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PERIODICITY AND STABILITY IN NONLINEAR NEUTRAL DYNAMIC EQUATIONS WITH INFINITE DELAY ON A TIME SCALE

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Abstract. Let $\mathbb{T}$ be a periodic time scale. We use a fixed point theorem due to Krasnoselskii to show that the nonlinear neutral dynamic equation with infinite delay

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + (Q(t, x(t-g(t)))))^{\Delta} + \int_{-\infty}^{t} D(t,u) f(x(u)) \Delta u, \; t \in \mathbb{T},$$

has a periodic solution. Under a slightly more stringent inequality we show that the periodic solution is unique using the contraction mapping principle. Also, by the aid of the contraction mapping principle we study the asymptotic stability of the zero solution provided that $Q(t,0) = f(0) = 0$. The results obtained here extend the work of Althubiti, Makhzoum and Raffoul [1].

1. Introduction

Delay dynamic equations arise from a variety of applications including in various fields of science and engineering. In particular, problems concerning qualitative analysis of delay dynamic equations have received the attention of many authors, see [1]–[6], [9], [10] and the references therein.

Recently, Althubiti, Makhzoum and Raffoul [1] discussed the periodicity and the stability for the neutral differential equation with infinite delay

$$\frac{d}{dt}x(t) = -a(t)x(t) + \frac{d}{dt}Q(t, x(t-g(t))) + \int_{-\infty}^{t} D(t,u) f(x(u)) du. \tag{1.1}$$

By employing the Krasnoselskii's fixed point theorem, the authors obtained existence of periodic solutions of (1.1). Also, the authors used the contraction

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mapping principle to show the existence of a unique periodic solution and the asymptotic stability of the zero solution of (1.1).

The neutral dynamic equation

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + (Q(t, x(t-g(t))))^{\Delta} + G\left(t, x(t), x(t-g(t))\right), \ t \in \mathbb{T}, \qquad (1.2)$$

has been investigated in [9]. By using the fixed point theorems, the existence, uniqueness and stability results have been established.

In this paper, we are interested in the analysis of qualitative theory of periodicity and stability in dynamic equations. Inspired and motivated by the works mentioned above and the papers [1]–[6], [9], [10] and the references therein, we consider the following nonlinear neutral dynamic equations with infinite delay

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + ((Q(t, x(t-g(t)))))^{\Delta} + \int_{-\infty}^{t} D\left(t, u\right) f\left(x(u)\right) \Delta u, \ t \in \mathbb{T}, \quad (1.3)$$

where $\mathbb{T}$ is a periodic time scale such that $0 \in \mathbb{T}$. In order for the function $x(t - g(t))$ to be well-defined over $\mathbb{T}$, we assume that $g : \mathbb{T} \to \mathbb{R}$ and that $id - g : \mathbb{T} \to \mathbb{T}$ is strictly increasing.

Our purpose here is to use the Krasnoselskii's fixed point theorem to show the existence of positive periodic solutions on time scales for equation (1.3). To reach our desired end we have to transform (1.3) into integral equation written as a sum of two mapping; one is a contraction and the other is compact. After that, we use Krasnoselskii's fixed point theorem, to show the existence of a positive periodic solution for equation (1.3). Also, we use the contraction mapping principle to show the existence of a unique periodic solution and the asymptotic stability of the zero solution provided that $Q\left(t, 0\right) = f\left(0\right) = 0$.

The organization of this paper is as follows. In Section 2, we present some preliminary material that we will need through the remainder of the paper. We will state some facts about the exponential function on a time scale as well as the Krasnoselskii fixed point theorem. For details on Krasnoselskii theorem we refer the reader to [11]. We present our main results on periodicity in Section 3. In Section 4 we state and prove a theorem concerning the stability of the zero solution of (1.3) for a general time scale. The results presented in this paper extend the main results in [1].

2. Preliminaries

The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing (see [2], [3], [5]–[10] and references therein). The theory of dynamic equations unifies the theories of differential equations and difference equations. We suppose that the reader is familiar with the basic concepts concerning the calculus on time scales for dynamic equations. Most of the following definitions, lemmas and theorems can be found in Bohner and Peterson books [7, 8].

A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t \in \mathbb{T}$ the forward jump operator σ, and the backward jump operator ρ, respectively, are defined as $\sigma\left(t\right) = \inf\left\{s \in \mathbb{T} : s > t\right\}$ and $\rho\left(t\right) = \sup\left\{s \in \mathbb{T} : s < t\right\}$. These operators allow elements in the time scale to be classified as follows. We say t is right scattered if $\sigma\left(t\right) > t$ and right dense if $\sigma\left(t\right) = t$. We say t is left scattered if $\rho\left(t\right) < t$

and left dense if $\rho(t) = t$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$, is defined by $\mu(t) = \sigma(t) - t$ and gives the distance between an element and its successor. We set $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, we define $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$. Otherwise, we define $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum m, we define $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$. Otherwise, we define $\mathbb{T}_k = \mathbb{T}$.

Let $t \in \mathbb{T}^k$ and let $f : \mathbb{T} \to \mathbb{R}$. The delta derivative of $f(t)$, denoted by $f^{\triangle}(t)$, is defined to be the number (when it exists), with the property that, for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\triangle}(t)[\sigma(t) - s] \right| \leq \epsilon |\sigma(t) - s|,$$

for all $s \in U$. If $\mathbb{T} = \mathbb{R}$ then $f^{\triangle}(t) = f'(t)$ is the usual derivative. If $\mathbb{T} = \mathbb{Z}$ then $f^{\triangle}(t) = \triangle f(t) = f(t+1) - f(t)$ is the forward difference of f at t.

A function f is right dense continuous (rd-continuous), $f \in C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$, if it is continuous at every right dense point $t \in \mathbb{T}$ and its left-hand limits exist at each left dense point $t \in \mathbb{T}$. The function $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}^k$ provided $f^{\triangle}(t)$ exists for all $t \in \mathbb{T}^k$.

We are now ready to state some properties of the delta-derivative of f. Note that $f^{\sigma}(t) = f(\sigma(t))$.

Theorem 2.1 ([7])**.** *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$ and let α be a scalar.*

(i) $(f + g)^{\triangle}(t) = f^{\triangle}(t) + g^{\triangle}(t)$.

(ii) $(\alpha f)^{\triangle}(t) = \alpha f^{\triangle}(t)$.

(iii) The product rules

$$(fg)^{\triangle}(t) = f^{\triangle}(t) g(t) + f^{\sigma}(t) g^{\triangle}(t),$$
$$(fg)^{\triangle}(t) = f(t) g^{\triangle}(t) + f^{\triangle}(t) g^{\sigma}(t).$$

(iv) If $g(t) g^{\sigma}(t) \neq 0$ then

$$\left(\frac{f}{g}\right)^{\triangle}(t) = \frac{f^{\triangle}(t) g(t) - f(t) g^{\triangle}(t)}{g(t) g^{\sigma}(t)}.$$

Our first two theorems concern the composition of two functions. The first theorem is the chain rule on time scales [7, Theorem 1.93].

Theorem 2.2 (Chain Rule)**.** *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $w : \tilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\Delta}(t)$ and $w^{\tilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^{\kappa}$, then*

$$(w \circ \nu)^{\Delta} = (w^{\tilde{\Delta}} \circ \nu)\nu^{\Delta}.$$

In the sequel we will need to differentiate and integrate functions of the form $f(t - g(t)) = f(\nu(t))$ where, $\nu(t) := t - g(t)$. Our second theorem is the substitution rule [7, Theorem 1.98].

Theorem 2.3 (Substitution)**.** *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\nu^{\Delta}(t)\,\Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\tilde{\Delta}s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+$ is given by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \log(1 + \mu(z)p(z))\Delta z\right), \tag{2.1}$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^\Delta = p(t)y,\ y(s) = 1$. Other properties of the exponential function are given in the following lemma, [7, Theorem 2.36].

Lemma 2.4. *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$,
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$,
(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$ *where,* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$,
(iv) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$,
(v) $e_p(t, s)e_p(s, r) = e_p(t, r)$,
(vi) $\left(\frac{1}{e_p(\cdot,s)}\right)^\Delta = -\frac{p(t)}{e_p^\sigma(\cdot,s)}$.

The notion of periodic time scales is introduced in Atici et al. [6] and Kaufmann and Raffoul [10]. The following two definitions are borrowed from [6] and [10].

Definition 2.5. We say that a time scale $\mathbb{T}$ is periodic if there exists a $p > 0$ such that if $t \in \mathbb{T}$ then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p with this property is called the period of the time scale.

Example 2.6. The following time scales are periodic.

(1) $\mathbb{T} = \bigcup_{i=-\infty}^{\infty}[2(i-1)h, 2ih],\ h > 0$ has period $p = 2h$.
(2) $\mathbb{T} = hZ$ has period $p = h$.
(3) $\mathbb{T} = \mathbb{R}$.
(4) $\mathbb{T} = \{t = k - q^m : k \in Z,\ m \in N_0\}$ where, $0 < q < 1$ has period $p = 1$.

Remark 2.7 ([10]). All periodic time scales are unbounded above and below.

Definition 2.8. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = np$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$.

If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

Remark 2.9 ([10]). If $\mathbb{T}$ is a periodic time scale with period p, then $\sigma(t \pm np) = \sigma(t) \pm np$. Consequently, the graininess function μ satisfies $\mu(t \pm np) = \sigma(t \pm np) - (t \pm np) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period p.

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of a periodic solution. For its proof we refer the reader to [11].

Theorem 2.10 (Krasnoselskii)**.** *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that A and B map $\mathbb{M}$ into $\mathbb{B}$ such that*

(i) $x, y \in \mathbb{M}$, implies $Ax + By \in \mathbb{M}$,
(ii) A is compact and continuous,
(iii) B is a contraction mapping.

Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$.

3. Existence of periodic solutions

We will state and prove our main result in this section as well as provide an example. Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} \neq \mathbb{R}$, $T = np$ for some $n \in \mathbb{N}$. By the notation $[a, b]$ we mean

$$[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$$

unless otherwise specified. The intervals $[a, b), (a, b]$, and (a, b) are defined similarly. Define $P_T = \{\varphi \in C(\mathbb{T}, \mathbb{R}) : \varphi(t + T) = \varphi(t)\}$ where, $C(\mathbb{T}, \mathbb{R})$ is the space of all real valued continuous functions on $\mathbb{T}$. Then P_T is a Banach space when it is endowed with the supremum norm

$$\|x\| = \sup_{t \in [0,T]} |x(t)|.$$

We will need the following lemma whose proof can be found in [10].

Lemma 3.1. *Let $x \in P_T$. Then $\|x^\sigma\|$ exists and $\|x^\sigma\| = \|x\|$.*

In this paper we assume that $a \in \mathcal{R}^+$ is continuous, $a(t) > 0$ for all $t \in \mathbb{T}$ and

$$a(t + T) = a(t),\ D(t + T, u + T) = D(t, u),\ (id - g)(t + T) = (id - g)(t), \tag{3.1}$$

where, id is the identity function on $\mathbb{T}$. We also require that $Q(t, x)$ and $f(x)$ are continuous and periodic in t and Lipschitz continuous in x. That is,

$$Q(t + T, x) = Q(t, x), \tag{3.2}$$

and there are positive constants E_1 and E_2 such that

$$|Q(t, x) - Q(t, y)| \leq E_1 \|x - y\|, \tag{3.3}$$

and

$$|f(x) - f(y)| \leq E_2 \|x - y\|. \tag{3.4}$$

Also, there is positive constant E_3 such that

$$\int_{-\infty}^{t} |D(t, u)|\, \Delta u \leq E_3. \tag{3.5}$$

Lemma 3.2. *Suppose (3.1), (3.2) hold. If $x \in P_T$, then x is a solution of equation (1.3) if and only if*

$$\begin{aligned} x(t) &= Q(t, x(t - g(t))) + (1 - e_{\ominus a}(t, t - T))^{-1} \\ &\times \int_{t-T}^{t} \left[-a(s) Q^\sigma(s, x(s - g(s))) + \int_{-\infty}^{s} D(s, u) f(x(u))\, \Delta u \right] e_{\ominus a}(t, s) \Delta s. \end{aligned} \tag{3.6}$$

Proof. Let $x \in P_T$ be a solution of (1.3). First we write (1.3) as

$$\{x(t) - Q(t, x(t-g(t)))\}^{\Delta} = -a(t)\{x^{\sigma}(t) - Q^{\sigma}(t, x(t-g(t)))\} - a(t)Q^{\sigma}(t, x(t-g(t))) + \int_{-\infty}^{t} D(t,u) f(x(u)) \Delta u.$$

Multiply both sides by $e_a(t,0)$ and then integrate from $t-T$ to t to obtain

$$\int_{t-T}^{t} [e_a(s,0)\{x(s) - Q(s, x(s-g(s)))\}]^{\Delta} \Delta s = \int_{t-T}^{t} \left[-a(s)Q^{\sigma}(s, x(s-g(s))) + \int_{-\infty}^{s} D(s,u) f(x(u)) \Delta u\right] e_a(s,0)\Delta s.$$

Consequently, we have

$$\begin{aligned} &e_a(t,0)\left(x(t) - a(t)Q(t, x(t-g(t)))\right) \\ &- e_a(t-T,0)\left(x(t-T) - a(t-T)Q(t-T, x(t-T-g(t-T)))\right) \\ &= \int_{t-T}^{t} \left[-a(s)Q^{\sigma}(s, x(s-g(s))) + \int_{-\infty}^{s} D(s,u) f(x(u)) \Delta u\right] e_a(s,0)\Delta s. \end{aligned}$$

After making use of (3.1), (3.2) and $x \in P_T$, we divide both sides of the above equation by $e_a(t,0)$ to obtain

$$\begin{aligned} x(t) &= Q(t, x(t-g(t))) + (1 - e_{\ominus a}(t, t-T))^{-1} \\ &\times \int_{t-T}^{t} \left[-a(s)Q^{\sigma}(s, x(s-g(s))) + \int_{-\infty}^{s} D(s,u) f(x(u)) \Delta u\right] e_{\ominus a}(t,s)\Delta s \end{aligned}$$

where, we have used Lemma 2.4 to simplify the exponentials. Since each step is reversible, the converse follows. This completes the proof. □

Define the mapping $H : P_T \to P_T$ by

$$\begin{aligned} &(H\varphi)(t) \\ &= Q(t, \varphi(t-g(t))) + (1 - e_{\ominus a}(t, t-T))^{-1} \\ &\times \int_{t-T}^{t} \left[-a(s)Q^{\sigma}(s, \varphi(s-g(s))) + \int_{-\infty}^{s} D(s,u) f(x(u)) \Delta u\right] e_{\ominus a}(t,s)\Delta s. \end{aligned} \tag{3.7}$$

To apply Theorem 2.10 we need to construct two mappings; one map is a contraction and the other map is compact. We express equation (3.7) as

$$(H\varphi)(t) = (B\varphi)(t) + (A\varphi)(t),$$

where, A and B are given by

$$(B\varphi)(t) = Q(t, \varphi(t-g(t))), \tag{3.8}$$

and

$$\begin{aligned} (A\varphi)(t) &= (1 - e_{\ominus a}(t, t-T))^{-1} \\ &\times \int_{t-T}^{t} \left[-a(s)Q^{\sigma}(s, \varphi(s-g(s))) + \int_{-\infty}^{s} D(s,u) f(\varphi(u)) \Delta u\right] e_{\ominus a}(t,s)\Delta s. \end{aligned} \tag{3.9}$$

Lemma 3.3. *Suppose (3.1)–(3.5) hold. Then $A : P_T \to P_T$, as defined by (3.9), is compact.*

Proof. We first show that $A : P_T \to P_T$. Evaluate (3.9) at $t+T$.

$$(A\varphi)(t+T) = (1 - e_{\ominus a}(t+T,t))^{-1} \int_t^{t+T} \Big[-a(s)Q^\sigma\left(s, \varphi(s-g(s))\right) + \int_{-\infty}^{s} D\left(s,u\right) f\left(\varphi\left(u\right)\right) \Delta u\Big] e_{\ominus a}(t+T,s)\Delta s. \tag{3.10}$$

Use Theorem 2.3 with $v = s - T$ and conditions (3.1)–(3.2) to get

$$\begin{aligned}(A\varphi)(t+T) &= (1 - e_{\ominus a}(t+T,t))^{-1} \\ &\quad\times \int_{t-T}^{t} \Big[-a(v+T)Q^\sigma\left(v+T, \varphi(v+T-g(v+T))\right) \\ &\quad+ \int_{-\infty}^{v+T} D\left(v+T,u\right) f\left(\varphi\left(u\right)\right) \Delta u\Big] e_{\ominus a}(t+T,v+T)\Delta v.\end{aligned}$$

From (2.1) and Theorem 2.3, we have $e_{\ominus a}(t+T,v+T) = e_{\ominus a}(t,v)$ and $e_{\ominus a}(t+T,t) = e_{\ominus a}(t,t-T)$. Thus (3.10) becomes

$$\begin{aligned}(A\varphi)(t+T) &= (1 - e_{\ominus a}(t,t-T))^{-1} \int_{t-T}^{t} \Big[-a(v)Q^\sigma\left(v, \varphi(v-g(v))\right) \\ &\quad+ \int_{-\infty}^{v} D\left(v,u\right) f\left(\varphi\left(u\right)\right) \Delta u\Big] e_{\ominus a}(t,v)\Delta v \\ &= (A\varphi)(t).\end{aligned}$$

That is, $A : P_T \to P_T$.

To see that A is continuous, we let $\varphi, \psi \in P_T$ with $\|\varphi\| \le C$ and $\|\psi\| \le C$ and define

$$\eta := \max_{t\in[0,T]} \left|(1 - e_{\ominus a}(t,t-T))^{-1}\right|, \ \rho := \max_{t\in[0,T]} |a(t)|, \ \gamma := \max_{u\in[t-T,t]} e_{\ominus a}(t,u). \tag{3.11}$$

Given $\varepsilon > 0$, take $\delta = \varepsilon/M$ with $M = \eta\gamma T(\rho E_1 + E_2E_3)$ where, E_1, E_2 and E_3 are given by (3.3)–(3.5) such that $\|\varphi - \psi\| < \delta$. Using (3.9) we get

$$\begin{aligned}\|A\varphi - A\psi\| &\le \eta\gamma \int_0^T \left[\rho E_1\|\varphi-\psi\| + E_2E_3\|\varphi-\psi\|\right] \Delta u \\ &\le M\|\varphi-\psi\| < \varepsilon.\end{aligned}$$

This proves that A is continuous.

We need to show that A is compact. Consider the sequence of periodic functions $\{\varphi_n\} \subset P_T$ and assume that the sequence is uniformly bounded. Let $R > 0$ be such that $\|\varphi_n\| \le R$, for all $n \in \mathbb{N}$. In view of (3.3) and (3.4) we arrive at

$$\begin{aligned}|Q(t,x)| &= |Q(t,x) - Q(t,0) + Q(t,0)| \\ &\le |Q(t,x) - Q(t,0)| + |Q(t,0)| \\ &\le E_1\|x\| + \alpha.\end{aligned}$$

Similarly,

$$\begin{aligned}|f(x)| &= |f(x) - f(0) + f(0)|\\ &\le |f(x) - f(0)| + |f(0)|\\ &\le E_2\|x\| + \beta,\end{aligned}$$

where, $\alpha = \sup_{t\in[0,T]} |Q(t,0)|$ and $\beta = |f(0)|$.

$$\begin{aligned}&|(A\varphi_n)(t)|\\ &= \Big|(1 - e_{\ominus a}(t, t-T))^{-1} \int_{t-T}^{t} \Big[- a(s)Q^\sigma(s, \varphi_n(s - g(s)))\\ &+ \int_{-\infty}^{s} D(s,u) f(\varphi_n(u)) \Delta u\Big] e_{\ominus a}(t,s)\Delta s\Big|\\ &\le \eta\gamma \int_{t-T}^{t} \Big[|a(s)| \, |Q^\sigma(s, \varphi_n(s - g(s)))| + \int_{-\infty}^{s} |D(s,u)| \, |f(\varphi_n(u))| \Delta u\Big] \Delta s\\ &\le \eta\gamma T \left[\rho(E_1 R + \alpha) + E_2 E_3 R + \beta\right] \equiv D.\end{aligned}$$

Thus the sequence $\{A\varphi_n\}$ is uniformly bounded. Now, it can be easily checked that

$$(A\varphi_n)^\Delta(t) = -a(t)(A\varphi_n)^\sigma(t) - a(t)Q^\sigma(t, \varphi(t - g(t))) + \int_{-\infty}^{t} D(t,u) f(\varphi_n(u)) \Delta u.$$

Consequently,

$$|(A\varphi_n)^\Delta(t)| \le D\rho + \rho(E_1 R + \alpha) + E_2 E_3 R + \beta,$$

for all n. That is $\left\|(A\varphi_n)^\Delta\right\| \le F$, for some positive constant F. Thus the sequence $\{A\varphi_n\}$ is uniformly bounded and equi-continuous. The Arzelà-Ascoli theorem implies that there is a subsequence $\{A\varphi_{n_k}\}$ which converges uniformly to a continuous T-periodic function φ^*. Thus A is compact. □

Lemma 3.4. *Let B be defined by (3.8) and*

$$E_1 \le \zeta < 1. \tag{3.12}$$

Then $B : P_T \to P_T$ is a contraction.

Proof. Trivially, $B : P_T \to P_T$. For $\varphi, \psi \in P_T$, we have

$$\begin{aligned}\|B\varphi - B\psi\| &= \sup_{t\in[0,T]} |B\varphi(t) - B\psi(t)|\\ &\le E_1 \sup_{t\in[0,T]} |\varphi(t - g(t)) - \psi(t - g(t))|\\ &\le \zeta\|\varphi - \psi\|.\end{aligned}$$

Hence B defines a contraction mapping with contraction constant ζ. □

Theorem 3.5. *Suppose the hypothesis of Lemma 3.2 holds. Suppose (3.1)–(3.5) hold. Let $\alpha = \sup_{t\in[0,T]} |Q(t,0)|$ and $\beta = |f(0)|$. Let J be a positive constant satisfying the inequality*

$$\alpha + E_1 J + \eta T\gamma \left[\rho(E_1 J + \alpha) + E_2 E_3 J + \beta\right] \le J. \tag{3.13}$$

Let $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq J\}$. Then (1.3) has a solution in M.

Proof. Define $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq J\}$. By Lemma 3.3, A is continuous and $A\mathbb{M}$ is contained in a compact set. Also, from Lemma 3.4, the mapping B is a contraction and it is clear that $B : P_T \to P_T$. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we have $\|A\varphi + B\psi\| \leq J$. Let $\varphi, \psi \in \mathbb{M}$ with $\|\varphi\|, \|\psi\| \leq J$. Then

$$\begin{aligned}
&\|A\varphi + B\psi\| \\
&\leq E_1\|\psi\| + \alpha + \eta\gamma \int_0^T \big[|a(u)|(\alpha + E_1\|\varphi\|) + E_2E_3\|\varphi\| + \beta\big]\Delta u \\
&\leq \alpha + E_1J + \eta T\gamma\left[\rho(E_1J + \alpha) + E_2E_3J + \beta\right] \leq J.
\end{aligned}$$

We now see that all the conditions of Krasnoselskii's theorem are satisfied. Thus there exists a fixed point z in $\mathbb{M}$ such that $z = Az + Bz$. By Lemma 3.2, this fixed point is a solution of (1.3). Hence (1.3) has a T-periodic solution. □

Theorem 3.6. *Suppose (3.1)–(3.5) hold. If*

$$E_1 + \eta\gamma T(\rho E_1 + E_2E_3) < 1,$$

then (1.3) has a unique T-periodic solution.

Proof. Let the mapping H be given by (3.7). For $\varphi, \psi \in P_T$, we have

$$\|H\varphi - H\psi\| \leq (E_1 + \eta\gamma T(\rho E_1 + E_2E_3))\,\|\varphi - \psi\|.$$

This completes the proof by invoking the contraction mapping principle. □

4. Stability

This section is mainly concerned with the asymptotic stability of the zero solution of (1.3). We assume that the functions Q and f are continuous, as before. Also, we assume that g is continuous and $g(t) \geq g^* > 0$, for all $t \in \mathbb{T}$ such that $t \geq t_0$ for some $t_0 \in \mathbb{T}$ and that $Q(t, 0) = f(0) = 0$ and Q and f obey the Lipschitz conditions (3.3) and (3.4). The techniques used in this section are adapted from the paper [9].

As before, we assume a time scale $\mathbb{T}$, that is unbounded above and below and that $0 \in \mathbb{T}$. Also, we assume that $g : \mathbb{T} \to \mathbb{R}$ and that $id - g : \mathbb{T} \to \mathbb{T}$ is strictly increasing.

To arrive at the correct mapping, we rewrite (1.3) as in the proof of Lemma 3.2, multiply both sides by $e_a(t, 0)$ and then integrate from 0 to t to obtain

$$\begin{aligned}
x(t) &= Q(t, x(t - g(t))) + \left[x(0) - Q\left(0, x(-g(0))\right)\right] e_{\ominus a}(t, 0) \\
&\quad + \int_0^t \left[-a(s)Q^\sigma\left(s, x(s - g(s))\right) + \int_{-\infty}^s D\left(s, u\right) f\left(\varphi\left(u\right)\right)\Delta u\right] e_{\ominus a}(t, s)\Delta s. \qquad (4.1)
\end{aligned}$$

Thus, we see that x is a solution of (1.3) if and only if it satisfies (4.1).

Let $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$ be a given Δ-differentiable bounded initial function. We say $x(t) := x(t, 0, \psi)$ is a solution of (1.3) if $x(t) = \psi(t)$ for $t \leq 0$ and satisfies (1.3) for $t \geq 0$. We say the zero solution of (1.3) is stable at t_0 if for each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $[\psi : (-\infty, t_0]_{\mathbb{T}} \to \mathbb{R}$ with $\|\psi\| < \delta]$ implies $|x(t, t_0, \psi)| < \varepsilon$.

Let $C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$ be the space of all rd-continuous functions from $\mathbb{T} \to \mathbb{R}$ and define the set S by

$$S = \{\varphi \in C_{rd} : \varphi(t) = \psi(t) \text{ if } t \leq 0, \varphi(t) \to 0 \text{ as } t \to \infty, \text{ and } \varphi \text{ is bounded}\}.$$

Then, $(S, \|\cdot\|)$ is a complete metric space where, $\|\cdot\|$ is the supremum norm. For the next theorem we impose the following conditions.

$$e_{\ominus a}(t, 0) \to 0, \text{ as } t \to \infty, \tag{4.2}$$

there is an $\alpha > 0$ such that

$$E_1 + \int_0^t \left[\,|a(s)|\, E_1 + E_2 E_3\right] e_{\ominus a}(t, s)\Delta s \leq \alpha < 1, \ t \geq 0, \tag{4.3}$$

$$t - g(t) \to \infty, \text{ as } t \to \infty, \tag{4.4}$$

Theorem 4.1. *If (3.3)–(3.5) and (4.2)–(4.4) hold, then every solution $x(t, 0, \psi)$ of (1.3) with small continuous initial function ψ, is bounded and goes to zero as $t \to \infty$. Moreover, the zero solution is stable at $t_0 = 0$.*

Proof. Define the mapping $P : S \to S$ by

$$(P\varphi)(t) = \psi(t) \text{ if } t \leq 0,$$

and

$$\begin{aligned}(P\varphi)(t) &= Q(t, \varphi(t - g(t))) + (\psi(0) - Q(0, \psi(-g(0))))\, e_{\ominus a}(t, 0) \\ &+ \int_0^t \left[-a(s)Q^\sigma(s, \varphi(s - g(s))) + \int_{-\infty}^s D(s, u)\, f(\varphi(u))\, \Delta u\right] e_{\ominus a}(t, s)\Delta s,\end{aligned}$$

if $t \geq 0$. It is clear that for $\varphi \in S$, $P\varphi$ is continuous. Let $\varphi \in S$ with $\|\varphi\| \leq K$, for some positive constant K. Let ψ be a small given continuous initial function with $\|\psi\| < \delta$, $\delta > 0$. Then,

$$\begin{aligned}\|P\varphi\| &\leq E_1 K + |\psi(0) - Q(0, \psi(-g(0)))|\, e_{\ominus a}(t, 0) \\ &\quad + \int_0^t \left[|a(u)|E_1 + E_2E_3\right] e_{\ominus a}(t, s)\Delta s K \\ &\leq (1 + E_1)\,\delta + E_1 K + \int_0^t \left[|a(s)|E_1 + E_2E_3\right] e_{\ominus a}(t, s)\Delta u K \\ &\leq (1 + E_1)\,\delta + \alpha K, \end{aligned} \tag{4.5}$$

which implies that, $\|P\varphi\| \leq K$, for the right δ. Thus, (4.5) implies that $(P\varphi)(t)$ is bounded. Next, we show that $(P\varphi)(t) \to 0$ as $t \to \infty$. The second term on the right side of $(P\varphi)(t)$ tends to zero, by condition (4.2). Also, the first term on the right side tends to zero, because of (4.4) and the fact that $\varphi \in S$. It is left to show that the integral term goes to zero as $t \to \infty$.

Let $\varepsilon > 0$ be given and $\varphi \in S$ with $\|\varphi\| \leq K$, $K > 0$. Then, there exists a $t_1 > 0$ so that for $t > t_1$, $|\varphi(t - g(t))| < \varepsilon$. Due to condition (4.2), there exists a $t_2 > t_1$ such that $t > t_2$ implies that $e_{\ominus a}(t, t_1) < \frac{\varepsilon}{\alpha K}$.

Thus for $t > t_2$, we have

$$\begin{aligned}
&\left| \int_0^t \left[-a(s)Q^\sigma \left(s, \varphi(s-g(s))\right) + \int_{-\infty}^s D\left(s,u\right) f\left(\varphi\left(u\right)\right) \Delta u \right] e_{\ominus a}(t,u)\Delta u \right| \\
&\leq K \int_0^{t_1} \big[|a(s)| E_1 + E_2 E_3 \big] e_{\ominus a}(t,s)\Delta s \\
&+ \varepsilon \int_{t_1}^t \big[|a(s)| E_1 + E_2 E_3 \big] e_{\ominus a}(t,u)\Delta s \\
&\leq K e_{\ominus a}(t,t_1) \int_0^{t_1} \big[|a(s)| E_1 + E_2 E_3 \big] e_{\ominus a}(t_1,s)\Delta s + \alpha\varepsilon \\
&\leq \alpha K e_{\ominus a}(t,t_1) + \alpha \varepsilon \\
&\leq \varepsilon + \alpha\varepsilon.
\end{aligned}$$

Hence, $\left(P\varphi\right)(t) \to 0$ as $t \to \infty$. It remains to show that P is a contraction under the supremum norm. Let $\zeta, \eta \in S$. Then

$$\begin{aligned}
|(P\zeta)(t) - (P\eta)(t)| &\leq \left\{ E_1 + \int_0^t \big[|a(s)| E_1 + E_2 E_3 \big] e_{\ominus a}(t,s)\Delta s \right\} \|\zeta - \eta\| \\
&\leq \alpha \|\zeta - \eta\|.
\end{aligned}$$

Thus, by the contraction mapping principle, P has a unique fixed point in S which solves (1.3), is bounded and tends to zero as t tends to infinity. The stability of the zero solution at $t_0 = 0$ follows from the above work by simply replacing K by ε. This completes the proof. □

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