CONVERGENCE ℓ-GROUPS WITH ZERO RADICAL

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Abstract. In this paper we investigate abelian convergence ℓ-groups with zero radical such that each bounded sequence has a convergent subsequence.

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Sequentially compact convergence groups were studied by Dikranjan [3]; cf. also the references given there.

All ℓ-groups (= lattice ordered groups) dealt with in the present paper are assumed to be abelian.

For convergence ℓ-groups we apply the same definitions and notation as in [6].

Let G be a convergence ℓ-group. The corresponding convergence will be denoted by α; thus if a sequence (x_n) converges to x in G, then we express this fact by writing $x_n \rightarrow_\alpha x$.

If every sequence in G has a converging subsequence, then G is said to be sequentially compact.

It turns out that the role of the notion of sequential compactness for convergence ℓ-groups is rather modest. Namely, G is sequentially compact if and only if $G = \{0\}$.

If every bounded sequence in G has a converging subsequence, then G will be called b-sequentially compact.

We use the notion of the radical of an ℓ-group as in Conrad [2] (the definition is recalled in Section 1 below); ℓ-groups with zero radical were investigated in [1] in connection with the lateral completion of ℓ-groups.

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In the present article we deal with the case when G satisfies the following conditions:

(a) the radical of G is zero;
(b) G is b-sequentially compact.

The symbols Z and R denote the additive group of all integers or of all reals, respectively, with the natural linear order.

The notion of o-convergence has the usual meaning; we apply the notation $x_n \rightarrow_{\alpha(o)} x$.

The ℓ-group G is said to satisfy the condition (F) if each bounded disjoint subset of G is finite (cf. [2]).

We prove the following results.

Let G be a convergence ℓ-group satisfying the Urysohn axiom.

(A) Suppose that G satisfies the conditions (a) and (b). Then G is a completely subdirect product of ℓ-groups G_i $(i \in I)$ such that
   (i) for each $i \in I$, G_i is isomorphic either to Z or to R;
   (ii) if $x_n \rightarrow_{\alpha} x$ holds in G and if $i \in I$, then for the natural projection p_i of G onto G_i the relation $p_i(x_n) \rightarrow_{\alpha(o)} p_i(x)$ is valid.

(B) Suppose that G is a completely subdirect product of ℓ-groups G_i $(i \in I)$ such that the conditions (i) and (ii) from (A) are satisfied. Further suppose that the condition (F) is valid. Then G is b-sequentially compact and its radical is zero.

By an example we show that the assumption on the validity of (F) cannot be cancelled in the above theorem.

1. Preliminaries; sequential precompactness

In what follows, $\mathbb{N}$ denotes the set of all positive integers. For the sake of completeness we recall the following definitions from [6].

Let G be an ℓ-group, $g \in G$ and $(g_n) \in G^{\mathbb{N}}$. If $g_n = g$ for each $n \in \mathbb{N}$, then we write $(g_n) = \operatorname{const} g$. For $(h_n) \in G^{\mathbb{N}}$ we set $(h_n) \sim (g_n)$ if there is $m \in \mathbb{N}$ such that $h_n = g_n$ for each $n \in \mathbb{N}$ with $n \geqslant m$.

The set $G^{\mathbb{N}}$ is an ℓ-group under the obvious definition of the partial order and of the operation $+$. Let α be a convex subsemigroup of the lattice ordered semigroup $(G^{\mathbb{N}})^+$ such that the following conditions are satisfied:

(I) If $(g_n) \in \alpha$, then each subsequence of (g_n) belongs to α.
(II') Let $(g_n) \in \alpha$ and $(h_n) \in (G^{\mathbb{N}})^+$. If $(h_n) \sim (g_n)$, then $(h_n) \in \alpha$.
(III) Let $g \in G$. Then $\operatorname{const} g$ belongs to α if and only if $g = 0$.

Under these conditions α is said to be a convergence on G.

For $(g_n) \in G^{\mathbb{N}}$ and $g \in G$ we put $g_n \to_\alpha g$ if and only if $(|g_n - g|) \in \alpha$. It is easy to verify that $g_n \to_\alpha 0$ if and only if $(g_n) \in \alpha$.

We denote by $\operatorname{conv} G$ the set of all convergences on G.

Let $\alpha(o)$ be the set of all sequences (g_n) in G^+ having the property that there exists $(h_n) \in (G^{\mathbb{N}})^+$ such that (i) $h_{n+1} \leqslant h_n$ for each $n \in \mathbb{N}$; (ii) $\bigwedge_{n\in\mathbb{N}} h_n = 0$; (iii) there is $m \in \mathbb{N}$ such that $h_n \geqslant g_n$ for each $n \in \mathbb{N}$ with $n \geqslant m$. Then $\alpha(o) \in \operatorname{conv} G$; $\alpha(o)$ is said to be the o-convergence in G.

Further let $\alpha(d)$ be the set of all $(x_n) \in (G^{\mathbb{N}})^+$ such that $(x_n) \sim \operatorname{const} 0$. Then clearly $\alpha(d) \in \operatorname{conv} G$; it is said to be the discrete convergence on G.

Let us remark that if $x_n \to_\alpha x$, $y_n \to_\alpha y$ and $\circ \in \{+, -, \wedge, \vee\}$, then

$$x_n \circ y_n \to_\alpha x \circ y;$$

also, if $(x_n) = \operatorname{const} x$, then $x_n \to_\alpha x$. (Cf. [6].)

The system $\operatorname{conv} G$ is partially ordered by the set-theoretical inclusion. The least element of $\operatorname{conv} G$ is $\alpha(d)$.

The convergence α is said to satisfy the Urysohn axiom if it fulfils

(II) Whenever (g_n) is a sequence in G^+ such that each subsequence of (g_n) has a subsequence belonging to α, then $(g_n) \in \alpha$.

The system of all elements of $\operatorname{conv} G$ which satisfy the Urysohn axiom will be denoted by $\operatorname{Conv} G$.

Let $0 \neq g \in G$. We denote by A_g the system of all convex ℓ-subgroups A of G such that $g \notin A$; further let R_g be the subgroup of G generated by the set $\bigcup A$ $(A \in A_g)$. The radical $R(G)$ of G is defined to be the set $\bigcap R_g$ $(0 \neq g \in G)$. (Cf. [2].)

A subset X of G^+ is said to be disjoint if $x \geqslant 0$ for each $x \in X$, and if $x_1 \wedge x_2 = 0$ whenever x_1 and x_2 are distinct elements of X.

Let $(G_i)_{i\in I}$ be an indexed system of ℓ-groups and let φ be an isomorphism of an ℓ-group G into the direct product $\prod_{i\in I} G_i$ such that, whenever $i \in I$ and $x^i \in G_i$, then there exists $g \in G$ with

$$\varphi(g)_i = x^i;$$
$$\varphi(g)_j = 0 \quad \text{for each } j \in I \setminus \{i\}.$$

Under these assumptions we say that φ is a completely subdirect product decomposition of the ℓ-group G. The notion of the completely subdirect product is due to Šik [7].

The condition defining the completely subdirect product decomposition can be expressed also by writing

$$\sum_{i\in I} G_i \subseteq \varphi(G) \subseteq \prod_{i\in I} G_i.$$

A sequence (x_n) in a convergence ℓ-group G is called a Cauchy sequence if, whenever (y_n) and (z_n) are subsequences of (x_n), then $y_n - z_n \to_\alpha 0$.

G is called sequentially precompact if each its sequence has a Cauchy subsequence. (Cf. [3] for the case of convergence groups.)

G will be said to be b-sequentially precompact if each its bounded sequence has a Cauchy subsequence.

1.1. Lemma. *Let G be a convergence ℓ-group, $0 < x \in G$, $x_n = nx$ for each $n \in \mathbb{N}$. Then the sequence (x_n) has no Cauchy subsequence.*

Proof. By way of contradiction, suppose that (y_n) is a Cauchy subsequence of (x_n). We have $y_{n+1} - y_n \geqslant x > 0$ for each $n \in \mathbb{N}$, hence the relation

$$y_{n+1} - y_n \to_\alpha 0$$

cannot hold and so we arrive at a contradiction. □

1.2. Corollary. *Let G be a convergence ℓ-group. Suppose that G is b-sequentially precompact. Then G is archimedean.*

Proof. If G is not archimedean, then there are $x, y \in G$ such that $0 < nx < y$ is valid for each $n \in \mathbb{N}$. Thus in view of 1.1, G is not sequentially precompact. □

1.3. Corollary. *Each b-sequentially compact convergence ℓ-group is archimedean.*

2. Congruence relations

Again, let G be a convergence ℓ-group with the convergence α.

A subset X of G is said to be closed with respect to α if, whenever $x_n \to_\alpha x$ and all x_n belong to X, then x belongs to X as well.

2.1. Lemma. *Let A be a convex ℓ-subgroup of G and let $g_1 \in G$. Then $g_1 + A$ is closed with respect to α if and only if A is closed with respect to α.*

Proof. This is an immediate consequence of the fact that the convergence is compatible with the operations $+$ and $-$. □

Let A be as in 2.1 and suppose that A is closed with respect to α. For each $x \in G$ and $X \subseteq G$ we put

$$\overline{x} = x + A, \quad \overline{X} = \{\overline{x} \colon x \in X\}.$$

Hence $\overline{G}$ is the factor ℓ-group of G corresponding to the ℓ-ideal A, i.e., $\overline{G} = G/A$. We set

$$\overline{\alpha} = \{(\overline{x}_n) \colon (x_n) \in \alpha\}.$$

2.2. Lemma. $\overline{\alpha} \in \operatorname{conv} \overline{G}$.

Proof. We have to verify that the conditions (I), (II') and (III) are satisfied for $\overline{\alpha}$.

i) Let $(\overline{g}_n) \in \overline{\alpha}$ and let $(\overline{h}_n)$ be a subsequence of $(\overline{g}_n)$. Hence there is $(x_n) \in \alpha$ such that $(\overline{g}_n) = (\overline{x}_n)$. Then $(\overline{h}_n) = (\overline{y}_n)$, where (y_n) is a subsequence of (x_n). We have $(y_n) \in \alpha$, therefore $(\overline{h}_n) \in \overline{\alpha}$.

ii) Let $(\overline{g}_n) \in \alpha$, $(\overline{h}_n) \in (\overline{G}^{\mathbb{N}})^+$, $\overline{g}_n \sim \overline{h}_n$. Further let (x_n) be as in (i). There is $m \in \mathbb{N}$ such that $\overline{h}_n = \overline{g}_n$ for each $n \in \mathbb{N}$ with $n \geqslant m$. Put $y_n = h_n$ for $n < m$ and $y_n = x_n$ otherwise. Then $(y_n) \sim (x_n)$, whence $(y_n) \in \alpha$. Clearly $(\overline{h}_n) = (\overline{y}_n)$. Thus $(\overline{h}_n) \in \overline{\alpha}$.

iii) Let $g \in G$, $(\overline{g}_n) = \operatorname{const} \overline{g}$.

Suppose that $(\overline{g}_n) \in \overline{\alpha}$. Hence there exists $(x_n) \in \alpha$ with $(\overline{g}_n) = (\overline{x}_n)$. Then $x_n \in g + A$ for each $n \in \mathbb{N}$. We have $x_n \to_\alpha 0$ and thus in view of 2.1 we obtain that $0 \in g + A$ yielding that $\overline{g} = \overline{0}$.

Conversely, suppose that $\overline{g} = \overline{0}$. Put $x_n = 0$ for each $n \in \mathbb{N}$. Then $(x_n) \in \alpha$ and $(\overline{x}_n) = (\overline{g}_n)$, whence $(\overline{g}_n) \in \overline{\alpha}$. □

Under the notation as above we always consider $\overline{G}$ to be a convergence ℓ-group with the convergence $\overline{\alpha}$.

For $X \subseteq G$ we denote by X^δ the polar of X (cf. [2]).

2.3. Lemma. *Let $X \subseteq G$. Then X^δ is closed with respect to α.*

Proof. Put $X^\delta = A$. Denote $X_1 = \{|x| \colon x \in X\}$. Then $X^\delta = X_1^\delta$ and $X_1 \subseteq G^+$. Hence without loss of generality we can suppose that $X \subseteq G^+$.

Let $a_n \in A$ for each $n \in \mathbb{N}$, $a_n \to_\alpha g$. Then $a_n \vee 0 \in A$, $a_n \vee 0 \to_\alpha g \vee 0$. Let $x \in X$. We have $x \wedge (a_n \vee 0) = 0$, whence $x \wedge (g \vee 0) = 0$ and thus $g \vee 0 \in A$.

Further, $-(a_n \wedge 0) \in A$, thus

$$x \wedge (-(a_n \wedge 0)) = 0$$

yielding that

$$x \wedge (-(g \wedge 0)) = 0,$$

hence $-(g \wedge 0) \in A$. Therefore $g \wedge 0 \in A$. Since A is a convex subset of G we get $g \in A$. □

2.4. Corollary. *Each direct factor of the ℓ-group is closed with respect to α.*

For an ℓ-subgroup A of G we denote

$$\alpha_A = \alpha \cap (A^{\mathbb{N}})^+.$$

Then applying the conditions (I), (II') and (III) we immediately obtain

2.5. Lemma. $\alpha_A \in \operatorname{conv} A$.

The ℓ-subgroup A is always regarded as a convergence ℓ-group with the convergence α_A.

Now suppose that the ℓ-group G is represented as a direct product

$$G = A \times B. \tag{1}$$

In view of 2.4, B is closed with respect to α; let us denote by $\overline{\alpha}$ the corresponding convergence on the ℓ-group G/B.

Each element $g \in G$ can be uniquely represented as $g = a + b$ with $a \in A$ and $b \in B$; if $g \geqslant 0$, then $a \geqslant 0$ and $b \geqslant 0$. Hence each element $g + B$ of G/B can be written as

$$a + b + B = a + B$$

with $a \in A$. If $a_1 \in A$ and $a_1 + B = a + B$, then $a - a_1 \in B$, whence $a = a_1$.

2.6. Proposition. *Let* (1) *be valid.*

a) *Let $(a_n) \in \alpha_A$. Then $(\overline{a}_n) \in \overline{\alpha}$.*

b) *Let $(\overline{g}_n) \in \overline{\alpha}$, $g_n = a_n + b_n$, $a_n \in A$, $b_n \in B$. Then $(a_n) \in \alpha_A$.*

P r o o f. a) Let $(a_n) \in \alpha_A$. Then $(a_n) \in \alpha$ and thus $(\overline{a}_n) \in \overline{\alpha}$.

b) Let $(\overline{g}_n) \in \overline{\alpha}$ and let a_n, b_n be as above. In view of the definition of $\overline{\alpha}$ there exists $(h_n) \in \alpha$ such that $(\overline{h}_n) = (\overline{g}_n)$. Let $h_n = a'_n + b'_n$, $a'_n \in A$, $b'_n \in B$. Then $(a'_n) \in (A^{\mathbb{N}})^+$ and for each $n \in \mathbb{N}$ we have

$$a'_n + B = a'_n + b'_n + B = \overline{h}_n = \overline{g}_n = a_n + b_n + B = a_n + B,$$

whence $a'_n = a_n$. Thus $0 \leqslant a_n \leqslant h_n$ for each $n \in \mathbb{N}$. Since α is a convex subset of $(G^{\mathbb{N}})^+$ we infer that $(a_n) \in \alpha$. Hence $(a_n) \in \alpha_A$. □

2.7. Lemma. *Let A be a convex ℓ-subgroup of G and let $(\overline{g}_n)$ be a bounded sequence in $\overline{G} = G/A$. Then there exists a bounded sequence (h_n) in G such that $\overline{h}_n = \overline{g}_n$ for each $n \in \mathbb{N}$.*

P r o o f. In view of the assumption there exist $x, y \in G$ such that $\overline{x} \leqslant \overline{g}_n \leqslant \overline{y}$ for each $n \in \mathbb{N}$. Put $h_n = (x_1 \vee g_n) \wedge y_1$, where $x_1 = x \wedge y$ and $y_1 = x \vee y$. Then

$$\overline{x}_1 = \overline{x}, \quad \overline{y}_1 = \overline{y}, \quad \overline{h}_n = \overline{g}_n, \quad x_1 \leqslant h_n \leqslant y_1$$

for each $n \in \mathbb{N}$. □

2.8. Lemma. *Suppose that G is b-sequentially compact and that A is an ℓ-ideal of G which is closed with respect to α. Then G/A is b-sequentially compact.*

P r o o f. This is an immediate consequence of the definition of $\overline{\alpha}$ and of 2.7. □

From 2.6 and 2.8 we obtain

2.8.1. Corollary. *Suppose that G is b-sequentially compact and that (1) is valid. Then A is b-sequentially compact.*

2.9. Lemma. *Let (1) be valid, $g_n \in G$, $g_n = a_n + b_n$ ($a_n \in A$, $b_n \in B$, $n \in \mathbb{N}$). Then the following conditions are equivalent:*

(i) *$(g_n) \in \alpha$;*

(ii) *$a_n \in \alpha_A$ and $b_n \in \alpha_B$.*

P r o o f. (i) Let $(g_n) \in \alpha$. Since $0 \leqslant a_n \leqslant g_n$ we obtain that $(a_n) \in \alpha$ and thus $(a_n) \in \alpha_A$. Similarly, $(b_n) \in \alpha_B$.

(ii) Let $(a_n) \in \alpha_A$ and $(b_n) \in \alpha_B$. Then (a_n), $(b_n) \in \alpha$ and thus $(g_n) = (a_n + b_n) \in \alpha$. □

By the obvious induction we can generalize the above result for the case

$$G = A_1 \times A_2 \times \ldots \times A_k. \tag{2}$$

2.10. Lemma. *Let (2) be valid. Then G is b-sequentially compact if and only if all A_i ($i = 1, 2, \ldots, k$) are b-sequentially compact.*

P r o o f. This follows from 2.6, 2.8.1 and 2.9. □

3. The case of linearly ordered groups

In this section we suppose that G is as above and that, moreover, G is linearly ordered.

3.1. Lemma. *Let $(g_n) \in \alpha$. Then $(g_n) \in \alpha(o)$.*

Proof. From $(g_n) \in \alpha$ we obtain that $g_n \geqslant 0$ for each $n \in \mathbb{N}$. The case $G = \{0\}$ being trivial we can suppose $G \neq \{0\}$. Let $0 < x \in G$. If the set $S_x = \{n \in \mathbb{N} \colon g_n \geqslant x\}$ is infinite then there exists a subsequence (h_n) of (g_n) such that $h_n \geqslant x$ for each $n \in \mathbb{N}$. Since $h_n \to_\alpha 0$ we would have $x_n \to_\alpha 0$, where $(x_n) = \operatorname{const} x$, which is a contradiction. Hence for each $0 < x \in G$ the set S_x is finite. This yields that for each $m \in \mathbb{N}$ the set $\{g_n \colon g_n \geqslant g_m\}$ has a greatest element; this will be denoted by g_m^0. Then $g_1^0 \geqslant g_2^0 \geqslant \ldots \geqslant 0$. Since each g_m^0 is equal to some g_n with $n \geqslant m$, we have $\bigwedge_{n \in \mathbb{N}} g_n^0 = 0$. Hence $(g_n) \in \alpha(o)$. □

As a corollary we obtain

3.2. Proposition. *If G is linearly ordered, then $\alpha(o)$ is the greatest element of* $\operatorname{conv} G$.

In general, if G fails to be linearly ordered, then $\operatorname{conv} G$ need not have the greatest element. For related questions cf. [5].

3.3. Proposition. (Harminc [4].) *Suppose that G is linearly ordered. Then*

(i) *$\alpha(o)$ belongs to* $\operatorname{Conv} G$;

(ii) *if α belongs to* $\operatorname{Conv} G$, *then either $\alpha = \alpha(d)$ or $\alpha = \alpha(o)$.*

In the remaining part of this section we assume that G is linearly ordered and b-sequentially compact. We also suppose that α belongs to $\operatorname{Conv} G$. In view of 1.4, G is archimedean. It is well-known that each archimedean linearly ordered group is isomorphic to an ℓ-subgroup of R. Hence without loss of generality we can assume that the ℓ-group G coincides with an ℓ-subgroup of R. We also assume that $G \neq \{0\}$.

There exists $x \in R$ with $x > 0$ such that the interval $[0, x]$ of R contains an element of G distinct from 0. Put $A = G \cap [0, x]$. We distinguish two cases:

a) The set A is finite.

b) The set A is infinite.

Firstly suppose that a) is valid. Then there exists an element g_1 in G such that g_1 covers the element 0. It is a routine to verify that in this case G is isomorphic to Z.

Further let us suppose that b) holds. Then for each $y \in R$ with $y > 0$ there exist distinct elements $g_1, g_2 \in G$ such that $0 < g_1 < g_2 \leqslant x$ and $g_2 - g_1 < y$.

This yields that there is a sequence (g_n) in G such that $g_1 > g_2 > \ldots > g_n > g_{n+1} > \ldots > 0$ and $\bigwedge_{n\in\mathbb{N}} g_n = 0$. No subsequence of (g_n) belongs to $\alpha(d)$. Thus, since G is b-sequentially compact, $\alpha \neq \alpha(d)$. Therefore in view of 3.3, $\alpha = \alpha(o)$.

The symbol $\alpha(o)$ means the o-convergence in G; now we will denote it by $\alpha(o, G)$ in order to distinguish it from the o-convergence in R, which will be denoted by $\alpha(o, R)$. It is clear that

$$\alpha(o, G) = (G^{\mathbb{N}})^+ \cap \alpha(o, R). \tag{3}$$

Suppose that there is $t \in R$ such that t does not belong to G. Then $t' = |t| > 0$ and $t' \notin G$. For each $n \in \mathbb{N}$ there exists $g_n \in G$ such that

$$0 < g_n < \frac{1}{n}, \quad g_n < t'.$$

Since G is archimedean there is $n' \in \mathbb{N}$ such that

$$n' g_n < t' < (n' + 1) g_n.$$

Denote $n' g_n = g_n^1$, $(n'+1) g_n = g_n^2$. Thus $g_n^1 < t' < g_n^2$ and $g_n^2 - g_n^1 < \frac{1}{n}$. From these relations we easily obtain that

$$g_n^1 \to_{\alpha(o,R)} t', \quad g_n^2 \to_{\alpha(o,R)} t'.$$

(g_n^1) is a bounded sequence in G. If (h_n) is a subsequence of (g_n^1), then

$$h_n \to_{\alpha(o,R)} t',$$

whence in view of (3), (h_n) is not convergent with respect to the o-convergence in G. Thus G is not b-sequentially compact and so we arrive at a contradiction. Therefore $G = R$.

Summarizing, we conclude:

3.4. Lemma. *Let G be a convergence ℓ-group with the convergence α such that* (i) *G is linearly ordered,* (ii) *G is b-sequentially compact, and* (iii) *α satisfies the Urysohn axiom. Then either*

a) *G is isomorphic to Z and $\alpha = \alpha(d)$,*

or

b) *G is isomorphic to R and α coincides with the o-convergence.*

4.1. Lemma. *Let G be an archimedean ℓ-group with zero radical. Then G is a completely subdirect product of linearly ordered groups.*

Proof. This is a consequence of Theorem 3.5 and Theorem 5.4 in [2]. □

Proof of (A).

Suppose that G is a convergence ℓ-group with the convergence α such that

a_1) the radical of G is zero;

a_2) G is b-sequentially compact;

a_3) the Urysohn condition is satisfied.

Then in view of a_2) and 1.4, the ℓ-group G is archimedean. Thus according to 4.1, the ℓ-group G is a completely subdirect product of linearly ordered groups A_i $(i \in I)$.

Each A_i is a direct factor of G. We consider the convergence $\alpha_i = \alpha_{A_i}$ on A_i. Then in view of 2.8.1, A_i is b-sequentially compact. Since α satisfies the Urysohn axiom, α_i satisfies this axiom as well. Thus according to 3.4, some of the conditions a) or b) from 3.4 holds. □

Proof of (B).

Suppose that the assumptions from (B) are satisfied. Thus in view of 3.4, all G_i are b-sequentially compact.

Let (g_n) be a bounded sequence in G. Using translations we see that without loss of generality it suffices to consider the case when $0 \leqslant g_n \leqslant g$ for some $g \in G$. Let $g_i = g(G_i)$. Then $\{g_i\}_{i \in I}$ is a disjoint subset of $[0, b]$. Put $I_1 = \{i \in I \colon g_i > 0\}$. The case $I_1 = \emptyset$ is trivial; suppose that $I_1 \neq \emptyset$. Since G satisfies the condition (F), the set I_1 is finite and we can write $I_1 = \{i_1, i_2, \ldots, i_k\}$. Thus $[0, b]$ is a subset of $G_{i_1} \times G_{i_2} \times \ldots \times G_{i_k} = B$. Now according to 2.10 there exists a subsequence (h_n) of (g_n) which is convergent with respect to α_B and hence this subsequence is convergent also with respect to α. Hence G is b-sequentially compact. From the definition of the radical we obtain that $R(G) = \{0\}$. □

The following example shows that the condition (F) in (B) cannot be omitted.

Let $G = \prod\limits_{i \in I} G_i$, where $I = \mathbb{N}$ and $G_i = Z$ for each $i \in I$. If $g \in G$, then the component of g in G_i will be denoted by $g(i)$. We consider the discrete convergence $\alpha(d) = \alpha$ on G. Then for each $i \in I$, α_{G_i} is the discrete convergence on G_i. Hence all assumptions of (B) except the validity of (F) are satisfied.

For $0 \leqslant x \in R$ we denote by $\operatorname{int} x$ (the integral part of x) the greatest integer y with $y \leqslant x$.

Let $n \in \mathbb{N}$. We define $g_n \in G$ as follows. For each $i \in I$ we put

$$g_n(i) = \operatorname{int}\Big(\frac{1}{n} i\Big).$$

Then we have $g_1 > g_2 > \ldots > g_0$, where g_0 is the zero element of G. Thus (g_n) is a bounded sequence in G. No subsequence of (g_n) is convergent with respect to α. Hence G fails to be b-sequentially compact.

We conclude by remarking that for each infinite cardinal k there exists a convergence ℓ-group G such that G is b-sequentially compact and $\operatorname{card} G = k$. Indeed, let I be a set of indices with $\operatorname{card} I = k$ and for each $i \in I$ let $G_i = Z$; put $G_0 = \prod_{i \in I} G_i$. We denote by G the ℓ-subgroup of G consisting of all $g \in G_0$ such that the set $\{i \in I : g(i) \neq 0\}$ is finite. (In other words, G is a weak direct product of ℓ-groups G_i.) Then G satisfies the assumptions of (B) if we put $\alpha = \alpha(d)$. Hence the convergence ℓ-group G is b-sequentially compact. It is clear that $\operatorname{card} G = k$.

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