_2$, identified with their representation matrices, and the Schatten classes S_p, $1 \leq p < \infty$ verify the property (i) (cf. [6]).

We remark that T_A given by $T_A(B) = A * B$, for $A \in S(X,Y)$, *preserves the diagonals* that is $P_n^Y = T_A P_n^X$, for all $n \geq 0$.

We are lead naturally to the following definition.

Definition 1.1. Let X, Y be two spaces with property (i). We denote by $B_d(X,Y)$ the space of all bounded linear operators $T : X \to Y$ preserving the diagonals.

Remark 1.2. Clearly $B_d(X,Y)$ is a closed subspace of $B(X,Y)$.
Using the previous identification between a space X of analytic functions and a space of infinite matrices, denoted also by X, we get for instance that the classical H^∞ may be regarded as the space of all upper triangular infinite Toeplitz matrices belonging to $B(\ell_2)$, which is a well-known result [14].

Next let X be a Banach space of analytic functions, which can be regarded as a space of infinite matrices. We say that X verifies condition (ii), if we have:

$$\lim_{\rho\to 1^-} f_\rho = f \in X,$$

where $f_\rho(z) = f(\rho z)$, for $|z| < 1$ and $0 \le \rho < 1$. We can describe the space $B_d(X,Y)$.

Proposition 1.3. *Let X be a Banach space of analytic functions on the unit disk verifying conditions (i) and (ii), and let Y be a Banach space of upper triangular infinite matrices (not necessarily Toeplitz matrices) verifying condition (i). Let moreover $T \in B_d(X,Y)$. There is an upper triangular infinite matrix A such that*

$$T(B) = \lim_{\rho\to 1^-} \sum_{n=0}^{\infty} (A_n * B_n)\rho^n,$$

for all $B \in X$, the limit being taken in the norm of Y.
Moreover,

$$|||A||| := \sup_{||B||_X\le 1} || \lim_{\rho\to 1^-} \sum_{n=0}^{\infty} (A_n * B_n)\rho^n ||_Y \le ||T|| < \infty.$$

Conversely, if $|||A||| < \infty$,

$$T(B) = \lim_{\rho\to 1^-} \sum_{n=0}^{\infty} (A_n * B_n)\rho^n$$

where $B \in X$, is an operator belonging to $B_d(X,Y)$, and $||T|| \le |||T|||$.

Proof. Let $T \in B_d(X,Y)$. If $E = (e_{ij})_{i,j}$, where $e_{ij} = 1$ for all $i, j \ge 1$, its diagonal E_n corresponds to the function z^n, with $|z| < 1$.
We put $A_n := T(E_n)$. Since $T \in B_d(X,Y)$, A_n is the nth diagonal of an upper triangular matrix A. If $B \in X$, all the entries on the diagonal B_n are equal to the same complex number, denoted by b_n.
By using condition (ii) we get

$$\lim_{\rho\to 1^-} ||B_\rho - B|| = 0,$$

consequently

$$T(B) = \lim_{\rho\to 1^-} T(B_\rho) = \lim_{\rho\to 1^-} T(\sum_{n=0}^{\infty} b_n\rho^n z^n).$$

Since

$$\sum_{n=0}^{\infty} ||B_n||_X\rho^n \le \left(\sum_{n=0}^{\infty} \rho^n\right) C(X)||B||_X \le \frac{C(X)}{1-\rho}||B||_X < \infty,$$

by condition (i) it follows that the series $\sum_{n=0}^{\infty} b_n\rho^n z^n$ converges with respect to the norm of X. Thus

$$T(B) = \lim_{\rho\to 1^-} \sum_{n=0}^{\infty} b_n\rho^n A_n = \lim_{\rho\to 1^-} \sum_{n=0}^{\infty} (A_n * B_n)\rho^n$$

for all $B \in X$. Moreover

$$|| \lim_{\rho\to 1^-} \sum_{n=0}^{\infty} (A_n * B_n)\rho^n ||_Y = ||T(B)||_Y \le ||T|| \cdot ||B||_X,$$

thus

$$|||A||| \le ||T|| < \infty.$$

The converse is immediate, since

$$T(B_k) = \lim_{\rho\to 1^-} (A_k * B_k)\, \rho^k = A_k * B_k$$

for all $k \ge 0$. Thus T preserves the diagonals, and moreover, T is a bounded linear operator, because $||T|| = |||A||| < \infty$. □

Definition 1.4. Let X, Y be Banach spaces of upper triangular infinite matrices verifying condition (i). An upper triangular matrix A is called Abel-Schur multiplier (with respect to the pair X, Y) if, for every $B \in X$, there exists

$$\langle A, B\rangle := \lim_{\rho\to 1^-} \sum_{n=0}^{\infty} (A_n * B_n)\rho^n$$

in Y and moreover,

$$|||A||| := \sup_{||B||_X \le 1} || \lim_{\rho\to 1^-} \sum_{n=0}^{\infty} (A_n * B_n)\rho^n ||_Y < \infty.$$

Krtinic [8] showed that, in the case $Y = B(\ell_2)$ or $Y = S_p$, $1 \le p < \infty$, the filter $(\sum_{n=0}^{\infty} A_n\rho^n)_{0\le\rho<1}$ converges to the matrix A if and only if

$$\lim_{n\to\infty} \sigma_n(A) = A,$$

where $\sigma_n(A) = \sum_{k=0}^{\infty}(1 - \frac{k}{n+1})A_k$.

Lemma 1.5. *Let X be a Banach space verifying conditions (i) and (ii), and Y with condition (i). Then every Schur multiplier $A \in S(X, Y)$ is an Abel-Schur multiplier and moreover,*

$$||A||_{S(X,Y)} = |||A|||.$$

Proof. Let $A \in S(X, Y)$ and $B \in X$. Then

$$||A_n||_{S(X,Y)} = \sup_{||B||_X\le 1} ||A_n * B||_Y = \sup_{||B||_X\le 1} ||A * B_n||_Y$$

$$\le ||A||_{S(X,Y)} \sup_{||B||\le 1} ||B_n||_X \le C(X)||A||_{S(X,Y)}$$

for all n. Thus

$$\sum_{n=0}^{\infty} ||A_n||_{S(X,Y)}||B_n||_X\rho^n \le C(X)||A||_{S(X,Y)} \sum_{n=0}^{\infty} ||B_n||_X\rho^n$$

$$\le C(X)^2||A||_{S(X,Y)}(1-\rho)^{-1} < \infty,$$

for all $B \in X$, $0 \le \rho < 1$. Consequently, $A * B_\rho = \sum_{n=0}^{\infty}(A_n * B_n)\rho^n$ converges in the norm of Y for any $0 \le \rho < 1$.

Since $\lim_{\rho\to 1^-} B_\rho = B$ in the norm of X, we have that $\lim_{\rho\to 1^-} A * B_\rho = A * B$ in Y, that is

$$\lim_{\rho\to 1^-} \sum_{n=0}^{\infty} (A_n * B_n)\rho^n = A * B \in Y$$

for all $B \in X$. Therefore A is an Abel-Schur multiplier and

$$|||A||| = \sup_{||B||_X \le 1} ||A * B||_Y = ||A||_{S(X,Y)} < \infty.$$

□

In view of Lemma 1.5 we ask ourselves if the converse is also true, that is if any Abel-Schur multiplier is a Schur multiplier. The negative answer is given by the following statement.

First, we introduce a different notation for the entries of an upper triangular infinite matrix $A = (a_{ij})_{i,j\ge 1}$, namely we denote by $a_k^l := a_{l,k+l}$ for $l \ge 1$ and $k \ge 0$.

We recall from [1] that $\mathcal{B}$ is the Bloch space of analytic functions, that is

$$\mathcal{B} = \{f : D \to \mathbb{C}; f \text{ is an analytic function such that } ||f||_{\mathcal{B}} := |f(0)| + \sup_{|z|<1} (1 - |z|)||f'(z)||_\infty < \infty\}$$

Example 1.6. There is a pair of Banach spaces X, Y of upper triangular infinite matrices and an Abel-Schur multiplier A for the pair X, Y such that $A \notin S(X, Y)$.

As in [1] let

$$\mathcal{I} = \{f : D \to \mathbb{C};\ f(z) = \sum_{n=0}^{\infty} a_n z^n,\ z \in D,\ \text{is an analytic function such that}$$

$$||f||_{\mathcal{I}} := |f(0)| + \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |f'(re^{i\theta})| d\theta r dr < \infty\}.$$

If we consider the space $\mathcal{I}$, identified as above with a space of upper triangular infinite Toeplitz matrices X, it is well-known [1], that X verifies conditions (i) and (ii). Let

$$Y = \{A;\ ||A||_Y := \sup_{n\ge 0; l\ge 1} |\sum_{k=0}^{n} a_k^l| < \infty\}.$$

It is easy to see that Y is a Banach space. Moreover, Y verifies condition (i) since $A \in Y$ implies that for all $n \ge 0$ we have

$$\sup_{k\ge 0} ||P_k^Y(A)||_Y = \sup_{k\ge 0,\, l\ge 1} |a_k^l| \le 2 \sup_{n\ge 0,\, l\ge 1} |\sum_{k=0}^{n} a_k^l| = 2||A||_Y,$$

that is $C(Y) \le 2$.

By using [1, Theorem 2.3, Theorem 2.4], it follows that any $f \in \mathcal{B}$ determines an Abel-Schur multiplier for X and Y, given by the Toeplitz matrix A_f. But [1, Theorem 2.5] shows that there are upper triangular Toeplitz matrices $A = (a_k^l) \in \mathcal{I}$ and $B = (b_k^l) \in \mathcal{B}$ such that the sequence $(\sum_{k=0}^{n} a_k^l b_k^l)_n$ is unbounded for all l. But this shows that $A * B \notin Y$.

2. An example of $B_d(X,Y)$

We denote by $\mathcal{B}(D,\ell_2)$ the space of all upper triangular matrices A such that

$$||A|| := |A_0| + \sup_{0\le r<1} (1-r)||A'_r||_{B(\ell_2)} < \infty,$$

where

$$A'_r := \sum_{k=1}^{\infty} kA_k r^{k-1}$$

for all $0 \le r < 1$, see [10], [11].

Let next $H^\infty(\ell_2)$ be the Banach space of all upper triangular infinite matrices $A \in B(\ell_2)$ equipped with the norm induced by $B(\ell_2)$.

Under these notations we can describe the space $B_d\big(\mathcal{I}, H^\infty(\ell_2)\big)$.

Theorem 2.1. *Let $T \in B_d(\mathcal{I}, H^\infty(\ell_2))$. Then there exists a unique upper triangular matrix $A \in \mathcal{B}(D,\ell_2)$ such that*

$$T(B) = \lim_{\rho\to 1^-} \sum_{n=0}^{\infty} (A_n * B_n)\rho^n$$

for all $B \in \mathcal{I}$, where the limit is taken in the operator norm. Moreover,

$$||A||_{\mathcal{B}(D,\ell_2)} \le 3||T||.$$

Proof. Let $A = T(E_n)$, for all $n \ge 0$. Here E_n was introduced in the proof of Proposition 1.3. Next, let $A = \sum_{n=0}^{\infty} A_n$, the series being a formal one, and let $0 \le \rho < 1$. Then we have

$$T(B_n) = T\left(\sum_{n=0}^{\infty} b_n \rho^n E_n\right),$$

where B is the Toeplitz matrix corresponding to the analytic function $g(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{I}$. But

$$\sum_{n=0}^{\infty} |b_n|\rho^n \le \left(\sup_n |b_n|\right) \frac{1}{1-\rho} \le 2||g||_{\mathcal{I}} \frac{1}{1-\rho},$$

where the last inequality follows by Cauchy's inequalities. Therefore

$$T(B_\rho) = T\left(\sum_{n=0}^{\infty} b_n E_n \rho^n\right) = \sum_{n=0}^{\infty} b_n \rho^n A_n.$$

Since $\mathcal{I}$ verifies condition (ii), by Proposition 1.3, it follows that

$$T(B) = \lim_{\rho\to 1^-} T(B_\rho) = \lim_{\rho\to 1^-} \sum_{n=0}^{\infty} (B_n * A_n)\rho^n$$

for all $B \in \mathcal{I}$. Therefore the matrix A is an Abel-Schur multiplier with respect to the pair $\mathcal{I}$, $H^\infty(\ell_2)$.

Next we show that $A \in \mathcal{B}(D,\ell_2)$ and $||A||_{\mathcal{B}(D,\ell_2)} \le 2||T||$.

First, clearly

$$A'_r = \sum_{k=0}^{\infty} kA_k r^{k-1} = T(B(r)),$$

where $B(r)$ is the Toeplitz matrix corresponding to the function $f(z) = \frac{z}{(1-rz)^2}$, for $0 \le r < 1$ and $|z| < 1$. It is known by [1] that $||f||_{\mathcal{I}} \le \frac{2}{1-r^2}$.
Therefore

$$||A'_r||_{B(\ell_2)} = ||T(B(r))||_{B(\ell_2)} \le ||T|| \cdot ||B(r)||_{\mathcal{I}} \le \frac{2}{1-r^2}||T||.$$

Consequently,

$$||A||_{\mathcal{B}(D,\ell_2)} = \sup_{0\le r<1}(1-r)||A'_r||_{B(\ell_2)} + ||A_0||_{B(\ell_2)} \le 3||T||.$$

Next, putting $B = E_n$, for all $n \ge 0$, we conclude that $A \in \mathcal{B}(D, \ell_2)$ is uniquely determined by T. □

We recall that the space of all Schur multipliers $S\big(B(\ell_2), B(\ell_2)\big)$ will be denoted by $M(\ell_2)$, [3].

Next we give a converse of Theorem 2.1 and at the same time, an extension of [1, Theorem 2.3] stated at the beginning of this paper.

Theorem 2.2. *Let $A \in \mathcal{B}(D, \ell_2)$ and let $B \in \mathcal{I}$ be a Toeplitz matrix. Then*

$$h(r) = \sum_{n=0}^{\infty}(A_n * B_n)r^n$$

is a continuous function on $[0, 1)$ and it can be extended by continuity to $[0, 1]$. Moreover, we have

$$||h(r)||_{H^\infty(\ell_2)} \le 5||A||_{\mathcal{B}(D,\ell_2)} \cdot ||B||_{\mathcal{I}}, \tag{2.1}$$

for all $r \in [0, 1)$.
Particularly, A is an Abel-Schur multiplier for the pair $\mathcal{I}$, $H^\infty(\ell_2)$, that is the limit

$$\lim_{\rho\to 1^-}\sum_{n=0}^{\infty}(A_n * B_n)\rho^n$$

exists in $H^\infty(\ell_2)$, for any $A \in \mathcal{B}(D, \ell_2)$, $B \in \mathcal{I}$.

Proof. Let $0 \le \rho < 1$. Then $A'_r = \sum_{n=1}^{\infty} nA_n r^{n-1}$ and

$$\frac{d}{dr}[r(B_r - B_0)] = \sum_{n=1}^{\infty}(n+1)B_n r^n,$$

where B is the Toeplitz matrix corresponding to the function $g \in \mathcal{I}$. Straightforward computations shows that

$$\sum_{n=1}^{\infty}(A_n * B_n)\rho^{n-1} = 2\int_0^1 \big[(1-r^2)A'_{\rho r}\big] * \big[\frac{d}{dr}[r(B_r - B_0)]\big]dr.$$

Since

$$||C * D||_{B(\ell_2)} \le ||C||_{B(\ell_2)}||D||_{M(\ell_2)},$$

we have

$$||\sum_{n=1}^{\infty}(A_n * B_n)\rho^n||_{B(\ell_2)} \leq 2 \sup_{0\leq r<1}(1-r^2)||A'_{\rho r}||_{B(\ell_2)} \int_0^1 ||B_r - B_0 + rB'_r||_{M(\ell_2)}dr$$

$$\leq 2 \sup_{0\leq s<1}(1-s^2)||A'_s||_{B(\ell_2)} \int_0^1 \left[||B_r - B_0||_{M(\ell_2)} + r||B'_r||_{M(\ell_2)}\right]dr$$

$$\leq 4||A||_{\mathcal{B}(D,\ell_2)} \left[\int_0^1 ||B_r - B_0||_{M(\ell_2)}dr + \int_0^1 r||B'_r||_{M(\ell_2)}dr\right].$$

But

$$||B_r - B_0||_{M(\ell_2)} = \left\|\int_0^r B'_t dt\right\|_{M(\ell_2)} \leq \int_0^r ||B'_t||_{M(\ell_2)}dt.$$

Thus

$$\int_0^1 ||B_r - B_0||_{M(\ell_2)}dr \leq \int_0^1 \int_0^r ||B'_t||_{M(\ell_2)}dt = \int_0^1 \left[\int_t^1 dr\right] ||B'_t||_{M(\ell_2)}dt$$

$$= \int_0^1 (1-t)||B'_t||_{M(\ell_2)}dt.$$

Then

$$||\sum_{n=1}^{\infty}(A_n * B_n)\rho^n||_{B(\ell_2)} \leq 4||A||_{\mathcal{B}(D,\ell_2)} \int_0^1 ||B'_t||_{M(\ell_2)}dt = 4||A||_{\mathcal{B}(D,\ell_2)}\cdot||B||_{\mathcal{I}}. \quad (2.2)$$

Here we used the well-known fact that the norm $||C||_{M(\ell_2)}$ coincide with the norm of the Borel measure associated to the Toeplitz matrix C, [4].

We show that h is a continuous function on $[0, 1)$. Let $0 \leq \rho_1, \rho_2 < 1$. Then, using (2.2)

$$||h(\rho_1) - h(\rho_2)||_{B(\ell_2)} = ||\sum_{n=0}^{\infty} A_n * [(B_n)_{\rho_1} - (B_n)_{\rho_2}]||_{B(\ell_2)}$$

$$\leq 5||A||_{\mathcal{B}(D,\ell_2)}||B_{\rho_1} - B_{\rho_2}||_{\mathcal{I}} \to 0$$

uniformly whenever $|\rho_1 - \rho_2| \to 0$, (see relation (2.9) in [1]). Therefore $h(\rho)$ is also continuous on $[0, 1]$, consequently relation (2.1) holds. □

Corollary 2.3. *The matrices $A \in \mathcal{B}(D, \ell_2)$ may be identified with the Abel-Schur multipliers for the pair $\mathcal{I}$, $H^\infty(\ell_2)$, with equivalent norms.*

As a consequence, by using [8], we have

$$\mathcal{B}(D, \ell_2) = S(\mathcal{I}, H^\infty(\ell_2)), \quad (2.3)$$

with equivalent norms. More generally, we can do the following.

Remark 2.4. For the pair $\mathcal{I}$, X, where X is $H^\infty(\ell_2)$, or a symmetric mononormalizing ideal of upper triangular matrices, the Schur multipliers and the Abel-Schur multipliers are the same.

3. Some applications of Corollary 2.3.

As an application of Corollary 2.3 we give a different proof of a theorem found independently in [13] and [12]. This is the necessity part of Theorem HL [9]. The sufficiency was proved in 1941 by Hardy and Littlewood [7].

Let us state Theorem HL.

Theorem HL. An analytic function $g \in (H^1, H^2)$ if and only if

$$M_2(r, g') \leq \frac{C}{1-r},\ 0 < r < 1.$$

Here

$$M_p^p(r, f) := \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta,\ 1 \leq p < \infty,$$

and H^p means the Hardy space of all analytic functions f on the unit disk D, such that

$$\sup_{0 \leq r < 1} M_p(r, f) < \infty.$$

Moreover, for two sequence spaces A, B, a sequence $\lambda = (\lambda_n)$ is called a *multiplier* from A to B if $(\lambda_n \alpha_n) \in B$ whenever $(\alpha_n) \in A$. The space of all multipliers from A to B is denoted by (A, B). Therefore if we identify a space X (respectively Y) of analytic functions with the space A (respectively B) of all sequences of Taylor's coefficients of those functions, then the space (X, Y) means the space of all multipliers (A, B). This is the meaning of the notation (H^1, H^2) in the statement of the above theorem.

Theorem 3.1. *Let the analytic function $g \in (H^1, H^2)$. Then we have*

$$M_2(r, g') \leq \frac{C}{1-r}$$

for all $0 < r < 1$, where $C > 0$ is a constant.

Proof. Let $g(z) = \sum_{n=0}^{\infty} c_n z^n \in (H^1, H^2)$, where $|z| < 1$. Next we consider the matrix

$$\begin{pmatrix} c_0 & c_1 & c_2 & c_3 & \dots & c_n & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}.$$

Since $\mathcal{I} \subset H^1$ [1], clearly the sequence $(c_n)_{n \geq 0}$ is a multiplier from $\mathcal{I}$ to H^2, therefore the matrix G is a Schur multiplier from $\mathcal{I}$ to $H^\infty(\ell_2)$.

Thus, by Corollary (2.3), $G \in \mathcal{B}(D, \ell_2)$, that is, by using the special form of the matrix G, we have

$$\sup_{0 < r < 1} (1-r) M_2(r, g') \leq C < \infty.$$

The proof is complete. □

Another application of Corollary (2.3) is an extension into matrix spaces framework of a celebrated theorem of Mateljevic and Pavlovic [9]. We use the beautiful ideas of the theory developed for the analytic functions by O. Blasco and M. Pavlovic [5]. More precisely, motivated by [5, Definition 2.3] we introduce the notion of *tensor product* of two Banach spaces of infinite matrices.

Let first $\mathcal{M}$ be the space of all upper triangular matrices A so that for any natural number k we have $||A||_{\mathcal{M},k} := ||P_k(A)||_{B(\ell_2)} < \infty$. The space $\mathcal{M}$ will be equipped with the semi-norms $||A||_{\mathcal{M},k}$, $k \in \mathbb{N}$.

Definition 3.2. Let X and Y be two Banach spaces of upper triangular infinite matrices verifying condition (i). Then we define the space $X \otimes Y$, to be the set of all upper triangular infinite matrices C that can be represented in the form $C = \sum_{n=0}^{\infty} A^{(n)} * B^{(n)}$, $A^{(n)} \in X$, $B^{(n)} \in Y$ so that the series converges in $\mathcal{M}$ and

$$\sum_{n=0}^{\infty} ||A^{(n)}||_X ||B^{(n)}||_Y < \infty. \tag{3.1}$$

The norm in $X \otimes Y$ is given by

$$||C||_{X \otimes Y} = \inf \sum_{n=0}^{\infty} ||A^{(n)}||_X ||B^{(n)}||_Y,$$

where the infimum is taken over all the above representations.

Next we get easily the following matrix version of [5, Theorem 2.3].

Theorem 3.3. *Let X, Y, Z be Banach matrix spaces verifying condition (i). Then*

$$S(X \otimes Y, Z) = S\big(X, S(Y, Z)\big).$$

We omit the details of the proof.

Now we use a beautiful result due to Blasco and Pavlovic, (see [5, Corollary 8.1]):

$$H^1 \otimes H^1 = \mathcal{I}.$$

More precisely, using (2.1), Theorem 3.3, and this last equality, we get:

$$B(D, \ell_2) = S\big(\mathcal{I}, H^\infty(\ell_2)\big) = S\big(H^1, S\big(H^1, H^\infty(\ell_2)\big)\big). \tag{3.2}$$

Next we give a definition motivated by the well-known fact, that $BMOA = (H^1, H^\infty)$, see [9].

Definition 3.4. We denote the space of Schur multipliers $S\big(H^1, H^\infty(\ell_2)\big)$ by $BMOA(\ell_2)$ and we equipped it by the usual norm on the space of Schur multipliers. .

It is easy to see that the subspace of all upper triangular Toeplitz matrices of $BMOA(\ell_2)$ coincides with the classical Banach space $BMOA$ of analytic functions on the unit disk.

Now, by using (3.2) we get the matrix extension of a well-known theorem of Mateljevic and Pavlovic [9]

Theorem 3.5.

$$S\big(H^1, BMOA(\ell_2)\big) = \mathcal{B}(D, \ell_2).$$

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