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CLOSED GRAPH THEOREMS FOR BORNOLOGICAL SPACES

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Abstract. The aim of this paper is that of discussing closed graph theorems for bornological vector spaces in a self-contained way, hoping to make the subject more accessible to non-experts. We will see how to easily adapt classical arguments of functional analysis over $\mathbb{R}$ and $\mathbb{C}$ to deduce closed graph theorems for bornological vector spaces over any complete, non-trivially valued field, hence encompassing the non-Archimedean case too. We will end this survey by discussing some applications. In particular, we will prove De Wilde's Theorem for non-Archimedean locally convex spaces and then deduce some results about the automatic boundedness of algebra morphisms for a class of bornological algebras of interest in analytic geometry, both Archimedean (complex analytic geometry) and non-Archimedean.

Introduction

This paper aims to discuss the closed graph theorems for bornological vector spaces in a self-contained exposition and to fill a gap in the literature about the non-Archimedean side of the theory at the same time. In functional analysis over $\mathbb{R}$ or $\mathbb{C}$ bornological vector spaces have been used since a long time ago, without becoming a mainstream tool. It is probably for this reason that bornological vector spaces over non-Archimedean valued fields were rarely considered. Over the last years, for several reasons, bornological vector spaces have drawn new attentions: see for example [1], [2], [3], [5], [15] and [22]. These new developments involve the non-Archimedean side of the theory too and that is why it needs adequate foundations. Among the classical books on the theory of bornological vector spaces, the only one considering also non-Archimedean base fields, in a unified fashion with the Archimedean case, is [10]. But that book cannot cover all the theory, and in particular, it lacks a unified treatment of the closed graph

theorems. This work comes out from the author's need for a reference for these theorems and also in the hope that in the future bornological vector spaces will gain more popularity and that this work may be useful for others.

The closed graph theorem for Banach spaces over $\mathbb{R}$ and $\mathbb{C}$ is one of the most celebrated classical theorems of functional analysis. Over the years, it has been generalized in many ways, for example to Fréchet and LF spaces as a consequence of the Open Mapping Theorems. Although it is a classical argument that the closed graph theorem can be deduced from the Open Mapping Theorem, people have understood that the closed graph theorem can be proved in an independent way, with argumentations that also work when the Open Mapping Theorem fails. The two most famous examples f this kind of proofs are given in [19] by Popa and [7] by De Wilde. In particular, De Wilde's Theorem is probably the most general closed graph theorem for locally convex spaces, and states the following:

Theorem 0.1. *(De Wilde's closed graph theorem)*
If E is an ultrabornological locally convex space and F is a webbed locally convex space over $\mathbb{R}$ or $\mathbb{C}$, then every linear map $f : E \to F$ which has bornologically closed graph with respect to the convex bornologies on E and F that are generated by all bounded Banach disks in E and in F, respectively, is continuous even if regarded as a mapping into the ultrabornologification F_{uborn} of F.

The terminology and the notation of the theorem will be explained in the course of this work. It is interesting to notice that, although De Wilde's theorem is a theorem for locally convex spaces we are naturally led to talk about bornologies and bounded maps. Popa's theorem, on the contrary, is a purely bornological statement which is the Archimedean case of our Theorem 2.7.

The content of the paper is the following: In the first section we give an overview of the basics of theory of bornological vector spaces. In particular, since we adopt the unusual attitude of discussing the Archimedean and non-Archimedean case of the theory at the same time, we spend some time in recalling basic definitions and discuss in details the notions from the theory of bornological vector spaces that will be used. In the second section we introduce the notion of bornological nets and then we give the main examples of bornological vector spaces endowed with nets. We then deduce our first closed graph theorem, which is the unified version of Popa's theorem (cf. [19]), stated as follows:

Theorem 0.2. *Let E and F be separated convex bornological vector spaces, where E is complete and F has a net compatible with its bornology. Then, every linear map $f : E \to F$ with bornologically closed graph is bounded.*

In the remaining part of the second section the notion of bornological net is generalized by the notion of bornological web and the closed graph theorem for webbed bornological vector spaces is proved quite easily as a consequence of the previous discussion. This latter theorem is the direct generalization, for all base fields, of the closed graph theorem proved by Gach in [8, Theorem 4.3]. In the last section we discuss some applications of the theorems we proved. We will see how one can deduce isomorphism theorems from closed graph theorems and following [8] we will see how De Wilde's closed graph theorem can be deduced.

We would like to remark that for non-Archimedean base fields we need to add some restrictions, that do not affect the Archimedean side of the theory. Our generalization of De Wilde's Theorem is the following:

Theorem 0.3. *If E is an ultrabornological locally convex space and F is a webbed locally convex space defined over a spherically complete field K, then every linear map $f : E \to F$ which has bornologically closed graph with respect to the convex bornologies on E and F that are generated by all bounded Banach disks in E and in F, respectively, is continuous even if regarded as a mapping into the ultrabornologification F_{uborn} of F.*

Therefore, in order to deduce De Wilde's theorem for non-Archimedean base fields, we need to suppose that the base field K is spherically complete, a condition which is always satisfied when K is Archimedean. We remark that this hypothesis on K is only used in Lemma 3.19 and in Lemma 3.20 in order to be able to adapt the methods of [8]; one might ask if it is possible to prove those lemmata without these restrictions. We do not address this problem in this work. Finally, in the last part of the paper we show how to use the closed graph theorems for bornological spaces to deduce that all algebra morphisms between dagger affinoid algebras, as defined in [1], are bounded. This application, and others coming in [3] and planned in future works, are our main motivations for the study presented in this article.

1. Bornological spaces and closed graph theorems

1.1. **Closed graph.** Let $u : E \to F$ be a map of sets, then the set

$$\Gamma(u) = \{(x, y) \in E \times F \mid y = u(x)\}$$

is called the graph of u. If E and F are Hausdorff topological spaces and u a continuous map, then $\Gamma(u)$ is a closed subspace of $E \times F$ endowed with the product topology. This is a basic property of Hausdorff topological spaces. If E and F are vector spaces over a field K and u is a linear map, then $\Gamma(u)$ is a vector subspace of $E \times F$. If E and F are separated bornological vector spaces over a complete, non-trivially valued field K and u is linear and bounded, then $\Gamma(u)$ is bornologically closed in $E \times F$ endowed with the product bornology (see below for what it means for a subset of a bornological vector space to be bornologically closed). This assertion is pretty easy to check as follows. Let $\big(x_n, u(x_n)\big)$ be a sequence of elements of $\Gamma(u)$ which converges bornologically to (x, y) in $E \times F$. Then, by definition of product bornology, $x_n \to x$ in E and $u(x_n) \to y$ in Y. Since u is bounded, the sequence $(u(x_n))$ converges bornologically to $u(x)$ and since Y is separated, we must have $y = u(x)$. Therefore $(x, y) \in \Gamma(u)$ and $\Gamma(u)$ is bornologically closed in $E \times F$.

The closed graph theorems are converses of the above statements for some special classes of bornological or topological K-vector spaces. Here we pursue the main ideas of [8] for which the bornological closed graph theorems are of a more fundamental nature extending it to encompass non-Archimedean base fields. The statements of our closed graph theorems assert that if $u : E \to F$ is a linear map which has bornologically closed graph then it is bounded when E and F

belong to some particular classes of bornological vector spaces: We will prove it when E is a complete bornological vector space and F a separated bornological vector space endowed with a net and then when F is endowed with a web. Both proofs are adaptations of results from [11] and [8] on bornological vector spaces over Archimedean base fields to any non-trivially valued, complete base field K, Archimedean or not.

1.2. **Bornologies.** Bornological vector spaces are well-studied objects of functional analysis over $\mathbb{R}$ and $\mathbb{C}$. They are not mainstream, as the theory of topological or locally convex vector spaces, but they are often useful in addressing problems for locally convex spaces and they have found a good amount of applications. Thus, during the years, a good amount of work has been done to study the properties of bornological vector spaces and algebras over $\mathbb{R}$ and $\mathbb{C}$: for example [11], [12], [13], [20], [15] discuss various aspects and applications of the theory. On the other hand, for non-Archimedean base fields the theory has never got much attention. The works which deal with bornological vector spaces over non-Archimedean base fields usually date back to many years ago and they often discuss the non-Archimedean base field case as a mere example, interesting for working out general theories and general theorems, pursuing a "Bourbaki" study of the subject; however, this side of the theory was seldom thought to have applications to "real" mathematics. An example of this kind of work is [10]. In recent years a renewed interest in the theory of bornological vector spaces may challenge this attitude: for example, on the Archimedean side of the theory, one has the work [15], [16], [17] showing the usefulness of bornological vector spaces in non-commutative geometry and representation theory and [1], [5], [2] where also the theory of non-Archimedean bornological vector spaces and algebras is used in analytic geometry and higher order local field theory. We will recall the basis of the theory of bornological vector spaces as far as needed for our scopes.

Definition 1.1. Let X be a set. A *bornology* on X is a collection $\mathcal{B}$ of subsets of X such that

(1) $\mathcal{B}$ is a covering of X, *i.e.* $\forall x \in X, \exists B \in \mathcal{B}$ such that $x \in \mathcal{B}$;
(2) $\mathcal{B}$ is stable under inclusions, *i.e.* $A \subset B \in \mathcal{B} \Rightarrow A \in \mathcal{B}$;
(3) $\mathcal{B}$ is stable under finite unions, *i.e.* for each $n \in \mathbb{N}$ and $B_1, ..., B_n \in \mathcal{B}$, $\bigcup\limits_{i=1}^{n} B_i \in \mathcal{B}$.

A pair $(X, \mathcal{B})$ is called a *bornological set*, and the elements of $\mathcal{B}$ are called *bounded subsets* of X (with respect to $\mathcal{B}$, if it is needed to specify). A family of subsets $\mathcal{A} \subset \mathcal{B}$ is called a *basis* for $\mathcal{B}$ if for any $B \in \mathcal{B}$ there exist $A_1, \ldots, A_n \in \mathcal{A}$ such that $B \subset A_1 \cup \cdots \cup A_n$. A *morphism* of bornological sets $\varphi : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is defined to be a bounded map $\varphi : X \to Y$, *i.e.* a map of sets such that $\varphi(B) \in \mathcal{B}_Y$ for all $B \in \mathcal{B}_X$.

It is natural to develop the theory of bornological algebraic structures in a similar fashion to what has been done for topological algebraic structures. Indeed,

it is enough to start with a bornological set and to ask that the structural morphisms are bounded maps. We refer to the first chapter of [1] for basic properties of bornological groups, rings, fields and modules.

From now on we fix a complete, non-trivially valued field K. We use the notations $K^\circ = \{x \in K | \ |x| \leq 1\}$, $K^\times = \{x \in K | x \neq 0\}$ and $|K^\times| = \{r \in \mathbb{R}_+ | \ \exists x \in K^\times, |x| = r\}$. Clearly K has a natural structure of bornological field given by subsets which are bounded with respect to its valuation.

Definition 1.2. A *bornological vector space* over K is a K-vector space E along with a bornology on the underlying set of E for which the maps $(\lambda, x) \mapsto \lambda x$ and $(x, y) \mapsto x + y$ are bounded.

We will only work with bornological vector spaces whose bounded subsets can be described using convex subsets, in the sense explained in the next definition, although many of the results of this paper can be proved without this assumption.

Definition 1.3. Let E be a K-vector space. A subset $B \subset E$ is called *balanced* if for every $\lambda \in K^\circ$ one has that $\lambda B \subset B$. A subset $B \subset E$ is called *absolutely convex* (or *disk*) if

(1) for K Archimedean, it is convex and balanced; where convex means that for every $x, y \in B$ and $t \in [0, 1]$ then $(1 - t)x + ty \in B$;

(2) for K non-Archimedean, it is a K°-submodule of E.

The definition of absolutely convex subsets of E is posed in two different ways, depending on whether K is Archimedean or non-Archimedean, although the formal properties are essentially the same in both cases. Moreover, using the theory of generalized rings (in the sense of Durov [6]) one can put the two situations on equal footing, but we are not interested in this issue in this work.

Definition 1.4. A bornological K-vector space is said to be *of convex type* if its bornology has a basis made of absolutely convex subsets.

We denote by $\mathbf{Born}_K$ the category of bornological vector spaces of convex type over K. Then, every semi-normed space over K can be endowed with the bornology induced by the bounded subsets with respect to the semi-norm and is manifestly of convex type. This association permits to see the category of semi-normed spaces over K as a full subcategory of $\mathbf{Born}_K$.

Remark 1.5. For every bornological vector space of convex type E there is an isomorphism

$$E \cong \varinjlim_{B \in \mathcal{B}_E} E_B$$

where $\mathcal{B}_E$ denotes the family of bounded absolutely convex subsets of E and E_B is the vector subspace of E spanned by the elements of B equipped with the *gauge semi-norm* (also called *Minkowski functional*) defined by B (cf. [10] Chaper 2, §2, $n^\circ 2$). Notice that all morphisms of this inductive system are monomorphisms.

Definition 1.6. A bornological vector space over K is said to be *separated* if its only bounded vector subspace is the trivial subspace $\{0\}$.

Remark 1.7. A bornological vector space of convex type over K is separated if and only if for each $B \in \mathcal{B}_E$, the gauge semi-norm on E_B is actually a norm.

Definition 1.8. A disk $B \subset E$ is said to be a *Banach disk* if E_B is a Banach space.

Definition 1.9. A bornological space E over K is said to be *complete* if

$$E \cong \varinjlim_{i \in I} E_i$$

for a filtered colimit of Banach spaces over K for which the system morphisms are all injective.

It can be shown that the definition of complete bornological vector space just given is equivalent to the request that the family of bounded disks $\mathcal{B}_E$ of E admits a final subfamily made of Banach disks, cf. [10, Proposition 7, page 96] .

1.3. **Bornological convergence and bornologically closed subsets.** The data of a bornology is enough for endowing a vector space with a notion of convergence.

Definition 1.10. Let E be a bornological K-vector space and $\{x_n\}_{n \in \mathbb{N}}$ a sequence of elements of E. We say that $\{x_n\}_{n \in \mathbb{N}}$ *converges (bornologically) to* 0 *in the sense of Mackey* if there exists a bounded subset $B \subset E$ such that for every $\lambda \in K^\times$ there exists an $n = n(\lambda)$ for which

$$x_m \in \lambda B, \forall m > n.$$

We say that $\{x_n\}_{n \in \mathbb{N}}$ converges (bornologically) to $a \in E$ if the sequence $\{x_n - a\}_{n \in \mathbb{N}}$ converges (bornologically) to zero.

An analogous definition can be given for the convergence of filters of subsets of E. We omit the details of this definition since it is not important for our scope.

Remark 1.11. The notion of bornological convergence on a bornological vector space of convex type $E \cong \varinjlim_{B \in \mathcal{B}_E} E_B$ can be restated in the following way: $\{x_n\}_{n \in \mathbb{N}}$ is convergent to zero in the sense of Mackey if and only if there exists a $B \in \mathcal{B}_E$ and $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in E_B$ and $x_n \to 0$ in E_B for the semi-norm of E_B.

Definition 1.12. Let E be a bornological vector space over K;

- a sequence $\{x_n\}_{n \in \mathbb{N}} \subset E$ is called *Cauchy-Mackey* if the double sequence $\{x_n - x_m\}_{n,m \in \mathbb{N}}$ converges to zero;
- a subset $U \subset E$ is called *(bornologically) closed* if every Mackey convergent sequence of elements of U converges (bornologically) to an element of U.

Definition 1.13. A bornological vector space is called *semi-complete* if every Cauchy-Mackey sequence is convergent.

The notion of semi-completeness is not as useful as the notion of completeness in the theory of topological vector spaces. We remark that any complete bornological vector space is semi-complete, but the converse is false.

It can be shown that the notion of bornologically closed subset induces a topology on E, but this topology is neither a vector space topology nor a group topology in general. Therefore, an arbitrary intersection of bornological closed subsets of a bornological vector space is bornologically closed. Hence, the following definition is well-posed.

Definition 1.14. Let $U \subset E$ be a subset of a bornological vector space. The closure of U is the smallest bornologically closed subset of E in which U is contained. We denote the closure of U by $\overline{U}$.

The concept of bornologically closed subspace fits nicely in the theory. For example a bornological vector space is separated, in the sense of Definition 1.6, if and only if $\{0\}$ is a bornologically closed subset. We have to warn the reader that the closure of a subset $X \subset E$ of a bornological subset is not always equal to the limit points of the convergent sequences of elements of X but, in general, strictly contains it.

1.4. **Duality between bornologies and topologies.** Let $\mathbf{Loc}_K$ denote the category of locally convex vector spaces over K. We recall the definitions of two functors from [10], $^t : \mathbf{Born}_K \to \mathbf{Loc}_K$ and $^b : \mathbf{Loc}_K \to \mathbf{Born}_K$. To a bornological vector space of convex type E we associate the topological vector space E^t in the following way: we equip the underlying vector space of E with a topology for which a basis of neighborhoods of 0 is given by *bornivorous subsets*, *i.e.* subsets that absorb all bounded subsets of E. Recall that a subset $B \subset E$ is said to be absorbed by $U \subset E$ if there exists a $\lambda \in K^\times$ such that $B \subset \lambda U$. The association $E \mapsto E^t$ is well-defined and functorial. Then, if E is a locally convex space, E^b is defined to be the bornological vector space obtained by equipping the underlying vector space of E with the *von Neumann* (also called canonical) bornology, whose bounded subsets are the subsets of E absorbed by all neighborhood of $0 \in E$. Also, this association is well-defined and functorial. In chapter 1 of [10] one can find many properties of these constructions. The main one is that the couple $((\cdot)^t, (\cdot)^b)$ forms an adjoint pair of functors.

We conclude this review of the theory by recalling some relations between bornological and topological vector spaces that we will use later in our proofs.

Proposition 1.15. *Let E be a locally convex space and consider the bornological vector space E^b. Then, if a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to 0 in the sense of Mackey in E^b it converges topologically in E.*

Proof. Let $B \subset E$ be a von Neumann bounded subset such that the condition of Mackey convergence to 0 for $\{x_n\}_{n\in\mathbb{N}}$ is satisfied. Given any neighborhood of zero $U \subset E$, then there must exist a $\lambda \in K^\times$ such that

$$\lambda B \subset U$$

and by Definition 1.10 there exists an $n = n(\lambda)$ such that

$$x_i \in \lambda B \subset U, \quad \forall i > n,$$

hence the sequence converges topologically.   □

Definition 1.16. Let E be a bornological vector space of convex type. We say that E is metrizable if $E \cong F^b$ for a metrizable locally convex space.

The following result is a well-known statement of the theory of bornological vector spaces over $\mathbb{R}$ or $\mathbb{C}$, see for example Proposition (3) of Section 1.4.3 of [11]. But as far as we know the non-Archimedean version of the result is hard to find in literature. The last lines of page 108 of [10], essentially affirm, without proof, the statement of next proposition. Here we offer a detailed proof.

Proposition 1.17. *Let E be metrizable bornological vector space of convex type. Then, a sequence $\{x_n\}_{n \in \mathbb{N}}$ converges bornologically in E if and only if it converges topologically in E^t.*

Proof. Thanks to Proposition 1.15 we only need to check that topological convergence implies bornological convergence. So, let $\{V_n\}_{n \in \mathbb{N}}$ be a countable base of absolutely convex neighbourhoods of 0 in E such that $V_{n+1} \subset V_n$ for every $n \in \mathbb{N}$. Let $A = \{x_n\}_{n \in \mathbb{N}}$ be a sequence in E which converges to 0 topologically.

Since the sequence A converges topologically to zero, then it is absorbed by every neighbourhood of zero. Therefore, for every $n \in \mathbb{N}$, there exists an $\alpha_n \in |K^\times|$ such that

$$A \subset \lambda_n V_n$$

for $\lambda_n \in K^\times$ with $|\lambda_n| = \alpha_n$. It follows that

$$A \subset \bigcap_{n=1}^{\infty} \lambda_n V_n.$$

Let $\{\beta_n\}_{n \in \mathbb{N}}$ be a sequence of strictly positive real numbers such that $\beta_n \in |K^\times|$ and $\beta_n \to 0$ as $n \to \infty$. Let $\gamma_n = \frac{\alpha_n}{\beta_n}$. Define

$$B = \bigcap_{n=1}^{\infty} \mu_n V_n$$

with $\mu_n \in K^\times$ and $|\mu_n| = \gamma_n$. B is clearly a bounded subset of E, because it is absorbed by all V_n. We are going to prove the following assertion: For every $\epsilon \in |K^\times|$ there is an integer $m \in \mathbb{N}$ for which $A \cap V_m \subset \lambda B$, for $\lambda \in K^\times$ with $|\lambda| = \epsilon$, from which the proposition will be then proved.

Since the sequence $\frac{\gamma_n}{\alpha_n} = \frac{1}{\beta_n}$ tends to ∞ for $n \to \infty$, there is an integer $p \in \mathbb{N}$ such that

$$\forall n > p, \ \frac{\gamma_n}{\alpha_n} > \frac{1}{\epsilon}.$$

Therefore

$$A \subset \bigcap_{n=1}^{\infty} \lambda_n V_n \Rightarrow A \subset \rho_n V_n,$$

for $\rho_n = \lambda \mu_n \in K^\times$ with $|\lambda| = \epsilon$ and $n > p$. But the set

$$\bigcap_{n=1}^{p} \rho_n V_n,$$

with again $\rho_n = \lambda\mu_n \in K^\times$ and $|\lambda| = \epsilon$, is a neighbourhood of 0. Hence, there exists an integer $m \in \mathbb{N}$ such that

$$V_m \subset \bigcap_{n=1}^{p} \rho_n V_n,$$

therefore

$$A \cap V_m \subset \bigcap_{n=1}^{\infty} \rho_n V_n = \bigcap_{n=1}^{\infty} \lambda\mu_n V_n = \lambda B,$$

proving the claim. $\square$

The last proposition has the following interesting consequence.

Corollary 1.18. *Let E be a metrizable locally convex space. Then E is complete topologically if and only if E^b is complete bornologically if and only if E^b is semi-complete.*

Proof. That the completeness of E implies the completeness of E^b is proved in of [10, Proposition 15], and as we have already remarked the completeness of E^b implies its semi-completeness. Then, Proposition 1.17 directly implies that the semi-completeness of E^b implies the completeness of E, proving the corollary. $\square$

2. THE CLOSED GRAPH THEOREM FOR BORNOLOGICAL SPACES

2.1. **The closed graph theorem for bornological spaces with nets.** In this section we prove the closed graph theorem for bornological vector spaces endowed with a net. Essentially, we adapt the proof of Popa's Theorem that one can find in [11] and we make it work over any complete, non-trivially valued field K.

Definition 2.1. Let F be a K-vector space. A *net* on F is a map $\mathcal{N} : \bigcup_{k\in\mathbb{N}} \mathbb{N}^k \to \mathscr{P}(F)$ such that

    (1) each $\mathcal{N}(n_1, \ldots, n_k)$ is a disk;
    (2) $\mathcal{N}(\varnothing) = F$;
    (3) for every finite sequence $(n_0, \ldots, n_k)$ one has

$$\mathcal{N}(n_0, \ldots, n_k) = \bigcup_{n\in\mathbb{N}} \mathcal{N}(n_0, \ldots, n_k, n).$$

Notice that condition (2) of the previous definition is used to give meaning to the formula

$$F = \mathcal{N}(\varnothing) = \bigcup_{n\in\mathbb{N}} \mathcal{N}(n).$$

If $s : \mathbb{N} \to \mathbb{N}$ is a sequence we use the notation

$$\mathcal{N}_{s,k} = \mathcal{N}(s(0), \ldots, s(k)).$$

Definition 2.2. Let F be a separated bornological K-vector space of convex type. We say that a net $\mathcal{N}$ on F is *compatible* with its bornology if

(1) for every sequence $s : \mathbb{N} \to \mathbb{N}$ there is a sequence of positive real numbers $b(s) : \mathbb{N} \to \mathbb{R}_{>0}$ such that for all $x_k \in \mathcal{N}_{s,k}$ and $a_k \in K$ with $|a_k| \leq b(s)_k$ the series

$$\sum_{k \in \mathbb{N}} a_k x_k$$

converges bornologically in F and $\sum_{k \geq n} a_k x_k \in \mathcal{N}_{s,n}$ for every $n \in \mathbb{N}$.

(2) For every sequences $\{\lambda_k\}_{k \in \mathbb{N}}$ of elements of K and $s : \mathbb{N} \to \mathbb{N}$ the subsets

$$\bigcap_{k \in \mathbb{N}} \lambda_k \mathcal{N}_{s,k}$$

are bounded in F.

We say that a separated bornological vector space *has a net* if there exists a net which is compatible with its bornology.

The bornological vector spaces most commonly used in functional analysis have nets.

Example 2.3. (1) Let F be a bornological Fréchet space. By this we mean that $F \cong E^b$ for a Fréchet space E. Then, F has a net compatible with its bornology. To show this, consider a base of neighborhoods $\{V_n\}_{n \in \mathbb{N}}$ of $0 \in F$. If the base field K is Archimedean it is well-known that one can define a net in the following way. For any k-tuple of integers $(n_1, \ldots, n_k)$ define

$$\mathcal{N}(n_1, \ldots, n_k) = n_1 V_{n_1} \cap \ldots \cap n_k V_{n_k}.$$

See [11, Section 4.4.4] for a detailed proof of this fact. Notice that one has to ensure that

$$F = \bigcup_{k \in \mathbb{N}} n_k V_{n_k}$$

which can always be obtained by choosing as $\{V_n\}_{n \in \mathbb{N}}$ a family of disks whose radii goes to zero sufficiently slow.

This construction does not work for non-Archimedean base fields because $|n_k| \leq 1$ and so condition (3) of Definition 2.1 is not satisfied. To fix it, pick an element $\alpha \in K$ such that $|\alpha| > 1$. This is always possible since K is non-trivially valued. Then, we define

$$\mathcal{N}(n_1, \ldots, n_k) = \alpha^{n_1} V_{n_1} \cap \ldots \cap \alpha^{n_k} V_{n_k},$$

and we check that this is a net compatible with the bornology of F. Since every neighborhood of 0 is absorbent then the condition of $\mathcal{N}$ to be a net is clearly satisfied when the radii of V_n are chosen to go to 0 slow enough. Let $\{n_k\}_{k \in \mathbb{N}}$ be a sequence of integers, define $v_k = |\alpha^{-2^k n_k}|$ and choose $\lambda_k \in K$ with $|\lambda_k| \leq v_k$. Then, the series

$$\sum_{k=0}^{\infty} \lambda_k x_k$$

converges bornologically in F for every $x_k \in \mathcal{N}(n_1, \ldots, n_k)$ because it converges for the metric of F, and by Proposition 1.17 the convergence

for the metric of F is equivalent to the bornological convergence in F. Moreover, for every k_0

$$\sum_{k=k_0}^{\infty} \lambda_k x_k \in \lambda_{k_0} \mathcal{N}(n_1, \ldots, n_k) \subset \mathcal{N}(n_1, \ldots, n_k),$$

because $|\lambda_k| < 1$ and for non-Archimedean base fields disks are additive sub-groups of F. Then, we need to check the second condition on compatibility of the net with the bornology. Given any sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ of elements of K, we can consider the set

$$A = \bigcap_{k=0}^{\infty} \lambda_k \mathcal{N}(n_1, \ldots, n_k) = \bigcap_{k=0}^{\infty} \lambda_k \alpha^{n_1} V_{n_1} \cap \ldots \cap \alpha^{n_k} V_{n_k}.$$

Thus, given any V_{n_r}, we have that

$$A \subset \lambda_r \alpha^{n_r} V_{n_r},$$

hence A is absorbed by any neighborhood of $0 \in F$, and hence it is bounded for the bornology of F.

(2) The previous example applies in the particular case of Banach spaces and therefore also to the base field K itself, regarded as a Banach space over K.

(3) Let $F = \varinjlim_{n \in \mathbb{N}} F_n$ be a monomorphic inductive limit such that F_n are separated bornological vector spaces which have nets compatible with the bornology, then we claim that F is a separated bornological vector space which has a compatible net. To show this, first we note that since the inductive limit is monomorphic

$$F = \bigcup_{n \in \mathbb{N}} F_n$$

and F is separated when it is equipped with the direct limit bornology, see [10, Proposition 6, page 49] for a proof of this fact. Then, let $\mathcal{N}_n$ be a net for F_n. We define

$$\mathcal{E}(n_1) = \mathcal{N}_1(n_1), \quad \mathcal{E}(n_1, \ldots, n_k) = \mathcal{N}_{n_1}(n_2, \ldots, n_k).$$

One can check directly that $\mathcal{E}$ is a net on F compatible with the bornology of F.

(4) From the previous example it follows that every complete bornological vector space with a countable base for its bornology has a net compatible with the bornology. In particular, regular LB spaces and regular LF spaces have nets for their von Neumann bornology.

(5) As an example of a bornological vector space which cannot be endowed with a net, one can consider an infinite dimensional Banach space E endowed with the bornology of pre-compact subsets, if the base field is locally compact, or the bornology of compactoid subsets if the base field is not locally compact. We denote E^c the vector space E equipped with this bornology, which is of convex type and complete (see [11, Examples

1.3, (9), (10)] for a proof of this fact for Archimedean base fields. The same argument works for any base field). It is also well-known that the identity map $E^c \to E^b$ is bounded, but the two bornologies do not coincide in general. This also means that the bornology of E^c cannot have a net because the fact that the identity map is bounded would imply that, if E^c has a net, we could apply the isomorphism theorem 3.1 to deduce that the identity map is bounded also in the other direction $E^b \to E^c$, which is false. Notice that, by previous examples, this also shows that the bornology of E^c has not a countable base.

Before going into the proof of the closed graph theorem we need to prove some lemmata.

Lemma 2.4. *Let E and F be K-vector spaces. Let $B \subset E$ be bounded, $C \subset F$ any subset and $f : E \to F$ a linear map. Then, C absorbs $f(B)$ if and only if $f^{-1}(C)$ absorbs B.*

Proof. This is a very basic property of linear maps. C absorbs $f(B)$ means that there exists a $\lambda \in K^\times$ such that

$$f(B) \subset \lambda C$$

hence

$$B \subset f^{-1}(f(B)) \subset f^{-1}(\lambda C) = \lambda f^{-1}(C).$$

On the other hand, if $B \subset \lambda f^{-1}(C)$ then

$$f(B) \subset f(\lambda f^{-1}(C)) \subset \lambda f(f^{-1}(C)) \subset \lambda C.$$

$\square$

Lemma 2.5. *Let E be a K-Banach space and F be a separated convex bornological vector space endowed with a net $\mathcal{N}$, not necessarily supposed to be compatible with its bornology. Let $f : E \to F$ be a linear map, then there exists a sequence $\{n_k\}_{k\in\mathbb{N}}$ of integers such that $f^{-1}(\mathcal{N}(n_1, \ldots, n_k))$ is not meagre in E for each $k \in \mathbb{N}$.*

Proof. By the definition of net $F = \bigcup_{n_1 \in \mathbb{N}} \mathcal{N}(n_1)$, so $E = \bigcup_{n_1 \in \mathbb{N}} f^{-1}(\mathcal{N}(n_1))$ and since E is a Baire space, it follows that there must exist a n_1 such that $f^{-1}(\mathcal{N}(n_1))$ is not meagre in E. Then, we can apply the same reasoning to the relation $\mathcal{N}(n_1) = \bigcup_{n_2 \in \mathbb{N}} \mathcal{N}(n_1, n_2)$ obtaining a $f^{-1}(\mathcal{N}(n_1, n_2))$, for some $n_2 \in \mathbb{N}$, which is not meagre in E, and inductively for any k we get a $f^{-1}(\mathcal{N}(n_1, \ldots, n_k))$ which is not meagre in E, for suitable $n_1, \ldots, n_k \in \mathbb{N}$. $\square$

The next lemma is the key technical lemma to prove the closed graph theorem for bornological spaces equipped with a net.

Lemma 2.6. *Let E be a K-Banach space and F be a separated convex bornological vector space endowed with a net $\mathcal{N}$ compatible with its bornology. Let $B \subset E$ denote the open unit ball of E. If $f : E \to F$ is a morphism with bornologically closed graph, then there exists a sequence $\{n_k\}_{k\in\mathbb{N}}$ of integers such that $f(B)$ is absorbed in each $\mathcal{N}(n_1, \ldots, n_k)$.*

Proof. By Lemma 2.5 we can produce a sequence $\{n_k\}_{k \in \mathbb{N}}$ of integers such that $f^{-1}\big(\mathcal{N}(n_1, \ldots, n_{k_0})\big)$ is not meagre in E for each $k_0 \in \mathbb{N}$. Applying Lemma 2.4, it is sufficient to show that for each fixed k_0 the set $f^{-1}\big(\mathcal{N}(n_1, \ldots, n_{k_0})\big)$ absorbs B.

We denote by $s : \mathbb{N} \to \mathbb{N}$ the sequence of integers obtained by applying Lemma 2.5. By the compatibility of $\mathcal{N}$ with the bornology on F there exists a sequence $b(s)$ of positive real numbers such that for each sequence a_k of elements of K with $|a_k| \leq b(s)_k$ and for all $x_k \in \mathcal{N}_{s,k}$, the series $\sum_{k \geq k_0} a_k x_k$ converges bornologically to an element in $\mathcal{N}_{s,k_0}$. Let $\epsilon > 0$. We can choose a_k such that

$$\sum_{k \in \mathbb{N}} |a_k| \leq \epsilon.$$

Let's denote $A_k = a_k f^{-1}(\mathcal{N}_{s,k})$. Since A_k is not meagre, then there is a point b_k in the interior of $\overline{A_k}$ and a radius $\rho_k > 0$ such that the open ball

$$D(b_k, \rho_k) = b_k + \mu_k B, \quad \mu_k \in K^\times, \quad |\mu_k| = \rho_k,$$

of radius ρ_k and centred in b_k is contained in $\overline{A_k}$. We can assume that $b_k \in A_k$. In fact, since $b_k \in \overline{A_k}$, then there exists $b_k' \in A_k$ such that $|b_k - b_k'|_E < \frac{\rho_k}{2}$. So,

$$b_k' + D(0, \frac{\rho_k}{2}) = (b_k' - b_k) + (b_k + D(0, \frac{\rho_k}{2})) \subset \overline{A_k}.$$

We may also suppose that $\rho_k \leq \frac{1}{k}$ without loss of generality. So for a fixed k_0, we have

$$D(0, \rho_{k_0}) \subset \overline{A_{k_0}} - b_{k_0} \subset \overline{A_{k_0}} - \overline{A_{k_0}}.$$

Thus, if K is Archimedean we can deduce that

$$D(0, \rho_{k_0}) \subset 2\overline{A_{k_0}}$$

and if K is non-Archimedean that

$$D(0, \rho_{k_0}) \subset \overline{A_{k_0}},$$

hence if we redefine ρ_{k_0} being half its value when K is Archimedean we can always suppose that

$$D(0, \rho_{k_0}) \subset \overline{A_{k_0}}.$$

We conclude the proof by showing that there exists $\gamma \in K^\times$, such that

$$\overline{f^{-1}(\mathcal{N}_{s,k_0})} \subset \gamma f^{-1}(\mathcal{N}_{s,k_0})$$

because then we can deduce that

$$\mu_{k_0} B = D(0, \rho_{k_0}) \subset \gamma A_{k_0} \Rightarrow B \subset \mu_{k_0}^{-1} \gamma a_{k_0} \mathcal{N}(n_1, \ldots, n_{k_0}), \quad \mu_{k_0} \in K, \quad |\mu_{k_0}| = \rho_{k_0}.$$

Pick an element $x \in \overline{f^{-1}(\mathcal{N}_{s,k_0})}$. There exists an element $y_{k_0} \in f^{-1}(\mathcal{N}_{s,k_0})$ such that $|x - y_{k_0}|_E \leq \rho_{k_0+1}$. Therefore

$$(y_{k_0} - x) + b_{k_0+1} \in D(b_{k_0+1}, \rho_{k_0+1}) \subset \overline{A_{k_0+1}}.$$

Then, we can find $y_{k_0+1} \in A_{k_0+1}$ such that $|x - y_{k_0} - y_{k_0+1} + b_{k_0+1}|_E \leq \rho_{k_0+2}$. So, by induction for every $N > k_0$ we can find elements $y_k \in \overline{A_k}$ such that

$$|x - \sum_{k \geq k_0}^{N} y_k + \sum_{k \geq k_0+1}^{N} b_k|_E \leq \rho_{N+1}.$$

Since $\rho_N \to 0$, the left-hand side converges to 0. Let's show that the series

$$\sum_{k \geq k_0}^{N} z_k + \sum_{k \geq k_0+1}^{N} c_k$$

where $z_k = f(y_k)$ and $c_k = f(b_k)$ converges to $f(x)$. By hypothesis $z_{k_0} \in \mathcal{N}_{s,k_0}$ and $z_k, c_k \in a_k \mathcal{N}_{s,k}$ for $k > k_0$, so by the compatibility of the net with the bornology of F the series

$$\sum_{k \geq k_0}^{\infty} z_k, \quad \sum_{k \geq k_0+1}^{\infty} c_k$$

converge bornologically in F. Moreover, since $\mathcal{N}_{s,k} \subset \mathcal{N}_{s,k_0}$ for each $k > k_0$, one has that

$$\sum_{k \geq k_0}^{\infty} z_k \in \mathcal{N}_{s,k_0} + \gamma' \mathcal{N}_{s,k_0},$$

$$\sum_{k \geq k_0+1}^{\infty} c_k \in \gamma' \mathcal{N}_{s,k_0},$$

for a $\gamma' \in K$ such that $|\gamma'| \leq \epsilon$. So

$$y = \sum_{k \geq k_0}^{\infty} z_k - \sum_{k \geq k_0+1}^{\infty} c_k \in \gamma \mathcal{N}_{s,k_0},$$

where $\gamma \in K^\times$ can be chosen to have absolute value $1 + 2\epsilon$ if K is Archimedean and 1 if K is non-Archimedean, because $\mathcal{N}_{s,k_0} + \gamma' \mathcal{N}_{s,k_0} = \mathcal{N}_{s,k_0}$ for $|\gamma'| < 1$ in this case. Since the graph of f is bornologically closed in $E \times F$, then we have that

$$0 = f(0) = f(x - \sum_{k \geq k_0}^{\infty} y_k + \sum_{k \geq k_0+1}^{\infty} b_k) = f(x) - f(\sum_{k \geq k_0}^{\infty} y_k + \sum_{k \geq k_0+1}^{\infty} b_k) = f(x) - y,$$

so $f(x) \in \gamma \mathcal{N}_{s,k_0}$ which implies that $x \in \gamma f^{-1}(\mathcal{N}_{s,k_0})$, proving the lemma. $\square$

Theorem 2.7. *Let E and F be separated convex bornological vector spaces, where E is complete and F has a net $\mathcal{N}$ compatible with its bornology. Then every linear map $f : E \to F$ with bornologically closed graph is bounded.*

Proof. Notice that $E \cong \varinjlim_{B \in \mathcal{B}_E} E_B$, where B runs through all bounded Banach disks, i.e. E can be described as a monomorphic inductive limit of a family of Banach spaces. In order to prove that f is bounded we only have to show that the compositions of f with the canonical maps $i_B : E_B \to E$ are bounded. Indeed, the graph of $f \circ i_B$ is bornologically closed, so we just need to show the theorem for E supposed to a Banach space. To see that the graph of $f \circ i_B$ is closed one

can consider a sequence $\{x_n\} \subset E_B$, converging to $x \in E$. Since i_B is bounded then $i_B(x_n) \to i(x)$, in the sense of Mackey, so $f(i_B(x_n)) \to f(i(x))$ in the sense of Mackey in F, because the graph of f is bornologically closed.

Therefore, suppose that E is a K-Banach space with unit ball $B \subset E$. By Lemma 2.6 there exists a sequence $\{n_k\}_{k \in \mathbb{N}}$ of integers such that $f(B)$ is absorbed in each $\mathcal{N}(n_1, \ldots, n_k)$. It follows that there exists a sequence $\{a_k\}_{k \in \mathbb{N}}$ of elements of k such that

$$f(B) \subset \bigcap_{k \in \mathbb{N}} a_k \mathcal{N}(n_1, \ldots, n_k)$$

and the latter subset is bounded in F, by the request of the net to be compatible with the bornology of F. So, we can conclude that $f(B)$ is bounded in F. $\qquad \square$

We will discuss some applications of this theorem in the last section. We now see some stability properties of bornological nets, for which there is not much literature available.

Proposition 2.8. *Bornological nets have the following stability properties:*

(1) *if $E = \varinjlim\limits_{i \in \mathbb{N}} E_i$ is a monomorphic direct limit of bornological vector spaces with nets then E has a net;*

(2) *if $(E, \mathcal{B})$ has a net and $\mathcal{B}'$ is another bornology on E such that the identity $(E, \mathcal{B}) \to (E, \mathcal{B}')$ is bounded, then $(E, \mathcal{B}')$ has a net;*

(3) *every closed subspace $F \subset E$ of a bornological vector space with a net has a net;*

(4) *if $E = \varprojlim\limits_{i \in \mathbb{N}} E_i$ is a countable projective limit of bornological spaces with nets, then E has a net;*

(5) *if $E = \bigoplus\limits_{i \in \mathbb{N}} E_i$ is a countable coproduct of bornological spaces with nets, then E has a net.*

Proof. (1) This claim has already been discussed in Example 2.3 (3).

(2) Indeed, one can use the same net given on $(E, \mathcal{B})$ on the space $(E, \mathcal{B}')$, which is easily seen to be compatible with $\mathcal{B}'$ too.

(3) If E is equipped with a net $\mathcal{N}$ compatible with its bornology and $F \subset E$ is bornologically closed, then the association

$$\mathcal{N}_F(n_1, \ldots, n_k) = \mathcal{N}(n_1, \ldots, n_k) \cap F$$

defines a net on F which is compatible with the bornololgy induced by E on F because F is bornologically closed.

(4) The projective limit of a diagram $I \to \mathbf{Born}_K$ is the linear subspace

$$P = \left\{ (x_i) \in \prod_{i \in I} E_i \,\middle|\, f_{i,j}(x_j) = x_i \text{ for all } f_{i,j} \right\},$$

endowed with the induced bornology. It is easy to check that if all spaces E_i are separated then P is bornologically closed in $\prod\limits_{i \in I} E_i$. Thus, by the previous point, it is enough to show that the product of any countable family

of bornological vector spaces with nets is a bornological vector space with
a net. So, let $\{(E_i, \mathcal{N}^{(i)})\}_{i \in \mathbb{N}}$ be a countable family of bornological vector
spaces endowed with nets.

The product bornology on $\prod_{i \in \mathbb{N}} E_i$ is separated, see [10, Proposition 5,
page 48]. For any n we fix bijections $f_n : \mathbb{N}^{n+1} \to \mathbb{N}$ and let $s : \mathbb{N} \to \mathbb{N}$
be any given sequence of positive integers. For any $k \in \mathbb{N}$ there exists
$(a_0^{(k)}, \ldots, a_k^{(k)}) \in \mathbb{N}^{k+1}$ such that

$$s(k) = f_k(a_0^{(k)}, \ldots, a_k^{(k)})$$

So, we define a family of sequences $\{s_n : \mathbb{N} \to \mathbb{N}\}_{n \in \mathbb{N}}$ by

$$s_n(k) = a_n^{(k+n)}$$

for each $k \in \mathbb{N}$. We set

$$\mathcal{N}_{s,k} \doteq \mathcal{N}_{s_0,k}^{(0)} \times \mathcal{N}_{s_1,k-1}^{(1)} \times \ldots \times \mathcal{N}_{s_k,0}^{(k)} \times \prod_{i>k} E_i$$

to get a well-defined map $\mathcal{N} : \bigcup_{i \in \mathbb{N}} \mathbb{N}^i \to \mathscr{P}(\prod_{i \in \mathbb{N}} E_i)$, by setting $\mathcal{N}(\varnothing) = \prod_{i \in \mathbb{N}} E_i$
and $\mathcal{N}(n_1, \ldots, n_k) = \mathcal{N}_{s,k}$ when we choose $s : \mathbb{N} \to \mathbb{N}$ such that $s(i) = n_i$
for each $i \in \mathbb{N}$.

Let's check that $\mathcal{N}$ is a net on $\prod_{i \in \mathbb{N}} E_i$. The properties (1) and (2) of
Definition 2.1 are obvious. So let's check the last condition.

First, we fix for any n a sequence $t^{(n)} : \mathbb{N} \to \mathbb{N}$ such that $t^{(n)}(i) = s(i) = n_i$ for all $i \leq k$, $t^{(n)}(k+1) = n$, and the other values of $t^{(n)}$ can be freely
chosen. Then, consider

$$x = (x_k)_{k \in \mathbb{N}} \in \mathcal{N}_{s,k} \times \prod_{i>k} E_i.$$

We have to show that there exists an n such that

$$x \in \mathcal{N}_{t^{(n)},k+1} \times \prod_{i>k+1} E_i.$$

Then, for any n

$$\mathcal{N}_{t^{(n)},k+1} = \mathcal{N}_{s_0,k+1}^{(0)} \times \mathcal{N}_{s_1,k}^{(1)} \times \ldots \times \mathcal{N}_{s_k,1}^{(k)} \times \mathcal{N}_{t_k^{(n)},0}^{(k+1)}$$

so that the first k components do not change as n varies, whereas

$$\mathcal{N}_{t_k^{(n)},0}^{(k+1)} = \mathcal{N}^{(k+1)}\big((a_n)_k^{(k)}\big)$$

for the $(k+1)$-tuple such $f_k\big((a_n)_0^{(k)}, \ldots, (a_n)_k^{(k)}\big) = n$. Thus, since f_k is a
bijection then $\limsup_{n \to \infty}(a_n)_k^{(k)} \to \infty$ and since $\mathcal{N}^{(k+1)}$ is a net we have that

$$\bigcup_{n \in \mathbb{N}} \mathcal{N}^{(k+1)}\big((a_n)_k^{(k)}\big) = E_{k+1},$$

which shows (3) of Definition 2.1.

Then, we prove that $\mathcal{N}$ is compatible with the bornology of $\prod\limits_{i\in\mathbb{N}} E_i$. For any $s : \mathbb{N} \to \mathbb{N}$ we define

$$b(s)_k \doteq \min\{b^{(i)}(s_i)_{k-i} | \ 0 \leq i \leq k\},$$

where $b^{(i)}$ is as in Definition 2.2 for $\mathcal{N}^{(i)}$. Fix a sequence μ_k of elements of K such that $|\mu_k| \leq b(s)_k$. It is an easy consequence of the definition of product bornology that a series $\sum\limits_{k\in\mathbb{N}} \mu_k x_k$, with $x_k \in \mathcal{N}_{s,k}$, converges in $\prod\limits_{i\in\mathbb{N}} E_i$ if and only if its components converge in E_i for each $i \in \mathbb{N}$. We denote with $\pi_n : \prod\limits_{i\in\mathbb{N}} E_i \to E_n$ the canonical projection, so for $l > n$,

$$\pi_n\left(\sum_{k=0}^{l}\mu_k x_k\right) = \sum_{k=0}^{n-1}\mu_k \pi_n(x_k) + \sum_{k=n}^{l}\mu_{(k-n)}^{(n)}\frac{\mu_k}{\mu_{(k-n)}^{(n)}}\pi_n(x_k),$$

where $\mu_{(k-n)}^{(n)} \in K$ is such that $|\mu_{(k-n)}^{(n)}| \leq b^{(n)}(s_n)_{(k-n)}$. Notice that we always have that

$$\left|\frac{\mu_k}{\mu_{(k-n)}^{(n)}}\right| \leq 1, \quad \forall k \geq n.$$

Hence, the series

$$\sum_{k=n}^{\infty}\mu_{(k-n)}^{(n)}\frac{\mu_k}{\mu_{(k-n)}^{(n)}}\pi_n(x_k)$$

converges because $\mathcal{N}^{(n)}$ is a net on E_n.

Then, we need to check that for any given $m \in \mathbb{N}$ we have that

$$\sum_{k\geq m}\mu_k x_k \in \mathcal{N}_{s,m}$$

and since

$$\mathcal{N}_{s,m} = \mathcal{N}_{s_0,m}^{(0)} \times \mathcal{N}_{s_1,m-1}^{(1)} \times \ldots \times \mathcal{N}_{s_m,0}^{(m)} \times \prod_{i>m} E_i$$

we need to check only the first $m + 1$ components. So, we need to check that

$$\pi_n\left(\sum_{k=m}^{\infty}\mu_k x_k\right) \in \mathcal{N}_{s_n,m-n}^{(n)}$$

for $0 \leq n \leq m$. But for the above formula

$$\pi_n\left(\sum_{k=m}^{\infty}\mu_k x_k\right) = \sum_{k=m}^{\infty}\mu_{(k-n)}^{(n)}\frac{\mu_k}{\mu_{(k-n)}^{(n)}}\pi_n(x_k)$$

which belongs to $\mathcal{N}_{s_n,m-n}^{(n)}$ because $\mathcal{N}^{(n)}$ is a net on E_n compatible with its bornology.

Last condition of Definition 2.2 is easy to check because a subset $B \subset \prod\limits_{i\in\mathbb{N}} E_i$ is bounded if and only if $\pi_i(B)$ is bounded in E_i.

(5) Finally the last assertion of the proposition is a consequence of (1) and (4), because for any family of bornological vector spaces $\{E_n\}_{n\in\mathbb{N}}$ one has the isomorphism

$$\bigoplus_{n\in\mathbb{N}} E_n \cong \varinjlim_{n\in\mathbb{N}} \prod_{i=1}^{n} E_i$$

and the inductive system is monomorphic.

$\square$

It is easy to see that the quotient of a bornological vector space endowed with a net is not necessarily endowed with a net. One can consider a Fréchet-Montel space E and a quotient E/F, for a closed subspace $F \subset E$ which is not Fréchet-Montel. Examples of this kind of spaces are well-known both for Archimedean and non-Archimedean K. So, the von Neumann bornology of E coincides with the compact(oid) bornology of E, but the quotient bornology of E/F is the compact(oid) one and does not coincide with the von Neumann one. Example 2.3 (4) implies that E/F does not admit a net compatible with its bornology.

2.2. **The closed graph theorem for bornological spaces with webs.** We conclude this section by discussing the most general version of the closed graph theorem for bornological vector spaces of this paper. It relies on the notion of bornological web and, building on the results proved so far, its proof is quite straightforward.

Definition 2.9. Let E be a separated bornological vector space of convex type over K. A pair $(\mathcal{V}, b)$ consisting of mappings $\mathcal{V} : \bigcup_{k\in\mathbb{N}} \mathbb{N}^k \to \mathcal{P}(E)$ and $b : \mathbb{N}^{\mathbb{N}} \to (|K^\times|)^{\mathbb{N}}$ is called a *bornological web* if all the following conditions hold:

(1) The image of $\mathcal{V}$ consists of disks.
(2) $\mathcal{V}(\varnothing) = E$.
(3) Given a finite sequence $(n_0, \ldots, n_k)$, then $\mathcal{V}(n_0, \ldots, n_k)$ is absorbed by

$$\bigcup_{n\in\mathbb{N}} \mathcal{V}(n_0, \ldots, n_k, n).$$

(4) For every $s : \mathbb{N} \to \mathbb{N}$ the series $\sum_{k\in\mathbb{N}} \lambda(s)_k x_k$, with $\lambda(s) \in K$, converges bornologically in E, whenever we choose $x_k \in \mathcal{V}\big(s(0), \ldots, s(k)\big)$ and $|\lambda(s)_k| = b(s)_k$.

As we did in previous section, we will use the shorthand notation $\mathcal{V}_{s,k} = \mathcal{V}\big(s(0), \ldots, s(k)\big)$. We define the following sets, which depend on b; $\forall s : \mathbb{N} \to \mathbb{N}, \forall n \in \mathbb{N}$ let

$$\widetilde{\mathcal{V}}_{s,n} = \Gamma\left(\mathcal{V}_{s,n} \cup \left\{\sum_{k\geq n+1} \lambda(s)_k x_k | \forall k \geq n+1 : x_k \in \mathcal{V}_{s,k}, |\lambda(s)_k| = b(s)_k\right\}\right) \qquad (2.1)$$

where Γ denotes the absolutely convex hull. Furthermore, let $\mathcal{B}_{(\mathcal{V},b)}$ denote the convex linear bornology on E which is generated by all subsets of the form

$$\bigcap_{k \in \mathbb{N}} \mu_k \widetilde{\mathcal{V}}_{s,k},$$

where the $\{\mu_k\}_{k \in \mathbb{N}}$ is an arbitrary K-valued sequence.

Definition 2.10. A separated bornological vector space of convex type E which is endowed with a bornology of the form $\mathcal{B}_{(\mathcal{V},b)}$ for a bornological web $(\mathcal{V}, b)$ on E is called a *webbed convex bornological space*.

The following is the closed graph theorem for webbed bornological vector spaces.

Theorem 2.11. *Let E and F be separated convex bornological vector spaces, where E is complete and F is endowed with a bornological web $(\mathcal{V}, b)$. Then, every linear map $f : E \to F$ with bornologically closed graph (for the bornology of F) is bounded for the bornology $\mathcal{B}_{(\mathcal{V},b)}$.*

Proof. First, as in Theorem 2.7 we can reduce the proof to the case when E is a Banach space, because $f : E = \varinjlim E_i \to F$ is bounded if and only if $f \circ \alpha_i$ is bounded for every i, where $\alpha_i : E_i \to \varinjlim E_i$ are the canonical maps.

By condition (3) of Definition 2.9 we can use a reasoning similar to the one given in Lemma 2.5 to produce a sequence $s : \mathbb{N} \to \mathbb{N}$ such that $f^{-1}(\mathcal{V}_{s,k})$ is not meagre in E for any k. We put $b_k = b(s)_k$, for all $k \in \mathbb{N}$. Since $(\mathcal{V}, b)$ satisfies condition (4) of Definition 2.9, the series $\sum\limits_{k \in \mathbb{N}} \lambda_k x_k$ converges bornologically in F, whenever we choose $x_k \in \mathcal{V}_k$ and $\lambda_k \in K$ with $|\lambda_k| = b_k$.

Next, let $D(r)$ denote the ball of radius r in E centred at zero. If we can show that $f(D(1))$ is absorbed by $\widetilde{\mathcal{V}}_{s,k}$, or equivalently by Lemma 2.4, that $D(1)$ is absorbed by $f^{-1}(\widetilde{\mathcal{V}}_{s,k})$, for all $k \in \mathbb{N}$, then $f(D(1)) \in \mathcal{B}_{(\mathcal{V},b)}$, and we are done.

Define $A_k = \lambda_k f^{-1}(\mathcal{V}_{s,k})$, for all $k \in \mathbb{N}$ and pick $\lambda_k \in K$ with $|\lambda_k| = b_k$. Since A_k is not meagre and consequently not nowhere dense, the interior of $\overline{A}_k$ is not empty. Hence, there exist $\overline{y}_k \in \overline{A}_k$ and $\rho_k < \frac{1}{k+1}$ such that $\overline{y}_k + D(2\rho_k) \subset \overline{A}_k$. Since $\overline{y}_k \in \overline{A}_k$, there is a $y_k \in A_k$ such that $y_k \in \overline{y}_k + D(\rho_k)$, thus

$$y_k + D(\rho_k) = (y_k - \overline{y}_k) + (\overline{y}_k + D(\rho_k)) \subset \overline{y}_k + D(2\rho_k) \subset \overline{A}_k.$$

So, $D(\rho_k) \subset \overline{A}_k - y_k$ which implies $B(\rho_k) \subset 2\overline{A}_k$ if K is Archimedean and $D(\rho_k) \subset \overline{A}_k$ if K is non-Archimedean.

So, fix $n \in \mathbb{N}$ and let $x \in \overline{f^{-1}(\widetilde{\mathcal{V}}_{s,n})}$. Then, there is a $u_n \in f^{-1}(\widetilde{\mathcal{V}}_{s,n})$ with $x - u_n \in D(\rho_{n+1})$,

$$x - u_n + y_{n+1} \in D(\rho_{n+1}) + y_{n+1} \subset \overline{A}_{n+1}.$$

Thus, there is a $u_{n+1} \in A_{n+1}$ with $(x - u_n + y_{n+1}) - u_{n+1} \in D(\rho_{n+2})$ and inductively we find $u_k \in A_k$, $k > n$, such that we have

$$x - \sum_{k=n}^{l} u_k + \sum_{k=n+1}^{l} y_k \in D(\rho_{l+1}),$$

for $l > n$. Hence, the series $x - \sum\limits_{k=n}^{\infty} u_k + \sum\limits_{k=n+1}^{\infty} y_k$ converges to 0, since $\rho_{l+1} \to 0$.
Define $v_k = f(u_k)$ and $z_k = f(y_k)$. Then $v_n \in \mathcal{V}_{s,n}$, $z_n \in \lambda_n \mathcal{V}_{s,n}$, and $\forall k > n$ one
has that $v_k, z_k \in \lambda_k \mathcal{V}_{s,k}$. It follows from (4) of Definition 2.9 that $\sum\limits_{k \in \mathbb{N}} v_k$ and $\sum\limits_{k \in \mathbb{N}} z_k$
converge bornologically in F and moreover

$$y = \sum_{k \geq n} v_k - \sum_{k \geq n+1} z_k = v_n + \sum_{k \geq n+1} v_k - \sum_{k \geq n+1} z_k \in \widetilde{\mathcal{V}}_{s,n} + \widetilde{\mathcal{V}}_{s,n} - \widetilde{\mathcal{V}}_{s,n}.$$

Then, since f has bornologically closed graph, we infer $0 = f(0) = f(x) - y$, i.e.
$f(x) = y$ which shows that

$$f(x) \in f(\widetilde{\mathcal{V}}_{s,n})$$

if K is non-Archimedean and

$$f(x) \in 3f(\widetilde{\mathcal{V}}_{s,n})$$

if K is Archimedean.
Therefore, we can deduce that

$$D(\rho_k) \subset 2\overline{A}_k = 2\overline{\lambda_k f^{-1}(\mathcal{V}_{s,k})} \subset 2\overline{\lambda_k f^{-1}(\widetilde{\mathcal{V}}_{s,k})} \subset 6\lambda_k f^{-1}(\widetilde{\mathcal{V}}_{s,k})$$

if K is Archimedean or

$$D(\rho_k) \subset \overline{A}_k = \overline{\lambda_k f^{-1}(\mathcal{V}_k)} \subset \overline{\lambda_k f^{-1}(\widetilde{\mathcal{V}}_{s,k})} \subset \lambda_k f^{-1}(\widetilde{\mathcal{V}}_{s,k})$$

if K is non-Archimedean, completing the proof. $\square$

We conclude this section by discussing some stability properties of bornological
webs.

Proposition 2.12. *Bornological webs have the following stability properties:*

(1) *if $E = \varinjlim\limits_{i \in \mathbb{N}} E_i$ is a monomorphic direct limit of webbed bornological vector spaces then E is webbed;*
(2) *if $(E, \mathcal{B})$ is webbed and $\mathcal{B}'$ is another bornology on E such that the identity $(E, \mathcal{B}) \to (E, \mathcal{B}')$ is bounded, then $(E, \mathcal{B}')$ is webbed;*
(3) *every closed subspace of a webbed bornological vector space is a webbed bornological vector space;*
(4) *every countable projective limit of webbed bornological vector spaces is a webbed bornological space;*
(5) *every countable coproduct of webbed bornological vector spaces is a webbed bornological vector space.*

Proof. The proof of this proposition is similar to the proof of Proposition 2.8. For
details we refer to [8, Theorem 4.11], where a full proof is given when the base
field is Archimedean. $\square$

3. Applications

In this last section we deduce some consequences from the theorems we have proved so far. We start by discussing the more classical ones: various forms of isomorphisms theorems and then we deduce De Wilde's theorem for an arbitrary base field. We conclude showing some applications to the theory of bornological algebras from [1] and [3]. We remark that one of the main differences in our exposition with respect to other works in literature is that we work over any complete non-trivially valued field K, treating on the same footing, for as much as it is possible, the Archimedean and the non-Archimedean sides of the theory.

3.1. Isomorphism theorems.

Theorem 3.1. *Let $f : E \to F$ be a bijective bounded morphism between separated bornological vector spaces with F complete and E with a net compatible with the bornology (resp. is webbed), then F is an isomorphism of bornological vector spaces.*

Proof. The map $f^{-1} : F \to E$ is a linear map between bornological vector spaces whose domain is complete and whose codomain has a net compatible with the bornology (respectively, is webbed). Then, the graph of f^{-1} coincides with the graph of f, up to swap domain with the codomain, thus it is a closed subset of $F \times E$. So, by Theorem 2.7 (resppectively, Theorem 2.11) f^{-1} is a bounded map. $\square$

For any locally convex space E we denote with $(E, \mathcal{B}_{\mathrm{Ban}})$ the vector space E endowed with the vector space bornology of convex type on E generated by all bounded Banach disks of E and $E_{\mathrm{uborn}} = (E, \mathcal{B}_{\mathrm{Ban}})^t$, where ^{t} is the functor which associate to every bornological vector space the topological vector space identified by the bornivorous subsets. E_{uborn} is called the *ultrabornologification* of E.

Definition 3.2. Let E be a locally convex space over K. E is called *bornological* if $E \cong (E^b)^t$. E is called *ultrabornological* if $E \cong E_{\mathrm{uborn}}$.

Proposition 3.3. *Let $f : E \to F$ be a bijective continuous morphism between locally convex spaces. Suppose that E is ultrabornological, E^b is endowed with a net or webbed, F is bornological and F^b is complete. Then, f^{-1} is continuous.*

Proof. It is a direct consequence of Theorem 3.1, Definition 3.2 and the fact that the functors ^{b} and ^{t} are adjoints. $\square$

Corollary 3.4. *Let $f : E \to F$ be a bijective continuous morphism and that both E and F are Fréchet spaces. Then f is an isomorphism.*

Proof. Fréchet spaces are bornological and ultrabornological. $\square$

Remark 3.5. Although one can use the closed graph theorems to deduce the isomorphism theorems, they cannot be used to deduce open mapping theorems for bornological spaces, i.e. under some hypothesis a surjective bounded map must be a quotient map.

We conclude this section by discussing a result that is not a consequence of the Bornological closed graph theorems we are discussing, but for which we think it is important to have a proof that extends the classical one given over $\mathbb{R}$ and $\mathbb{C}$ to any valued base field. This result is Buchwalters theorem, which is an analogue of the open mapping theorem for bornological spaces. The interest of this theorem relies on the fact that such kind of results are very rare for bornological spaces. We need a definition and a couple of lemmata, which are adaptations of [21], section 1.5.

Definition 3.6. Let E be a vector space and B_1, B_2 be two Banach disks of E. We say that B_1 and B_2 are *compatible* if their intersection is a Banach disk.

Lemma 3.7. *Two Banach disks B_1 and B_2 of a vector space E are compatible if $B_1 + B_2$ does not contain a non-zero vector subspace.*

Proof. We shall write $E_1 = E_{B_1}$ and $E_2 = E_{B_2}$. We let also $E_1 + E_2$ be the semi-normed space absorbed by $B_1 + B_2$ with the Minkowski functional associated to $B_1 + B_2$. We have the short exact sequence

$$0 \to E_1 \cap E_2 \to E_1 \oplus E_2 \to E_1 + E_2 \to 0,$$

where the first morphism maps $x \in E_1 \cap E_2$ to $(x, -x) \in E_1 \oplus E_2$ and the second morphism maps $(x, y) \in E_1 \oplus E_2$ to $x + y \in E_1 + E_2$. The space $E_1 + E_2$ is normed if and only if the kernel of the map $E_1 \oplus E_2 \to E_1 + E_2$ is a closed subspace and the kernel is closed if and only if its unit ball is a Banach disk. The unit ball of the kernel is the image of the unit ball of $E_1 \cap E_2$ by an injective map. $\qquad\square$

Lemma 3.8. *(Grothendieck's lemma) Let E be a vector space, B a Banach disk in E and $\{B_n\}_{n \in \mathbb{N}}$ an increasing sequence of Banach disks of E such that $B = \bigcup\limits_{n=0}^{\infty} B_n$. Then B is absorbed by some B_n.*

Proof. It is an immediate consequence of Lemma 3.7 that the disk B is compatible with all the B_n for all $n \in \mathbb{N}$, since one has that $B_n \subset B$. The space E_B is a Banach space therefore it is a Baire space. According to the Baire's theorem, for some $n \in \mathbb{N}$, $\overline{B_n}$ has a non-empty interior, where the closure is taken in E_B. As B_n is absolutely convex, $\overline{B_n}$ contains a ball $D(\alpha)$ of radius $\alpha > 0$ in the Banach space E_B. Let $x_0 \in D(\alpha) \subset \overline{B_n}$. We can find $y_0 \in B_n$ and $x_1 \in D(\alpha) \subset \overline{B_n}$ such that $x_0 = y_0 + \lambda x_1$, with $\lambda \in K^\times$ and $|\lambda| \leq \frac{1}{2}$. Next we choose $y_1 \in B_n$ and $x_2 \in D(\alpha) \subset \overline{B_n}$ such that $x_1 = y_1 + \lambda x_2$, etc. For each $j \in \mathbb{N}$, we see that

$$x_0 = \sum_{l=0}^{l} \lambda^j y_l + \lambda^{j-1} x_j.$$

The series $\sum\limits_{l=0}^{\infty} \lambda^j y_l$ converges in the Banach space E_{B_n} and in that space, the norm of the sum is at most equal to 2. So the sum $\sum\limits_{l=0}^{\infty} \lambda^j y_l$ belongs to $\lambda^{-1} B_n$. In the Banach space E_B, $\lambda^{-j} x_k \to 0$. Thus $x_0 \in \lambda^{-1} B_n$ and $D(\alpha) \subset \lambda^{-1} B_n$. $\qquad\square$

Theorem 3.9. *(Buchwalters theorem) Let E be a complete bornological vector space whose bornology has a countable basis and F be a complete bornological vector space. Let $f : E \to F$ be a surjective bounded linear map. Then f is a strict epimorphism.*

Proof. Let $\{B_n\}_{n \in \mathbb{N}}$ be a basis of the bornology of E. Without lost of generality, we assume that the B_n are Banach disks and that for all $n \in \mathbb{N}$, $B_n \subset B_{n+1}$. Let C be a bounded Banach disk in F. Since f is bounded, the subsets $f(B_n)$ are bounded Banach disks and $F = \bigcup_{n=0}^{\infty} f(B_n)$. Lemma 3.7 implies that for all $n \in \mathbb{N}$, the subset $f(B_n) \cap C$ is a Banach disk as $f(B_n) + C$, being bounded in F, does not contain any non-zero subspace. Moreover, $C = \bigcup_{n=0}^{\infty} (f(B_n) \cap C)$. Then Grothendiecks lemma shows that C is absorbed by one of the sets $f(B_n)$. It follows that there exist $\lambda \in K^{\times}$ and $n \in \mathbb{N}$ such that $C \subset \lambda f(B_n)$, which yields $C \subset f(\lambda B_n)$. Thus, we showed that the map f is a strict epimorphism. $\qquad\square$

3.2. **De Wilde's Theorem.** Before proving our generalization of the De Wilde's theorem, let's see how Theorem 2.11 generalizes Theorem 2.7.

Proposition 3.10. *Let $(E, \mathcal{B})$ be a separated convex bornological vector space and $\mathcal{N}$ a net on E which is compatible with $\mathcal{B}$. Then, for every $b : \mathbb{N}^{\mathbb{N}} \to |K^{\times}|^{\mathbb{N}}$ satisfying (1) of Definition 2.2, the couple $(\mathcal{N}, b)$ is a bornological web on E such that*

$$\mathcal{B}_{(\mathcal{N}, b)} \subset \mathcal{B}$$

where $\mathcal{B}_{(\mathcal{N}, b)}$ is defined as in Definition 2.10.

Proof. The first three conditions of the definition of a bornological web are direct consequences of the definition of a net. The last one is imposed by hypothesis and by condition (1) of Definition 2.2. So, we need only to check that $\mathcal{B}_{(\mathcal{N}, b)} \subset \mathcal{B}$, and this follows directly from condition (2) of Definition 2.2. $\qquad\square$

Corollary 3.11. *Theorem 2.7 is a consequence of 2.11.*

Proof. Direct consequence of Proposition 3.10. $\qquad\square$

Then, we need to introduce the topological version of the notion of web.

Definition 3.12. Let E be a Hausdorff locally convex space. A map $\mathcal{W} : \bigcup_{k \in \mathbb{N}} \mathbb{N}^k \to \mathscr{P}(E)$ is called a *topological web* if

(1) the image of $\mathcal{W}$ consists of absolutely convex sets;
(2) $\mathcal{W}(\varnothing) = E$;
(3) Given a finite sequence $(n_0, \ldots, n_k)$, then $\mathcal{W}(n_0, \ldots, n_k)$ is absorbed by

$$\bigcup_{n \in \mathbb{N}} \mathcal{W}(n_0, \ldots, n_k, n).$$

(4) for every finite sequence $(n_0, \ldots, n_k, n_{k+1})$ one has

$$\mathcal{W}(n_0, \ldots, n_k, n_{k+1}) + \mathcal{W}(n_0, \ldots, n_k, n_{k+1}) \subset \mathcal{W}(n_0, \ldots, n_k).$$

A separated locally convex space E that carries a topological web is called a *webbed locally convex space*. Moreover, we say that $\mathcal{W}$ is *completing* if the following condition is satisfied: For every $s : \mathbb{N} \to \mathbb{N}$ and for every choice of $y_k \in \mathcal{W}\big(s(1), \dots, s(k)\big)$ the series

$$\sum_{k \in \mathbb{N}} y_k$$

converges topologically in E.

Remark 3.13. Notice that condition (4) of the last definition when K is non-Archimedean reduces to

$$\mathcal{W}(n_0, \dots, n_k, n_{k+1}) \subset \mathcal{W}(n_0, \dots, n_k),$$

because in this case $\mathcal{W}(n_0, \dots, n_k, n_{k+1}) + \mathcal{W}(n_0, \dots, n_k, n_{k+1}) = \mathcal{W}(n_0, \dots, n_k, n_{k+1})$, and reduces to the condition

$$2\mathcal{W}(n_0, \dots, n_k, n_{k+1}) \subset \mathcal{W}(n_0, \dots, n_k),$$

when K is Archimedean. Hence, from here on we define the constant

$$\lambda = \begin{cases} 2 & \text{if } K \text{ is Archimedean} \\ 1 & \text{if } K \text{ is non-Archimedean} \end{cases}.$$

Also, for topological webs we use the notation introduced in previous sections:

$$\mathcal{W}_{s,k} \doteq \mathcal{W}\big(s(0), \dots, s(k)\big), \text{ where } s : \mathbb{N} \to \mathbb{N}.$$

We need some technical lemmata. The following is the generalization of [14, Proposition 5.2.1].

Lemma 3.14. *Let E be a locally convex space and $\mathcal{W}$ a topological web on E, then $\mathcal{W}$ is completing if and only if for any open 0-neighborhood $U \subset E$ and any $s : \mathbb{N} \to \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $\mathcal{W}_{s,k} \subset U$.*

Proof. Suppose that $\mathcal{W}$ is completing. Then, every sequence of elements $y_k \in \mathcal{W}_{s,k}$ must be a zero sequence. Suppose that there exists a 0-neighborhood such that for every $k \in \mathbb{N}$ one has $\mathcal{W}_{s,k} \not\subset U$. If this would be true, we can construct a sequence $y_k \in \mathcal{W}_{s,k} - U$ which cannot be a zero sequence, thus $\mathcal{W}$ cannot be completing.

For the converse implication, consider $s : \mathbb{N} \to \mathbb{N}$ and $y_k \in \mathcal{W}_{s,k}$ for each $k \in \mathbb{N}$. So, for each 0-neighborhood $U \subset E$ we can find a $k_0 \in \mathbb{N}$ such that $\mathcal{W}_{s,k_0} \subset U$, therefore $\mathcal{W}_{s,k} \subset U$ for all $k \geq k_0$. Applying inductively (4) of Definition 3.12 we get that for any $m, k \in \mathbb{N}$, with $k \geq k_0$

$$\sum_{n=1}^{p} y_{k+n} \in \mathcal{W}_{s,k+1} + \dots + \mathcal{W}_{s,k+p} \subset \mathcal{W}_{s,k} \subset U.$$

This shows that the sequence of partial sums $\sum_{n=1}^{p} y_{k+n}$ for $p \to \infty$, is a zero sequence for the topology of E. $\qquad\square$

Lemma 3.15. *Let E be a Hausdorff locally convex space which is endowed with a topological web $\mathcal{W}$. Then, the map $\mathcal{V} : \bigcup_{k \in \mathbb{N}} \mathbb{N}^k \to \mathscr{P}(E)$ defined by*

$$\mathcal{V}(n_0, \ldots, n_k) \doteq \frac{1}{\lambda^k} \mathcal{W}(n_0, \ldots, n_k),$$

is again a topological web on E, where $\lambda \in K$ is as in Remark 3.13. Moreover, if $\mathcal{W}$ is completing then also $\mathcal{V}$ is completing.

Proof. $\mathcal{V}$ clearly satisfies the first three conditions of Definition 3.12, so we check the fourth one.

$$\lambda V(n_0, \ldots, n_k, n_{k+1}) = \frac{1}{\lambda^{k-1}} \mathcal{W}(n_0, \ldots, n_k, n_{k+1}) \subset \frac{1}{\lambda^k} W(n_0, \ldots, n_k) = \mathcal{V}(n_0, \ldots, n_k).$$

Finally, let $\mathcal{W}$ be completing. Then, since the sets $\mathcal{W}(n_0, \ldots, n_k)$ are absolutely convex they are in particular balanced, so

$$\mathcal{V}(n_0, \ldots, n_k) = \frac{1}{\lambda^k} \mathcal{W}(n_0, \ldots, n_k) \subset \mathcal{W}(n_0, \ldots, n_k),$$

therefore the completing condition for $\mathcal{W}$ implies the completing condition for $\mathcal{V}$. $\square$

From now on we consider only completing topological webs.

Definition 3.16. Let E be a K-vector space. A semi-norm p on E is called *polar* if

$$p = \sup\{|f| \mid f \in E^*, |f| \leq p\}$$

where E^* denotes the algebraic dual of E. A locally convex space E is said to be polar if its topology can be defined by polar semi-norms.

Remark 3.17. If K is spherically complete (hence also for $\mathbb{R}$ and $\mathbb{C}$), all locally convex spaces are polar, cf. [18, Theorem 4.4.3].

Notation 3.18. Let E be a locally convex space and $X \subset E$ any subset, we define

$$X^\circ \doteq \{f \in E' \mid |f(x)| \leq 1, \forall x \in E\}$$

$$X^{\circ\circ} \doteq \{x \in E \mid |f(x)| \leq 1, \forall f \in X^\circ\}$$

where E' is the continuous dual of E.

In next lemma we use the notation $S_K = l_K^1$ if K is Archimedean and $S_K = c_K^0$ if K is non-Archimedean, where l_K^1 is the K-Banach space of summable sequences and c_K^0 is the K-Banach space of zero sequences.

Lemma 3.19. *Let E be a polar locally convex space over a spherically complete field K. Let $\{y_k\}_{k \in \mathbb{N}}$ be a sequence of elements of E such that $\sum_{k \in \mathbb{N}} \mu_k y_k$ converges for each possible choice of $\mu_k \in K^\circ$. Then, $B = (\{y_k\}_{k \in \mathbb{N}})^{\circ\circ}$ is a Banach disk in E.*

Proof. By hypothesis the series $\sum\limits_{k\in\mathbb{N}}\mu_k y_k$ is convergent when we choose $\{\mu_k\}_{k\in\mathbb{N}} \in S_K$. We denote by $D \subset S_K$ the unit ball and with $e_k \in S_K$, for each $k \in \mathbb{N}$, the elements of the canonical Schauder base.

The map $T : S_K \to E$ defined by $\{\mu_k\} \mapsto \sum\limits_{k\in\mathbb{N}}\mu_k y_k$ sends $D \subset S_K$ into $B = (\{y_k\}_{k\in\mathbb{N}})^{\circ\circ}$. Moreover, T is adjoint to the map

$$E' \to c_K^0 : u \mapsto \{\langle u, y_k \rangle\}_{k\in\mathbb{N}}$$

so T is $\big(\sigma(S_K, c_K^0), \sigma(E, E')\big)$-continuous, where σ stands for the usual notation for weak topologies. The fact that $T(D)$ is $\sigma(E, E')$-bounded implies that $T(D)$ is $\sigma(E, E')$-compact for K Archimedean (this follows from the Bourbaki-Alaoglu theorem) and $\sigma(E, E')$-c-compact for K non-Archimedean, because we are supposing K spherically complete (for K non-Archimedean and spherically complete we can apply of [18, Theorem 5.4.2 and Theorem 6.1.13] to deduce weak-c-compactness from weak-boundedness). Hence, $T(D)$ is weakly closed also for K non-Archimedean because we can apply [18, Theorem 6.1.2 (iii)]. Thus, since $Te_k = y_k$ for each k and $T(D)$ is absolutely convex, the bipolar theorem implies $T(D) = B$ (see Theorem 5.2.7 for the non-Archimedean version of the bipolar theorem). Moreover, B is a bounded Banach disk, because E is polar and we can apply [18, Theorem 5.4.5]. $\square$

So, from now on until the end of this section K will be supposed to be spherically complete.

Lemma 3.20. *If $(E, \mathcal{W})$ is a webbed locally convex space, then also E_{uborn} is a webbed locally convex space.*

Proof. We shall show that if a topological web $\mathcal{W}$ is given on E, then the topological web $\mathcal{V}$ associated to $\mathcal{W}$ as in Lemma 3.15 is a topological web for E_{uborn}. The only non-trivial thing to check is that $\mathcal{V}$ is completing for the topology of E_{uborn}. So, we reproduce here the argument of [14, Theorem 13.3.3] adapting it for any base field.

Consider $s : \mathbb{N} \to \mathbb{N}$ and $x_k \in \mathcal{V}_{s,k}$ for each $k \in \mathbb{N}$. Then, by condition (4) of Definition 2.9 we know that

$$\lambda \mathcal{V}_{s,k+1} \subset \mathcal{V}_{s,k} \Rightarrow \lambda^{k-1} \mathcal{V}_{s,k+1} \subset \mathcal{W}_{s,k}.$$

Thus, for each x_k we can write

$$x_k = \frac{y_k}{\lambda^{k-1}}, \quad \text{with } y_k \in \mathcal{W}_{s,k}$$

and since $\mathcal{W}_{s,k}$ are balanced, the assumption that $\mathcal{W}$ is completing implies that the series $\sum\limits_{k\in\mathbb{N}}\mu_k y_k$ converges for the topology of E for each possible choice of $\mu_k \in K^\circ$. We can apply Lemma 3.19 to deduce that $B = (\{y_k\}_{k\in\mathbb{N}})^{\circ\circ}$ is a bounded Banach disk of E. This implies that $\sum\limits_{k\in\mathbb{N}} x_k$ converges in E_B and so also in E_{uborn}. $\square$

Lemma 3.21. *Let E be a locally convex space over K. If E is webbed, then $(E, \mathcal{B}_{\mathrm{Ban}})$ is a webbed convex bornological space with a bornological web $(\mathcal{V}, b)$ that*

can be chosen in such a way that $\mathcal{B}_{(\mathcal{V},b)}$ is finer than the von Neumann bornology of E_{uborn}.

Proof. Let $\mathcal{W}$ be a topological web on E. By Lemma 3.15

$$\mathcal{V}(n_0,\ldots,n_k) \doteq \frac{1}{\lambda^k}\mathcal{W}(n_0,\ldots,n_k)$$

is another topological web of E. For $s : \mathbb{N} \to \mathbb{N}$ define $b(s) : \mathbb{N} \to |K^\times|$ to be constant with value 1. We claim that $(\mathcal{V}, b)$ is a bornological web for $(E, \mathcal{B}_{\mathrm{Ban}})$. The first three conditions of bornological web are clear, so only the last one needs to be checked. Given a sequence of elements $x_k \in \mathcal{V}_{s,k}$, we can define

$$x_k = \frac{y_k}{\lambda^{k-1}}, \quad \text{with } y_k \in \mathcal{W}_{s,k}$$

and apply Lemma 3.19 to the sequence $\{y_k\}_{k\in\mathbb{N}}$ for obtaining that $\sum_{k\in\mathbb{N}} x_k$ converges in E_B, where $B = (\{y_k\}_{k\in\mathbb{N}})^{\circ\circ}$. So $\{y_k\}_{k\in\mathbb{N}}$ is a zero sequence in E_B which implies that $\sum_{k\in\mathbb{N}} x_k$ converges for the topology of E_{uborn}.

In order to prove that the $(\mathcal{V}, b)$ is finer than the von Neumann bornology of E_{uborn}, we notice that $(\mathcal{V}, b)$ is also a bornological web for $(E_{\mathrm{uborn}})^b$, since the bounded Banach disks that generate $\mathcal{B}_{\mathrm{Ban}}$ are all bounded subsets for E_{uborn}. Next, let $\{\mu_k\}_{k\in\mathbb{N}}$ be a sequence with values in $|K^\times|$. For every choice of $x_k \in \mathcal{V}_{s,k}$, $k \in \mathbb{N}$, the value of

$$\sum_{k>n+1}^{\infty} x_k$$

belongs to $\overline{\mathcal{V}_{s,n}}$. Hence, $\tilde{\mathcal{V}}_{s,n} \subset \overline{\mathcal{V}_{s,n}}$ (where the tilde symbol refers to the construction of formula 2.1), and $\bigcap_{k\in\mathbb{N}} \mu_k \overline{\mathcal{V}_{s,k}}$ is bounded, because for any given closed and absolutely convex 0-neighbourhood U there is an index $n \in \mathbb{N}$ such that $\mathcal{V}_{s,n} \subset U$, by Lemma 3.14. Hence, $\mu_n \overline{\mathcal{V}_{s,n}}$, and consequently $\bigcap_{k\in\mathbb{N}} \mu_k \overline{\mathcal{V}_{s,k}}$, is absorbed by U. Thus, we proved that $\mathcal{B}_{(\mathcal{V},b)}$ is finer than the canonical bornology of E_{uborn}. $\square$

Remark 3.22. Since $\lambda = 1$ for K non-Archimedean, last lemma proves that a topological web for a locally convex space E is automatically a topological web for E_{uborn} in this case and this essentially follows from the fact that for non-Archimedean base fields a sequence is summable if and only if it is a zero sequence.

Lemma 3.23. *Let E be a webbed locally convex space over K. Then $(E_{\mathrm{uborn}}, \mathcal{B}_{\mathrm{Ban}})$ carries a bornological web $(\mathcal{V}, b)$ such that the corresponding convex bornology $\mathcal{B}_{(\mathcal{V},b)}$ is contained in the von Neumann bornology of E_{uborn}.*

Proof. By Lemma 3.20 the ultrabornologification E_{uborn} of a webbed locally convex space E is webbed. Then, Lemma 3.21 applied to E yields the assertion. $\square$

Finally, we are able to deduce the main result of this section, our generalization of De Wilde's theorem.

Theorem 3.24. *If E is an ultrabornological locally convex space and F is a webbed locally convex space over K, then every linear map $f : E \to F$ which has*

bornologically closed graph with respect to $(E, \mathcal{B}_{\mathrm{Ban}})$ *and* $(F, \mathcal{B}_{\mathrm{Ban}})$, *is continuous even if regarded as a map* $f : E \to F_{\mathrm{uborn}}$.

Proof. We consider $(E, \mathcal{B}_{\mathrm{Ban}})$ as domain space of f and $(F_{\mathrm{uborn}}, \mathcal{B}_{\mathrm{Ban}})$ as the codomain of f. Notice that the family of bounded Banach disks of F and of F_{uborn} coincide and that, by Lemma 3.20, F_{uborn} carries a bornological web $(\mathcal{V}, b)$ such that $\mathcal{B}_{(\mathcal{V},b)}$ is finer than the canonical bornology of F_{uborn}. By hypothesis f has bornologically closed graph with respect to $(E, \mathcal{B}_{\mathrm{Ban}})$ and $(F, \mathcal{B}_{\mathrm{Ban}})$, therefore we can apply Theorem 2.11 in order to see that f is bounded, which also implies that f is bounded if regarded as a map from $(E, \mathcal{B}_{\mathrm{Ban}})$ to $(F_{\mathrm{uborn}})^b$. Since E is supposed to be ultrabornological, we get that $f : E \to F_{\mathrm{uborn}}$ is continuous. $\qquad\square$

Remark 3.25. The proof of De Wilde's theorem presented here closely follow the proof given in [8], adapting it in order to treat on the same footing both the Archimedean and the non-Archimedean base fields case. The main difference is the spherically completeness assumption on K which is automatic in [8].

3.3. **Applications to bornological algebras.** In this last section we show the applications which gave us the main motivations for writing down the proofs of the theorems discussed so far. The material of this section is mainly taken from [1].

Definition 3.26. A bornological K-vector space A equipped with a bilinear associative function $A \times A \to A$, called *multiplication map*, is said to be a *bornological algebra* if the multiplication map is bounded. We always suppose that A has an identity and that the multiplication is commutative. A morphism of bornological algebras is a bounded linear map that preserves multiplication and maps 1 to 1.

Our next proposition is a generalization of [4, Proposition 3.7.5/1], which holds for Banach algebras.
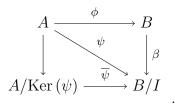
Proposition 3.27. *Let* A, B *be bornological algebras over* K *for which the underlying bornological vector space of* A *is complete and the one of* B *is a webbed bornological vector space and let* $\phi : A \to B$ *be an algebra morphism. Suppose that in* B *there is a family of ideals* $\mathcal{I}$ *such that*

(1) *each* $I \in \mathcal{I}$ *is bornologically closed in* B *and each* $\phi^{-1}(I)$ *is bornologically closed in* A;
(2) *for each* $I \in \mathcal{I}$ *one has* $\dim_K B/I < \infty$;
(3) $\bigcap_{I \in \mathcal{I}} I = (0)$.

Then, ϕ *is bounded.*

Proof. Let $I \in \mathcal{I}$ and let $\beta : B \to B/I$ denote the quotient epimorphism and $\psi = \beta \circ \phi$. Let $\overline{\psi} : A/\mathrm{Ker}\,(\psi) \to B/I$ denote the canonical injection, which gives

the following commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ \phi\ } & B \\
\downarrow & \searrow^{\psi} & \downarrow^{\beta} \\
A/\mathrm{Ker}\,(\psi) & \xrightarrow[\ \overline{\psi}\ ]{} & B/I
\end{array}
$$

.

We have that $\mathrm{Ker}\,(\psi) = \phi^{-1}(I)$, therefore, since by hypothesis B/I is finite dimensional, also $A/\mathrm{Ker}\,(\psi)$ is finite dimensional. Thus, both B/I and $A/\mathrm{Ker}\,(\psi)$ are finite dimensional separated bornological algebras, when they are equipped with the quotient bornology. Therefore, their underlying bornological vector spaces are isomorphic to the direct product of a finite number of copies of K. So, $\overline{\psi}$ is bounded and this fact implies the boundedness of ψ.

Consider a sequence $\{a_n\}_{n \in \mathbb{N}} \subset A$ such that $\lim\limits_{n \to \infty} a_n = 0$, bornologically. Then

$$
\beta\big(\lim\limits_{n \to \infty} \phi(a_n)\big) = \lim\limits_{n \to \infty} (\beta \circ \phi)(a_n)
$$

since β is bounded, and therefore

$$
\lim\limits_{n \to \infty} (\beta \circ \phi)(a_n) = \lim\limits_{n \to \infty} \psi(a_n) = \psi(\lim\limits_{n \to \infty} a_n) = 0
$$

which implies that $\phi(\lim\limits_{n \to \infty} a_n) \in I$. Since this must be true for any $I \in \mathcal{I}$ and $\bigcap\limits_{I \in \mathcal{I}} I = (0)$ we deduce that $\phi(\lim\limits_{n \to \infty} a_n) = 0$. This implies that the graph of ϕ is bornologically closed because for any sequence $\{a_n\}_{n \in \mathbb{N}}$ in A such that $\lim\limits_{n \to \infty} a_n = a$ one has that

$$
\lim\limits_{n \to \infty} \big(a_n, \phi(a_n)\big) = \big(a, \phi(a)\big) \in \Gamma(\phi).
$$

Now we can apply Theorem 2.11 to infer that ϕ is bounded. $\qquad\square$

Let $\rho = (\rho_1, ..., \rho_n) \in \mathbb{R}^n_+$ be a polyradius. We denote by $W^n_K(\rho)$ the algebra of overconvergent (also called germs of) analytic functions on the polycylinder of polyradius ρ. One can check that there is a bijection

$$
W^n_K(\rho) \cong \varinjlim_{r > \rho} T^n_K(r)
$$

where $T^n_K(r)$ denotes the algebra of strictly convergent analytic functions on the polycylinder of polyradius r. Since $T^n_K(r)$ are K-Banach algebras and by the identity theorem for analytic functions the system morphisms of the inductive system are monomorphism, then $W^n_K(\rho)$ has a canonical structure of complete bornological algebra. When $\rho = (1, \ldots, 1)$ we simply write W^n_K. For a detailed discussion of the algebras $W^n_K(\rho)$, their properties and their relations with the classical affinoid algebras and the algebras of germ of analytic functions on compact Stein subsets of complex analytic spaces the reader can refer to [1, Chapter 3].

Definition 3.28. A *strict K-dagger affinoid algebra* is a complete bornological algebra which is isomorphic to a quotient W^n_K/I, for an ideal $I \subset W^n_K$.

A (non-strict) K-*dagger affinoid algebra* is a bornological algebra which is isomorphic to a quotient

$$\frac{W_K^n(\rho)}{I}$$

for an arbitrary polyradius ρ.

Remark 3.29. It is easy to check that the underlying bornological vector space of a K-dagger affinoid algebra is an LB-space, hence in particular it is a webbed bornological vector space.

From Proposition 3.27 we can deduce the following result.

Proposition 3.30. *Every morphism between dagger affinoid algebras is bounded.*

Proof. If $\phi : A \to B$ is an algebra morphism between strict K-dagger affinoid algebras then we can apply Proposition 3.27 choosing as family $\mathcal{I}$ the family of all powers of maximal ideals of B. The only non-trivial fact to check for applying Proposition 3.27 is the requirement that all the elements of $\mathcal{I}$ must be bornologically closed in B and their preimages bornologically closed in A. But in [1, Section 3.2] it is proved that all ideals of dagger affinoid algebras are bornologically closed, hence it follows that ϕ is bounded.

The non-strict case can be reduced to the strict case noticing that any non-strict dagger affinoid algebra can be written as a direct limit of strict ones and that every algebra morphism can be written as a morphism of direct systems of algebras, as explained in [1, Section 3.2]. Therefore, every morphism between non-strict dagger affinoid algebras can be written as a direct limit of bounded ones, hence it is bounded. □

We conclude this overview of applications of the bornological closed graph theorem by saying that the last proposition can be generalized to encompass a more general class of bornological algebras used in analytic geometry: Stein algebras and (at least a big subclass of) quasi-Stein algebras, both dagger and non-dagger. In Proposition 3.30 we did not use the full strength of Theorem 2.11, as weaker versions of it suffice to deduce the proposition. But for the generalizations just mentioned we will need definitely to invoke Theorem 2.11, as the underlying bornological vector spaces of the algebras involved are of a quite exotic nature. The arguments for showing the boundedness of algebra morphisms for these classes of bornological algebras become more elaborate and do not fit in this discussion. The reader can refer to [3] for such a study.

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