COMPARISON THEOREMS FOR DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

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Abstract. We are interested in comparing the oscillatory and asymptotic properties of the equations $L_n\,[x(t)-P(t)\,x(g(t))]+\delta\, f(t,x(h(t)))=0$ with those of the equations $M_n\,[x(t)-P(t)\,x(g(t))]+\delta\, Q(t)q(x(r(t)))=0$.

Keywords: neutral differential equations, oscillatory (nonoscillatory) solution, property $\mathcal{A}$, property $\mathcal{B}$, quasi-derivatives

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1. Introduction

We consider neutral differential equations of the form

$$L_n\left[x(t)-P(t)\,x(g(t))\right]+\delta\, f\big(t,x(h(t))\big)=0, \tag{A}$$

where $n\geqslant 2, \delta=+1$ or-1 and the operator L_n is defined recursively by

$$L_0\,u(t)=u(t),\quad L_k\,u(t)=\frac{1}{a_k(t)}\,[L_{k-1}\,u(t)]',\quad k=1,2,\dots,n,\ a_n=1.$$

The following conditions are assumed to hold throughout the paper:

(a) $a_i\in C\big[[t_0,\infty),(0,\infty)\big], t_0\geqslant 0$ and $\int_{t_0}^{\infty} a_i(t)\,\mathrm{d}t=\infty$, $i=1,2,\dots,n-1$;

(b) $P\in C\big[[t_0,\infty),\mathbb{R}\big]$ and satisfies $|P(t)|\leqslant\lambda$ on $[t_0,\infty)$ for some constant $\lambda<1$;

(c) $g\in C\big[[t_0,\infty),(0,\infty)\big]$ is increasing, $g(t)<t$ for $t\geqslant t_0$ and $\lim_{t\to\infty} g(t)=\infty$;

(d) $h\in C\big[[t_0,\infty),(0,\infty)\big]$ and $\lim_{t\to\infty} h(t)=\infty$;

(e) $f \in C\big[[t_0,\infty)\times\mathbb{R},\mathbb{R}\big]$ is nondecreasing in x for each $t \geqslant t_0$ and $\operatorname{sgn} f(t,x) = \operatorname{sgn} x$ for $(t,x)\in[t_0,\infty)\times\mathbb{R}$.

By a solution of (A) we mean a continuous function $x(t)\colon [T_x,\infty)\to\mathbb{R}$, $T_x \geqslant t_0$ such that $x(t)-P(t)\,x(g(t))$ has continuous quasi-derivatives $L_i[x(t)-P(t)\,x(g(t))]$, $0\leqslant i\leqslant n$, and $x(t)$ satisfies (A) for all sufficiently large $t\geqslant T_x$. Our attention is restricted to those solutions $x(t)$ of (A) which satisfy

$$\sup\{|x(t)|\colon t\geqslant T\}>0,\ \text{ for any } T\geqslant T_x.$$

Such a solution is said to be a proper solution. We make the standing hypothesis that (A) possesses proper solutions. A proper solution of (A) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory.

In recent years there has been a growing interest in the oscillation theory of functional differential equations of neutral type (see, for example, the papers [3–6, 8–10]). One of the first attempts at a systematic investigation of oscillatory properties of higher order neutral equations was the work of Ladas and Sficas [6].

The purpose of this paper is to obtain comparison theorems for (A). The results from the paper [1] are extended to neutral differential equations.

2. Classification of nonoscillatory solutions

We classify the possible nonoscillatory solutions of (A) in a similar way as in the paper [5].

Let $x(t)$ be a nonoscillatory solution of (A). From (A) and (e) it follows that the function

$$y(t)=x(t)-P(t)\,x(g(t)) \tag{1}$$

has to be eventually of constant sign, so that either

$$x(t)\,y(t)>0 \tag{2}$$

or

$$x(t)\,y(t)<0 \tag{3}$$

for all sufficiently large t. Assume first that (2) holds. Then the function $y(t)$ satisfies $\delta y(t)\,L_n y(t)<0$ eventually and the well-known Kiguradze's lemma (see [5]) implies that there is an integer $\ell\in\{0,1,\dots,n\}$ and a $t_1\geqslant t_0$ such that $(-1)^{n-\ell-1}\delta=1$ and for every $t\geqslant t_1$

$$\begin{aligned} y(t)\,L_i y(t)&>0, \quad 0\leqslant i\leqslant \ell,\\ (-1)^{i-\ell}y(t)\,L_i y(t)&>0, \quad \ell\leqslant i\leqslant n \end{aligned} \tag*{$(4)_\ell$}$$

holds.

A function $y(t)$ satisfying $(4)_\ell$ is said to be a nonoscillatory function of degree ℓ. The set of all solutions $x(t)$ of (A) satisfying (2) and $(4)_\ell$ will be denoted by $\mathcal{N}_\ell^+$. Now assume that (3) holds. Then $y(t)$ satisfies $(-\delta)y(t)\,L_n y(t) < 0$ for all large t and so it is a function of degree ℓ for some $\ell \in \{0,1,\dots,n\}$ with $(-1)^{n-\ell}\delta = 1$. The totality of nonoscillatory solutions $x(t)$ of (A) which satisfy (3) and $(4)_\ell$ will be denoted by $\mathcal{N}_\ell^-$. Consequently, if we denote by $\mathcal{N}$ the set of all possible nonoscillatory solutions of (A), then (see [5])

$$
\begin{aligned}
&\mathcal{N} = \mathcal{N}_1^+ \cup \mathcal{N}_3^+ \cup \dots \cup \mathcal{N}_{n-1}^+ \cup \mathcal{N}_0^- \ \text{ for } \delta = 1 \ \text{ and } n \text{ even},\\
&\mathcal{N} = \mathcal{N}_0^+ \cup \mathcal{N}_2^+ \cup \dots \cup \mathcal{N}_{n-1}^+ \qquad\quad \text{ for } \delta = 1 \ \text{ and } n \text{ odd},\\
&\mathcal{N} = \mathcal{N}_0^+ \cup \mathcal{N}_2^+ \cup \dots \cup \mathcal{N}_{n}^+ \qquad\qquad \text{ for } \delta = -1 \ \text{ and } n \text{ even},\\
&\mathcal{N} = \mathcal{N}_1^+ \cup \mathcal{N}_3^+ \cup \dots \cup \mathcal{N}_{n}^+ \ \ \cup \mathcal{N}_0^- \ \text{ for } \delta = -1 \ \text{ and } n \text{ odd}.
\end{aligned} \tag{5}
$$

The class $\mathcal{N}_0^-$ must be removed from (5) provided if $P(t)$ is either oscillatory or eventually negative, because in this case equation (A) cannot possess a nonoscillatory solution $x(t)$ satisfying (3).

It is now clear that the oscillation of all proper solutions of (A) is equivalent to the situation in which $\mathcal{N} = \emptyset$.

Definition 1. Equation (A) is said to have property $\mathcal{A}$ if for $\delta = 1$ and n even all proper solutions are oscillatory while for $\delta = 1$ and n odd $\mathcal{N} = \mathcal{N}_0^+$.

Definition 2. Equation (A) is said to have property $\mathcal{B}$ if for $\delta = -1$ and n even $\mathcal{N} = \mathcal{N}_0^+ \cup \mathcal{N}_n^+$ while for $\delta = -1$ and n odd $\mathcal{N} = \mathcal{N}_n^+$.

3. Comparison Theorems

We are interested in comparing the oscillatory and asymptotic properties of equations (A) with those of the equations

$$
M_n\left[x(t) - P(t)\,x(g(t))\right] + \delta\, Q(t) q\big(x(r(t))\big) = 0, \tag{B}
$$

where $n \geqslant 2$, $\delta = +1$ or -1,

$$
M_0\, u(t) = u(t), M_k\, u(t) = \frac{1}{b_k(t)}\,[M_{k-1}\, u(t)]',\ k = 1,2,\dots,n,\ b_n = 1
$$

and the following conditions are fulfilled:

$(\mathrm{a})_1$ $b_i \in C\big[[t_0,\infty),(0,\infty)\big]$, $t_0 \geqslant 0$ and $\int_{t_0}^{\infty} b_i(t)\,\mathrm{d}t = \infty$, $i = 1,2,\dots,n-1$;

$(\mathrm{d})_1$ $Q, r \in C\big[[t_0,\infty),(0,\infty)\big]$ and $\lim_{t\to\infty} r(t) = \infty$;

(e)$_1$ $q \in C[\mathbb{R},\mathbb{R}]$ is nondecreasing, $xq(x) > 0$ for $x \neq 0$ and

$$xy\,q(xy) \geqslant Kxy\,q(x)q(y) \text{ for each } x, y \ (0 < K = \text{constant});$$

(f) $h(t) \geqslant r(t)$ for $t \geqslant t_0$;
(g) $a_i(t) \geqslant b_i(t)$ for $t \geqslant t_0, 1 \leqslant i \leqslant n-1$.

The following notation will be needed:
$g^{-1}(t)$ is the inverse function of $g(t)$;

$$s = \max\left\{1, K\,q\left(\tfrac{1}{\lambda}\right)\right\};$$

$$\alpha(t) = \int_{t_0}^{t} a_1(z_1) \int_{t_0}^{z_1} a_2(z_2) \dots \int_{t_0}^{z_{n-2}} a_{n-1}(z_{n-1})\,\mathrm{d}z_{n-1} \dots \mathrm{d}z_1;$$

$$\beta(t) = \int_{t_0}^{t} a_1(z_1) \int_{t_0}^{z_1} a_2(z_2) \dots \int_{t_0}^{z_{n-3}} a_{n-2}(z_{n-2})\,\mathrm{d}z_{n-2} \dots \mathrm{d}z_1;$$

$$\mathbb{R}_0 = (-\infty, 0) \cup (0, \infty);$$

$$C(\mathbb{R}) = \{F : \mathbb{R} \to \mathbb{R} \mid F \text{ is continuous and } x\,F(x) > 0 \text{ for } x \neq 0\};$$

$$C_p(\mathbb{R}_0) = \{F \in C(\mathbb{R}) \mid F \text{ is of bounded variation on every iterval } [a,b] \subset \mathbb{R}_0\}.$$

Lemma 1. [9] *Suppose that $x(t)$ is a nonoscillatory solution of equation* (B).
i) *Let $P(t)$ be eventually positive and let $x(t)\,y(t) > 0$ ($y(t)$ is defined by* (1)*). Then $x(t)$ is a member of $\mathcal{N}_\ell^+$ if and only if $y(t)$ is a solution of degree ℓ of*

$$\left\{\delta\,M_n\,y(t) + Q(t)\,q\big(y(r(t))\big)\right\}\operatorname{sgn} y(t) \leqslant 0, \tag{6}$$

whereby

$$|y(t)| \leqslant |x(t)| \text{ for large } t. \tag{7}$$

ii) *Let $P(t)$ be eventually positive and let $x(t)\,y(t) < 0$. Then $x(t)$ is a member of $\mathcal{N}_0^-$ if and only if $v(t) = -y(t)$ is a solution of degree 0 of*

$$\left\{-\,\delta\,M_n v(t) + S\,Q(t)\,q\big(v(g^{-1}(r(t)))\big)\right\}\operatorname{sgn} v(t) \leqslant 0, \tag{8}$$

where $0 < S = K\,q\left(\frac{1}{\lambda}\right) =$ constant, whereby

$$\frac{1}{\lambda}\left|v(g^{-1}(t))\right| \leqslant |x(t)| \text{ for large } t. \tag{9}$$

iii) *Suppose that $P(t)$ is eventually negative or that $P(t)$ is oscillatory and satisfies*

$$P(t)\,P(g(t)) \geqslant 0 \text{ for large } t. \tag{10}$$

Then $x(t)$ is a member of $\mathcal{N}_\ell^+$ with $\ell \geqslant 1$ if and only if $y(t)$ is a solution of degree ℓ of

$$\Big\{\delta\, M_n\, y(t) + M\, Q(t)\, q\big(y(r(t))\big)\Big\} \operatorname{sgn} y(t) \leqslant 0, \tag{11}$$

where $0 < M = K\, q(1-\lambda) =$ constant, whereby

$$|x(t)| \geqslant (1-\lambda)\, |y(t)| \quad \text{for large } t. \tag{12}$$

Lemma 2. [7] *Suppose $F \in C(\mathbb{R})$. Then $F \in C_p(\mathbb{R}_0)$ if and only if $F(x) = G(x)\, H(x)$ for all $x \in \mathbb{R}_0$, where $G\colon \mathbb{R}_0 \to (0,\infty)$ is nondecreasing on $(-\infty, 0)$ and nonincreasing on $(0,\infty)$ and $H\colon \mathbb{R}_0 \to \mathbb{R}$ is nondecreasing on $\mathbb{R}_0$.*

Remark. G, H are called a pair of continuous components of F.

We also assume that there exists a continuous function $Z\colon [t_0,\infty) \to [0,\infty)$ and $F \in C_p(\mathbb{R}_0)$ such that

$$f(t,x) \operatorname{sgn} x \geqslant Z(t)\, F(x) \operatorname{sgn} x \quad \text{for } (t,x) \in [t_0,\infty) \times \mathbb{R}. \tag{13}$$

In the following two comparison theorems we compare equation (A) with the special cases of equation (B), namely, when $M_n = L_n$ and $h = r$.

Theorem 1. *Let $\delta = 1$. Suppose that (13) holds and let G and H be a pair of continuous components of F with H being the nondecreasing one.*

i) *Assume that $P(t)$ is eventually negative or that $P(t)$ is oscillatory and satisfies (10). Then the conditions*

$$H((1-\lambda)\, x) \operatorname{sgn} x \geqslant q(x) \operatorname{sgn} x \quad \text{for } x \in \mathbb{R}, \tag{14}$$

$$b_i(t) \equiv a_i(t), \; 1 \leqslant i \leqslant n-1, \tag{15}$$

$$h(t) = r(t), \tag{16}$$

$$Z(t)\, G\big(\pm (1-\lambda)\, c\, \alpha(h(t))\big) \geqslant M\, Q(t) \quad \text{for every large } c > 0 \text{ and all large } t \tag{17}$$

(where $M = K\, q\,(1-\lambda)$) imply that equation (A) has property $\mathcal{A}$ if equation (B) has property $\mathcal{A}$.

ii) *Assume that $P(t)$ is eventually positive. Then the conditions (15), (16)*

$$H(x) \operatorname{sgn} x \geqslant q(x) \operatorname{sgn} x \quad \text{for } x \in \mathbb{R}, \tag{18}$$

$$Z(t)\, G\big(\pm\, c\, \alpha(h(t))\big) \geqslant s\, Q(t) \quad \text{for every large } c > 0 \text{ and all large } t \tag{19}$$

imply that equation (A) has property $\mathcal{A}$ if equation (B) has property $\mathcal{A}$.

P r o o f. We present the proof for n even.

i) According to (5), $\mathcal{N}_\ell^+, \ell \in \{1,3,\dots,n-1\}$ and $\mathcal{N}_0^-$ are the possible classes of nonoscillatory solutions of (A) with $\delta = 1$ and even n. In the case when $P(t)$ is eventually negative or oscillatory, $\mathcal{N}_0^-$ is necessarily empty. Suppose that $\mathcal{N}_\ell^+ \neq 0$ for some $\ell \in \{1,3,\dots,n-1\}$ and let $x \in \mathcal{N}_\ell^+$ be a solution of (A). Without loss of generality we may assume that x is eventually positive. Then from $(4)_\ell$ we observe that

$$L_{n-1}y(t) > 0 \text{ and } L_n y(t) < 0 \quad \text{for all large } t.$$

Thus,

$$L_{n-1}y(t) \leqslant c_1, \quad c_1 > 0$$

and hence there exists a $c > 0$ such that

$$y(t) \leqslant c\,\alpha(t) \text{ for all large } t$$

and in view of (d) we have

$$y(h(t)) \leqslant c\,\alpha(h(t)) \text{ for all sufficiently large } t.$$

Now, by conditions (e), (12), (13), (14), (17) and Lemma 2 we get

$$\begin{aligned} f\big(t, x(h(t))\big) &\geqslant f\big(t, (1-\lambda)\,y(h(t))\big) \geqslant Z(t)\,F\big((1-\lambda)\,y(h(t))\big)\\ &= Z(t)\,G\big((1-\lambda)\,y(h(t))\big)\,H\big((1-\lambda)\,y(h(t))\big)\\ &\geqslant Z(t)\,G\big((1-\lambda)\,c\,\alpha(h(t))\big)\,H\big((1-\lambda)\,y(h(t))\big)\\ &\geqslant M\,Q(t)\,H\big((1-\lambda)\,y(h(t))\big) \geqslant M\,Q(t)q\big(y(h(t))\big) \end{aligned} \tag{20}$$

and hence the function y which is of degree ℓ is a solution of the differential inequality (11), in which (15) and (16) hold.

On the other hand, Lemma 1 implies that differential inequality (11), in which (15) and (16) hold, has a solution of degree $\ell \geqslant 1$ if and only if equation (B) with $M_n = L_n$ and $h = r$, namely, the equation

$$L_n\big[x(t) - P(t)\,x(g(t))\big] + \delta\,Q(t)\,q\big(x(h(t))\big) = 0, \tag{21}$$

has a solution of degree ℓ. We supposed $1 \leqslant \ell \leqslant n-1$ and this contradicts the hypothesis that equation (21) is oscillatory.

ii) Let $\mathcal{N}_\ell^+ \neq \emptyset$ for some $\ell \in \{1,3,\dots,n-1\}$. Without loss of generality we may assume that x is eventually positive. Therefore similarly as above, by conditions (e), (7), (13), (18), (19) and Lemma 2 we get

$$\begin{aligned} f\big(t, x(h(t))\big) &\geqslant f\big(t, y(h(t))\big) \geqslant Z(t)\,F\big(y(h(t))\big)\\ &= Z(t)\,G\big(y(h(t))\big)\,H\big(y(h(t))\big)\\ &\geqslant Z(t)\,G\big(c\,\alpha(h(t))\big)\,H\big(y(h(t))\big)\\ &\geqslant Q(t)\,H\big(y(h(t))\big) \geqslant Q(t)q\big(y(h(t))\big). \end{aligned} \tag{22}$$

One can see that the function y which is of degree $\ell \in \{1,3,\dots,n-1\}$ is a solution of the differential inequality (6) in which $M_n = L_n$ and $r = h$.

Applying Lemma 1 we conclude that (21) has a solution of degree ℓ. This is a contradiction.

Suppose that $\mathcal{N}_0^- \neq \emptyset$. In this case $x(t)\,y(t) < 0$. Because $0 < \lambda < 1$ and H is nondecreasing, from (18) we obtain

$$H\left(\frac{1}{\lambda}\,x\right)\operatorname{sgn} x \geqslant q(x)\,\operatorname{sgn} x. \tag{23}$$

Next, without loss generality, we may assume that x is eventually positive. Then, because $\ell = 0$, we observe from $(4)_\ell$ that

$$L_0 y(t) < 0 \text{ and } L_1 y(t) > 0 \text{ for all large } t.$$

Thus,

$$y(t) \geqslant -c, \qquad c > 0,$$

or

$$-y(t) = v(t) \leqslant c \text{ for all large } t.$$

Now, by conditions (a), (e), (9), (13), (19), (23) and Lemma 2 we get

$$\begin{aligned} f\big(t, x(h(t))\big) &\geqslant f\Big(t, \frac{1}{\lambda}\,v\big(g^{-1}(h(t))\big)\Big) \geqslant Z(t)\,F\Big(\frac{1}{\lambda}\,v\big(g^{-1}(h(t))\big)\Big) \\ &= Z(t)\,G\Big(\frac{1}{\lambda}\,v\big(g^{-1}(h(t))\big)\Big)\,H\Big(\frac{1}{\lambda}\,v\big(g^{-1}(h(t))\big)\Big) \\ &\geqslant Z(t)\,G(c\,\alpha(h(t))\,H\Big(\frac{1}{\lambda}\,v\big(g^{-1}(h(t))\big)\Big) \\ &\geqslant S\,Q(t)\,H\Big(\frac{1}{\lambda}\,v\big(g^{-1}(h(t))\big)\Big) \geqslant S\,Q(t)q\Big(v\big(g^{-1}(h(t))\big)\Big) \end{aligned}$$

for sufficiently large t. Therefore similarly as above, applying Lemma 1 we get a contradiction. The proof in the case when n is odd is similar and will be omitted.

□

Theorem 2. *Let $\delta = -1$. Suppose that* (13) *holds and let G and H be a pair of continuous components of F with H being the nondecreasing one.*

i) *Assume that $P(t)$ is eventually negative or that $P(t)$ is oscillatory and satisfies* (10). *If* (14), (15), (16) *and*

$$Z(t)\,G(\pm(1-\lambda)\,c\,\beta(h(t))) \geqslant M\,Q(t) \text{ for every large } c > 0 \text{ and all large } t \tag{25}$$

hold, then equation (A) *has property* $\mathcal{B}$ *if equation* (B) *has property* $\mathcal{B}$.

ii) *Assume that* $P(t)$ *is eventually positive. Then the conditions* (15), (16), (18) *and*

$$(26) \qquad Z(t)\,G\big(\pm c\,\beta(h(t))\big) \geqslant s\,Q(t) \text{ for every large } c>0 \text{ and all large } t$$

imply that equation (A) *has property* $\mathcal{B}$ *if equation* (B) *has property* $\mathcal{B}$.

Proof of Theorem 2 is similar to that of Theorem 1 and will be omitted. □

The following theorems are intended to relax conditions (15), (16) in the previous result.

Theorem 3. *Let* $\delta = 1$ *and let* G, H *be a pair of continuous components of* F *with* H *being the nondecreasing one. Suppose that* (13), (14) *hold.*

i) *Assume that* $P(t)$ *is eventually negative or that* $P(t)$ *is oscillatory and satisfies* (10). *Then the condition* (17) *implies that equation (A) has property* $\mathcal{A}$ *if equation* (B) *has property* $\mathcal{A}$.

ii) *Assume that* $P(t)$ *is eventually positive. Then the condition* (19) *implies that equation* (A) *has property* $\mathcal{A}$ *if equation* (B) *has property* $\mathcal{A}$.

Proof. Let n be even. i) Let equation (B) have propery $\mathcal{A}$. By Lemma 1 inequality (11) has property $\mathcal{A}$ and by Theorem 1 in [11] inequality (11) with $M_n = L_n$ and $r = h$ has property $\mathcal{A}$ as well. Theorem 1 now shows that equation (A) has property $\mathcal{A}$.

The proof in the other cases can be done in an analogous way, so we omit it. □

Theorem 4. *Let* $\delta = -1$ *and let* G, H *be a pair of continuous components of* F *with* H *being the nondecreasing one. Suppose that* (13), (14) *hold.*

i) *Assume that* $P(t)$ *is eventually negative or that* $P(t)$ *is oscillatory and satisfies* (10). *Then the condition* (25) *implies that equation* (A) *has property* $\mathcal{B}$ *if equation* (B) *has property* $\mathcal{B}$.

ii) *Assume that* $P(t)$ *is eventually positive. Then the condition* (26) *implies that equation* (A) *has property* $\mathcal{B}$ *if equation* (B) *has property* $\mathcal{B}$.

Proof of Theorem 4 is similar to that of Theorem 3 and we omit it. □

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