QUASICONTINUITY AND RELATED PROPERTIES OF FUNCTIONS AND MULTIVALUED MAPS

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Abstract. The main results presented in this paper concern multivalued maps. We consider the cliquishness, quasicontinuity, almost continuity and almost quasicontinuity; these properties of multivalued maps are characterized by the analogous properties of some real functions. The connections obtained are used to prove decomposition theorems for upper and lower quasicontinuity.

Keywords: uniform space, multivalued map, quasicontinuity

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Throughout the paper (X,T) or simply X is a topological space, $\operatorname{Cl} A$, $\operatorname{Int} A$ are the closure and the interior of a set $A \subset X$, respectively.

A set $A \subset X$ is said to be
– semi-open, if $A \subset \operatorname{Cl}(\operatorname{Int} A)$, [10];
– preopen, if $A \subset \operatorname{Int}(\operatorname{Cl} A)$;
– semi-preopen, if $A \subset \operatorname{Cl}\big(\operatorname{Int}(\operatorname{Cl} A)\big)$, [1].

The union of any family of semi-open (preopen, semi-preopen) sets is a set of the same type, [1, 10]. The intersection of an open set and a semi-open (preopen, semi-preopen) one is again semi-open (preopen, semi-preopen).

Let X, Y be topological spaces. A function $f\colon X \to Y$ is called quasicontinuous (almost continuous, almost quasicontinuous) at a point $x \in X$ if for each neighbourhood V of $f(x)$ we have $x \in \operatorname{Cl}\big(\operatorname{Int} f^{-1}(V)\big)$, $\big(x \in \operatorname{Int}(\operatorname{Cl} f^{-1}(V))$, $x \in \operatorname{Cl}\big(\operatorname{Int}\big(\operatorname{Cl} f^{-1}(V)\big)\big)\big)$, [10, 8, 2]. Equivalently, f is quasicontinuous (almost continuous, almost quasicontinuous) at $x \in X$ if for each neighborhood V of $f(x)$ there is a semi-open (preopen, semi-preopen) set A satisfying $x \in A \subset f^{-1}(V)$. Let us observe

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that the quasicontinuity of f at x is equivalent to the condition $\operatorname{Int}\bigl(U \cap f^{-1}(V)\bigr) \neq \emptyset$ for every neighbourhoods U, V of x and $f(x)$, respectively, [9].

A function f is called quasicontinuous (almost continuous, almost quasicontinuous) if it has this property at each point. It follows from definitions that continuity implies quasicontinuity and almost continuity; moreover, each of these properties implies almost quasicontinuity but these implications cannot be replaced by equivalences, [14].

Quasicontinuity and almost continuity are independent properties. Furthermore, almost quasicontinuous functions need not have the Baire property.

E x a m p l e 1. Let $\mathbb{R}$ be the space of real numbers with the natural topology, $B \subset \mathbb{R}$ a Bernstein set and f the characteristic function of B. Then f has not the Baire property. Nonetheless, since B and $\mathbb{R} \setminus B$ are dense sets [15], the function f is almost continuous.

Now let (Y, ϱ) be a pseudometric space. A function $f \colon X \to Y$ is said to be cliquish at a point $x_0 \in X$ if

(1) for each $\varepsilon > 0$ and each neighbourhood U of x_0 there exists a nonempty open set $W \subset U$ with $\varrho\bigl(f(x), f(x')\bigr) < \varepsilon$ for x, $x' \in W$, ([18] for metric spaces).

By $A(f)$ we denote the set of all points at which f is cliquish. A function f is called cliquish if $A(f) = X$.

The symbols $C(f)$ and $E(f)$ are used to denote the sets of all points at which a function f is continuous or quasicontinuous, respectively. Then

(2) $C(f) \subset E(f) \subset A(f) = \operatorname{Cl}\bigl(A(f)\bigr)$;

(3) $A(f) \setminus C(f)$ is of the first category;

(4) if X is a Baire space, then $f \colon X \to (Y, \varrho)$ is cliquish iff $X \setminus C(f)$ is of the first category;

(5) a function $f \colon X \to (Y, \varrho)$ is continuous iff it is cliquish and almost continuous.

The statements (2)–(5) are proved in [11, 13, 6, 17] for a metric space (Y, d), but by the same arguments they are valid for pseudometric ones.

In the sequel we will consider maps with values in uniform spaces. For a uniform space (Y, ν) we denote by P_ν a saturated family of pseudometrics inducing ν. For any $y \in Y$, M, $M_1 \subset Y$, $\varrho \in P_\nu$ and $r > 0$ we denote

$$B(y, \varrho, r) = \{z \in Y \colon \varrho(y, z) < r\}, \qquad B(M, \varrho, r) = \bigcup \{B(y, \varrho, r) \colon y \in M\},$$
$$\varrho(y, M) = \inf_{z \in M} \varrho(y, z), \qquad \varrho(M_1, M) = \sup_{y \in M_1} \varrho(y, M).$$

It is easy to verify

(6) if M_1 is compact, then $\varrho(M_1, M) < r$ iff $M_1 \subset B(M, \varrho, r)$.